Math1024 Answer to Homework 8

Exercise 4.2.20 (2)

$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \dfrac{n^{\frac{p}{n}}}{\log n} = 0$. The series diverges.

Exercise 4.2.20 (4)

$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{(a^n(1+(\frac{b}{a})^n))^p} = \lim_{n\to\infty} a^p(1+(\frac{b}{a})^n)^{\frac{p}{n}} = \begin{cases} a^p, & a \geq b \\ b^p, & b > a \end{cases}$

Thus when $a \geq b$, the series converges if $a^p < 1$ and when $a < b$, the series converges if $b^p < 1$.

Exercise 4.2.20 (10)

$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty}(1+\dfrac{a}{n})^{\frac{2n-n^2}{n}} = \lim_{n\to\infty} \dfrac{1}{(1+\dfrac{a}{n})^{n-2}} = \dfrac{1}{e^a} < 1$. The series converges.

Exercise 4.2.21(3)

We need to use Raabe test here to determine the convergence of the series:

$$\lim_{n\to\infty} n(1-|\frac{a_{n+1}}{a_n}|) = \lim_{n\to\infty} \frac{2n}{3n+4} = \frac{2}{3} < 1$$

.

So the series diverges.

Exercise 4.2.22 (5)

$$\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = \lim_{n\to\infty} \frac{a_1+b_1n+c_1n^2}{a_2+b_2n+c_2n^2} = \frac{c_1}{c_2}$$

Thus the series converges when $\dfrac{c_1}{c_2} < 1$ and diverges when $\dfrac{c_1}{c_2} > 1$.

Exercise 4.2.23 (3)

By $\dfrac{a_{n+1}}{a_n} = \dfrac{(n+1)(2n+2)(2n+1)}{(3n+3)(3n+2)(3n+1)}r \to \dfrac{4}{27}r$ and the ratio test, the series converges for $|r| < \dfrac{27}{4}$ and diverges for $|r| > \dfrac{27}{4}$. Moreover, if $|r| = \dfrac{27}{4}$, then $\left|\dfrac{a_{n+1}}{a_n}\right| = \dfrac{(n+1)(2n+2)(2n+1)}{(3n+3)(3n+2)(3n+1)}\dfrac{27}{4} > 1$. Therefore $|a_n|$ is increasing and cannot converge to 0. Therefore the series also diverges.

We conclude that the series converges if and only if $|r| < \dfrac{27}{4}$.

Exercise 4.2.23 (8)

By $\dfrac{a_{n+1}}{a_n} = \dfrac{(2n+2)(2n+1)n^{2n}}{(n+1)^{2n+2}}r \to \dfrac{4}{e^2}r$ and the ratio test, the series converges for $|r| < \dfrac{e^2}{4}$ and diverges for $|r| > \dfrac{e^2}{4}$. Moreover, if $|r| = \dfrac{e^2}{4}$, then

We conclude that the series converges if and only if $|r| < \dfrac{e^2}{4}$.

Exercise 4.3.1 (3)

The series is $\sum(-1)^{n+1}a_n = \sum(-1)^{n+1}\dfrac{2\cdot 5\cdot 8\cdots(3n-1)}{4\cdot 7\cdot 10\cdots(3n+1)}$. By $\dfrac{a_n}{a_{n-1}} = \dfrac{3n-1}{3n+1}$, we see a_n decreases. By

$$\frac{a_n}{a_{n-1}} = 1 - \frac{2}{3n+1} < 1 - \frac{\frac{1}{2}}{n} + o\left(\frac{1}{n}\right) = \frac{\sqrt{n-1}}{\sqrt{n}},$$

and $\lim \dfrac{1}{\sqrt{n}} = 0$, we conclude that $\lim a_n = 0$. Then by the Leibniz rule, we conclude that the series converges.

Exercise 4.3.2 (1)

Denote $k_n = \frac{a(a+1^r)\cdots(a+n^r)}{b(b+1^r)\cdots(b+n^r)}$, necessary conditions for series,

$$\sum_{n=0}^{\infty}(-1)^n k_n$$

to converge require $k_n \to 0$.

$$k_n = \prod_{t=0}^{n}\frac{a+t^r}{b+t^r} = \prod_{t=0}^{n}(1+\frac{a-b}{b+t^r}),$$

First, we exclude the naive cases that some factors equal zero, i.e., $a = -n_0^r$ for some n_0. It is clear under such circumstance, the series converge absolutely.(only finitely many terms are nonzero.) Also we assume $b \neq -n^r$ for any n.

Then, $k_n \to 0$ requires: $b > a$ if $r > 0$; $|a+1| < |b+1|$ if $r = 0$; $|a| < |b|$ or $a = -b$ if $r < 0$.

1. $r > 0$: $0 < 1 + \frac{a-b}{b+t^r} < 1$ finally, since we are discussing the convergence of the series, the first finitely many terms in each k_n make no contribution, we can therefore assume without loss of generosity $0 < 1 + \frac{a-b}{b+t^r} < 1$ for all t.

   $$\log k_n = \sum_{t=0}^{n}\log(1+\frac{a-b}{b+t^r})$$

   $k_n \to 0$ is equivalent to $-\log k_n \to \infty$.

   $$-\log k_n = \sum_{t=0}^{n}\log(1+\frac{b-a}{a+t^r})$$

   $\log(1+\frac{b-a}{a+t^r}) \approx \frac{b-a}{a+t^r} \approx t^{-r}$ as $t \to \infty$, here the symbol $\approx$ means comparable. We can thereby deduce equivalence of $k_n \to 0$ and $r \leq 1$. This is also the necessary and sufficient condition for the series to converge conditionally.

   If $r = 1$, $-\log k_n \approx \log n$ or $k_n \approx \frac{1}{n}$, the series does not converge absolutely. If $r < 1$, $-\log k_n \approx n^{1-r}$ or $k_n = e^{-n^{1-r}}$, the series converges absolutely.

2. $r = 0$: It is a geometric series which converges if $|a+1| < |b+1|$ both conditionally and absolutely.

3. $r < 0$:

(a) $a = -b$, assume without loss of generosity, $\frac{2b}{b+t^r} - 1 > 0$ for all t. $k_n = \prod_{t=0}^{n}(1 - \frac{2b}{b+t^r}) = (-1)^n \prod_{t=0}^{n}(\frac{2b}{b+t^r} - 1)$, we get $b > 0$ to make $\frac{2b}{b+t^r} - 1 < 1$, then

$$\log \prod_{t=0}^{n}(\frac{2b}{b+t^r} - 1) = \sum_{t=0}^{n} \log(\frac{2b}{b+t^r} - 1).$$

We know

$$\log(\frac{2b}{b+t^r} - 1) = \log((1 + \frac{2b}{b+t^r} - 2)) \approx \frac{2b}{b+t^r} - 2 = \frac{-2t^r}{b+t^r} \approx -t^r.$$

Since $(-1)^n k_n = (-1)^{2n} \prod_{t=0}^{n}(\frac{2b}{b+t^r} - 1) = \prod_{t=0}^{n}(\frac{2b}{b+t^r} - 1)$, we know under this situation, the series is positive, conditional convergence is the same as absolutely convergence.

Similar to the last section of case 1, the series converges for $r > -1$ absolutely.

(b) $|a| < |b|$, we may first assume $0 < \frac{a}{b} < 1$. Since $\frac{a+t^r}{b+t^r} \to \frac{a}{b}$ as $t \to \infty$, we have $0 < \frac{a+t^r}{b+t^r} \leq (1 - \epsilon)$ for t sufficiently large. Hence the series is controlled by a convergent geometric series, the series is convergent absolutely. The case when $-1 < \frac{a}{b} < 0$ is similar.

In summary,

Conditional convergence: $0 < r \leq 1, b > a$ or $r = 0, |a+1| < |b+1|$ or $-1 < r < 0, a = -b$ or $r < 0, |a| < |b|$.

Absolute convergence: $0 < r < 1, b > a$ or $r = 0, |a+1| < |b+1|$ or $-1 < r < 0, a = -b$ or $r < 0, |a| < |b|$.

Exercise 4.3.3 (2)

We have

$$\frac{n^2 + \sin n}{(-1)^n n^3 + n + 2} - (-1)^n \frac{1}{n} = \frac{n \sin n - (-1)^n (n+2)}{n((-1)^n n^3 + n + 2)} = b_n.$$

By comparing with $\sum \frac{1}{n^3}$, it is easy to see that $\sum b_n$ absolutely converges. Therefore the (absolute or conditional) convergences of $\sum \frac{n^2 + \sin n}{(-1)^n n^3 + n + 2}$ and $\sum (-1)^n \frac{1}{n}$ are the same. Since $\sum (-1)^n \frac{1}{n}$ conditionally converges, we conclude that $\sum \frac{n^2 + \sin n}{(-1)^n n^3 + n + 2}$ conditionally converges.

Exercise 4.3.3 (5)

The series is $\sum (-1)^n a_n$ with $a_n = \frac{r^n}{n^p (\log n)^q}$. By $\sqrt[n]{|a_n|} \to |r|$, we know the series absolutely converges for $|r| < 1$ and diverges for $|r| > 1$.

For $|r| = 1$, by Exercise 4.2.8 (2), the series absolutely converges if and only if $p > 1$ or $p = 1$ and $q > 1$.

It remains to consider $r = -1$. By the Leibniz rule, the series converges for $p > 0$ or $p = 0$ and $q > 0$.

In summary, the series absolutely converges in the following cases: (1) $|r| < 1$; (2) $|r| = 1$, $p > 1$; (2) $|r| = 1$, $p = 1$, $q > 1$. The series conditionally converges in the following cases: (1) $r = -1$, $p > 0$; (2) $r = -1$, $p = 0$, $q > 0$.

Exercise 4.3.3 (9)

1) If $a \neq 0$, then $a > 0$. When n is large enough, $an^2 + bn + c > 0$.

If $a > 1$, then

$$\frac{|a_n|}{\frac{1}{n^2}} = \frac{n_{n+p}}{|an^2 + bn + c|} = \left(\frac{n}{\sqrt{an^2 + bn + c}}\right)^n \cdot \frac{n^{p+2}}{an^2 + bn + c)^q} \to 0.$$

the series absolutely converges.

If $0 < a < 1$, then

$$|a_n| = \left(\frac{n}{\sqrt{an^2 + bn + c}}\right)^n \cdot \frac{n^p}{an^2 + bn + c)^q} \to \infty$$

so the series is diverge.

If $a = 1$, for any p, q, we can select k such that $p + k = 2q$, then

$$\frac{|a_n|}{\frac{1}{n^2}} = \frac{n_{n+p}}{|an^2 + bn + c|} = \left(\frac{n}{\sqrt{an^2 + bn + c}}\right)^n \cdot \frac{n^{p+k}}{(an^2 + bn + c)^q} \to 1.$$

so the series absolutely converges if and only if $2q - p > 1$. Note that under condition $2q - p \leqslant 1$,

$$|a_n| = \left(\frac{n}{\sqrt{an^2 + bn + c}}\right)^n$$

decreasingly goes to a constant.

When $p > 2q$, a_n doesn't go to 0, so the series diverges. When $0 < 2q - p < 1$, n is large enough,

$$\frac{n^{p+k}}{(an^2 + bn + c)^q}$$

decreasingly goes to 0. A positive decreasing sequence multiplies the other one positive decreasing sequence will result a decreasing sequence, then a_n decreasingly goes to 0, then by Dirichlet test, the series conditionally converges.

2) If $a = 0$, then $b > 0$.

If $b > 1$, when n is large enough

$$|a_n| = \frac{n^{n+p}}{(bn + c)^{\frac{n}{2}+q}} = \left(\frac{n}{\sqrt{bn + c}}\right)^n \cdot \frac{n^p}{(bn + c)^q}$$

so for any $p, q, c > 0$,

$$\lim_{n\to\infty} \frac{|a_n|}{\frac{1}{n^2}} = 0$$

Similarily as in the case of $a \neq 0$, when $0 < b < 1$, $|a_n| \to \infty$, the series dieveges.

When $b = 1$, the series absolutely converges if and only if $q - p > 1$. If $q - p \leqslant 1$,

$$\frac{1}{\sqrt{(b + \frac{c}{n})^n}}$$

decreasingly goes to constant $\frac{1}{\sqrt{e^c}}$. The same reason will show that when $0 < q - p \leqslant 1$, the series converges conditionally, and when $q \leqslant p$, the series diverges.

Exercise 4.3.6

$\forall q > p$,

$$\sum \frac{a_n}{n^q} = \sum \frac{a_n}{n^p} \cdot \frac{1}{n^{q-p}}.$$

Since $\sum \frac{a_n}{n^p}$ converges, it is bounded. So by Dirichlet test, $\sum \frac{a_n}{n^q}$ converges.

Exercise 4.3.7 (2)

This exercise is similar to the Example 4.2.7 and 4.3.5.

If $0 < a < \frac{\pi}{2}$, then there exists subsequence $\{n_k\}$ such that $k\pi - \frac{\pi}{4} < n_k a < k\pi + \frac{\pi}{4}$, then

$$\frac{|\cos na|}{n+b} > \frac{1}{\sqrt{2}} \frac{1}{n_k + b}$$

whose right hand side diverges, so the original series is not absolutely convergent.

For $a \notin (0, \frac{\pi}{2})$, we can find $b \in (0, \frac{\pi}{2})$ such that $a + b$ or $a - b$ is integer multiple of π, then $|\cos na| = |\cos nb|$, then still we get that the series diverges.

But the series conditionally converges by Dirichlet test as follows. Obviously $\frac{1}{n+b}$ decreasingly goes to 0. Let's show the partial sum

$$\left| \sum_{k=1}^{m} (-1)^k \cos na \right|$$

is bounded.

Let

$$S_n = \cos a + \cos 2a + \cdots + \cos na.$$

Note that

$$\begin{aligned}
&\sin \frac{a}{2}(\cos a + \cos 2a + \cdots + \cos na \\
&= \sin\left(a + \frac{a}{2}\right) - \sin\left(a - \frac{a}{2}\right) + \sin\left(2a + \frac{a}{2}\right) - \sin\left(2a - \frac{a}{2}\right) + \cdots \\
&+ \sin\left(na + \frac{a}{2}\right) - \sin\left(na - \frac{a}{2}\right) \\
&= \sin\left(na + \frac{a}{2}\right) - \sin\left(a - \frac{a}{2}\right)
\end{aligned}$$

that is,

$$S_n = \frac{\sin\left(na + \frac{a}{2}\right) - \sin\left(a - \frac{a}{2}\right)}{\sin\frac{1}{a}},$$

and

$$|S_n| \leqslant \frac{1}{\left|\sin\frac{1}{a}\right|}.$$

Exercise 4.3.8(1)

When n is odd,

$$\frac{1}{|n^2 + (-1)^n n^2|} = \frac{1}{|n - n^2|} = \frac{1}{n^2 - n} = \frac{1}{n(n-1}.$$

When n is even,

$$\frac{1}{|n^2 + (-1)^n n^2|} = \frac{1}{n + n^2} < \frac{1}{n^2} < \frac{1}{n(n-1}.$$

Therefore we always have

$$\sum \frac{1}{|n^2 + (-1)^n n^2|} < \sum \frac{1}{n(n-1)}.$$

So the series is absolutely convergent.

Exercise 4.3.8(4)

Note that the series starts from $n = 2$, always $\sqrt{n} + (-1)^n > 0$

$$\frac{\left|\frac{(-1)^{\frac{n(n-1)}{2}}}{\sqrt{n}+(-1)^n}\right|}{\frac{1}{\sqrt{n}}} = \frac{\sqrt{n}}{\sqrt{n} + (-1)^n} \to 1$$

So the series is not absolutely converges.

Take out the odd terms, that is $n = 2k + 1$, we get a series

$$\sum O_k = \sum \frac{(-1)^k}{\sqrt{2k+1} - 1}.$$

Take out the even terms, that is $n = 2k$, we get a series

$$\sum E_k = \sum \frac{(-1)^k}{\sqrt{2k} + 1}.$$

By Dirichlet test, both of these two series conditionally converges

For the original series, let S_N be its partial sum.Then

$$S_{2k} = \sum_{m=1}^{k} (O_m + E_m),$$

and

$$S_{2k+1} = S_{2k} + O_{k+1}$$

Hence the following limits exists and equal

$$\lim S_{2k+1} = \lim S_{2k}$$

Then the original series converges.

EXERCISE 4.3.9(1)

We have

$$\sin\sqrt{n^2+a}\pi = (-1)^n \sin(\sqrt{n^2+a}\pi - n\pi) = (-1)^n \sin\frac{a\pi}{\sqrt{n^2+a}+n}.$$

Since $\sin\frac{a\pi}{\sqrt{n^2+a}+n}$ is decreasing and converge to 0, the series $\sum \sin\sqrt{n^2+a}\pi$ converges. Moreover, we have

$$\lim |na_n| = \lim\left|n\frac{a\pi}{\sqrt{n^2+a}+n}\right| = \frac{a\pi}{2}.$$

By the divergence of $\sum\frac{1}{n}$. the series $\sum |a_n|$ diverges unless $a = 0$. Therefore the series conditionally converges.

EXERCISE 4.3.10

Please see the attached picture.

EXERCISE 4.3.12

By Dirichlet test,

$$\sum \frac{(-1)^n}{\sqrt{n}}$$

converges.

The new series is

$$\sum_{k=1}^{\infty}\left(\frac{1}{\sqrt{2k+1}} - \frac{1}{\sqrt{4k+2}} - \frac{1}{\sqrt{4k+4}}\right)$$

Note that

$$\frac{1}{\sqrt{2k+1}} - \frac{1}{\sqrt{4k+2}} - \frac{1}{\sqrt{4k+4}} = \frac{1}{\sqrt{2k+1}}\left(1 - \frac{1}{\sqrt{2}} - \sqrt{\frac{2k+1}{4k+1}}\right)$$

and

$$1 - \frac{1}{\sqrt{2}} - \sqrt{\frac{2k+1}{4k+1}}$$

is decreasing to a negative constant. So when k is large enough, the new series is a negative series. By compare test,

$$\frac{\frac{1}{\sqrt{2k+1}}\left(1-\frac{1}{\sqrt{2}}-\sqrt{\frac{2k+1}{4k+1}}\right)}{\frac{1}{\sqrt{k}}} \to \frac{1}{\sqrt{2}}\left(1-\frac{1}{\sqrt{2}}-\frac{1}{2}\right) \neq 0$$

so the series diverges.

Exercise 4.3.14

Let the partial sum of $\sum \frac{(-1)^n}{\sqrt{n}}$ is A_n, and set $A_0 = 0$

1. Square arrangement:

The partial sum is

$$\begin{aligned} S &= \sum_{k=1}^{n} (a_k(A_k + A_{k-1})) \\ &= \sum_{k=1}^{n} ((A_k - A_{k-1})(A_k + A_{k-1}) \\ &= \sum_{k=1}^{n} (A_k^2 - A_{k-1}^2) \\ &= A_n^2 - A_0^2 = A_n^2 \to l^2, \end{aligned}$$

when $l \to \infty$.

2. Diagonal arrangement:

The series is

$$\sum_n c_n$$

where

$$c_n = (-1)^{n-1} \left(\frac{1}{1 \cdot \sqrt{n}} + \frac{1}{\sqrt{2(n-1)}} + \cdots + \frac{1}{\sqrt{(n-1)2}} + \frac{1}{\sqrt{n} \cdot 1} \right)$$

Since every term in the bracket is larger than $\frac{1}{n}$, $|c_n| > 1$, so the series diverges.