Math1024 Answer to Homework 9

Exercise 4.4.3 (2)

$\frac{1}{\lim \sqrt[n]{n^p}} = 1$. The series converges for $|x-1| < 1$, diverges for $|x-1| > 1$. The radius of converges is 1.

Exercise 4.4.3 (3)

$\frac{1}{\lim \sqrt[n]{n^p}} = 1$. The series converges for $|2x-1| < 1$, diverges for $|2x-1| > 1$. The radius of converges is $\frac{1}{2}$.

Exercise 4.4.3 (9)

$\lim \left| \frac{a_n}{a_{n+1}} \right| = \sqrt{\frac{(n+1)!}{n!}} = \sqrt{n+1} \to \infty$.
$R = +\infty$.

Exercise 4.4.3 (10)

$\lim \frac{\frac{((n+1)!)^2}{(2n+2)!}}{\frac{(n!)^2}{(2n)!}} = \lim \frac{(n+1)^2}{(2n+2)(2n+1)} = \frac{1}{4}$. The radius of convergence is 4.

Exercise 4.4.3 (14)

$$\frac{a_{n+1}}{a_n} = \frac{2^{n+1}|x|^{(n+1)^2-1}}{2^n|x|^{n^2-1}} = 2|x|^{2n+1}$$

Only when $|x| < 1$, the series absolutely converges, so the radius of convergence is 1.

Exercise 4.4.4 (2)

By $\lim \sqrt[n]{\left(\frac{n+1}{n}\right)^{n^2}} = \lim \left(1 + \frac{1}{n}\right)^n = e$, the radius of convergence is e^{-1}.

Exercise 4.4.5

$$\frac{a_{n+1}}{a_n} = \frac{\frac{|x|^{2n+3}}{(n+1)!(n+2)!2^{2n+3}}}{\frac{|x|^{2n+1}}{(n)!(n+1)!2^{2n+1}}} = \frac{|x|}{4(n+1)(n+2)} \to 0(n \to \infty)$$

So the radius of convergence is ∞.

Exercise 4.4.6

$$\frac{a_{n+1}}{a_n} = \frac{\frac{|x|^{3n+3}}{(3n+2)!!!(n+2)!(3n+3)!!!}}{\frac{|x|^{3n}}{(3n-1)!!!(3n)!!!}} = \frac{|x|^3}{(3n+2)(3n+3)} \to 0(n \to \infty)$$

So the radius of convergence is ∞.

Exercise 4.4.7

$\sum(a_n+b_n)x^n$ converges for $|x|\le \min\{R,R'\}$.
$\sum(a_n-b_n)x^n$ converges for $|x|\le \min\{R,R'\}$.
$\sum(-1)^n a_n x^n$ has the same radius R as $\sum a_n x^n$.
$\sum a_n(2x-1)^n$ converges for $|2x-1|<R$ and diverges for $|2x-1|>R$. The radius of convergence is $\frac{R}{2}$.
$\sum a_n x^{2n}$ converges for $|x^2|<R$ and diverges for $|x^2|>R$. The radius of convergence is $\sqrt{R}$.
For $\sum a_n x^{n+2}$,

$$\frac{|a_{n+1}||x|^{n+3}}{|a_n||x|^{n+2}} = \frac{a_{n+1}}{a_n}|x|$$

then the radius of convergence is R.
$\sum a_{2n}x^n$ absolutely converges when

$$\lim \sqrt[n]{|a_{2n}|}|x| < 1$$

Wand when the limit >1, the series diverges. so the radius of convergence is R^2.
$\sum a_{n+2}x^n = x^2\sum a_n x^n$ has the same radius R as $\sum a_n x^n$.
For $\sum a_n x^{n^2}$, if $R\ne 0,+\infty$, when $|x|<1$, $\sqrt[n]{|a_n|}|x|^n\to 0$, and when $|x|>1$, $\sqrt[n]{|a_n|}|x|^n\to\infty$, so the radius of convergence is 1.
For $\sum a_{n^2}x^n$, also by compare test, the radius of convergence is

$$\lim \frac{1}{\sqrt[n]{|a_{n^2}|}}$$

So when $R>1$, the radius is ∞; when $R=1$, the radius is 1; when $R<1$, the radius is 0.
For $\sum a_{2n}x^{2n}$,

$$\frac{|a_{2n+2}||x|^{2n+2}|}{|a_{2n}||x|^{2n}} = \frac{|a_{2n+2}|}{|a_{2n+1}|}\cdot\frac{|a_{2n+1}|}{|a_{2n}|}\cdot|x|^2 \to \frac{|x|^2}{R^2}$$

so the radius of convergence is R.
For $\sum a_{n^2}x^{n^2}$, by the root test

$$\sqrt[n]{|a_{n^2}|}|x|^n = |a_{n^2}|^{\frac{1}{n^2}\cdot n}|x|^n$$

You can check when $R^{-1}|x|<1$, the limit above is 0; when $R^{-1}|x|=1$, the limit above is 1; when $R^{-1}|x|>1$, the limit above is ∞. So the radius of convergence is R.

Exercise 4.4.9 (3)

Substituting x by x^2 in the Taylor expansion of $\sin x$, we get

$$\sin x^2 = x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \cdots + (-1)^n\frac{x^{2n+1}}{(2n+1)!} + \cdots .$$

The radius of convergence is $+\infty$.

Exercise 4.4.9 (6)

$$\sin 2x = \sin\left(2\left(x-\frac{\pi}{2}\right)+\pi\right) = -\sin 2\left(x-\frac{\pi}{2}\right)$$
$$= -\left(x-\frac{\pi}{2}\right) + \frac{2^3}{3!}\left(x-\frac{\pi}{2}\right)^3 - \frac{2^5}{5!}\left(x-\frac{\pi}{2}\right)^5 + \cdots + (-1)^{n+1}\frac{2^{2n+1}}{(2n+1)!}\left(x-\frac{\pi}{2}\right)^{2n+1} + \cdots .$$

The radius of convergence is $+\infty$.

Exercise 4.4.12

The Airy function in Exercise 4.4.6 is

$$A(x) = 1 + \sum_{n=1}^{\infty} \frac{x^{3n}}{\frac{(3n)!}{\prod_{k=1}^{n}(3k-2)}} = 1 + \sum_{n=1}^{\infty} \frac{x^{3n}\prod_{k=1}^{n}(3k-2)}{(3n)!}$$

Then it's easy to find that

$$A''(x) = \sum_{n=1}^{\infty} \frac{x^{3n-2}\prod_{k=1}^{n}(3k-2)}{(3n-2)!} \quad \text{for all } x.$$

and

$$A''(x) = xA(x).$$

Exercise 4.4.14(2)

By taking derivative terms by term and multiplying by x with respect to the following series, we get for $|x| < 1$,

$$\sum_{n=0}^{+\infty} x^n = \frac{1}{1-x},$$
$$\sum_{n=0}^{+\infty} nx^{n-1} = \left(\frac{1}{1-x}\right)' = \frac{1}{(1-x)^2},$$
$$\sum_{n=0}^{+\infty} nx^n = \frac{x}{(1-x)^2},$$
$$\sum_{n=0}^{+\infty} n^2x^{n-1} = \left(\frac{x}{(1-x)^2}\right)' = \frac{3-x}{(1-x)^3},$$
$$\sum_{n=0}^{+\infty} n^2x^n = \frac{x(3-x)}{(1-x)^3},$$
$$\sum_{n=0}^{+\infty} n^3x^{n-1} = \left(\frac{x(3-x)}{(1-x)^3}\right)' = \frac{3+7x-2x^2}{(1-x)^4},$$
$$\sum_{n=0}^{+\infty} n^3x^n = \frac{3x+7x^2-2x^3}{(1-x)^4}$$

Exercise 4.4.14 (5)

By $\left(\sum_{n=0}^{\infty}\dfrac{x^{2n+1}}{2n+1}\right)' = \sum_{n=0}^{\infty} x^{2n} = \dfrac{1}{1-x^2}$ for $|x|<1$, we get

$$\sum_{n=0}^{\infty}\frac{x^{2n+1}}{2n+1} = \int_0^x \frac{dt}{1-t^2} = \frac{1}{2}\log\frac{1+x}{1-x} \quad \text{for} |x|<1.$$

Exercise 4.4.15 (2)

$\sum_{n=1}^{\infty}(-1)^n\dfrac{x^{2n}}{(2n+1)!} = \dfrac{1}{x}\sum_{n=1}^{\infty}(-1)^n\dfrac{x^{2n+1}}{(2n+1)!} = \dfrac{1}{x}(\sin x - x).$ for $x\in(-\infty,\infty)$

Exercise 4.4.16 (2)

The power series $\dfrac{\sin x}{x} = \sum_{n=0}^{\infty}\dfrac{(-1)^n x^{2n}}{(2n+1)!}$ converges for all x. Integrating term by term, we get

$$\int_0^x \frac{\sin t}{t}dt = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)(2n+1)!}$$

for all x.

Exercise 4.4.16 (5)

The derivative $(\log(x+\sqrt{1+x^2}))' = \dfrac{1}{\sqrt{1+x^2}}$. For $|x|<1$, we have

$$\begin{aligned}\sqrt{1+x^2} &= 1 + \frac{1}{2}x^2 + \frac{1}{2!}\frac{1}{2}\frac{-1}{2}x^4 + \frac{1}{3!}\frac{1}{2}\frac{-1}{2}\frac{-3}{2}x^6 + \cdots \\ &= 1 - \sum_{n=1}^{\infty}(-1)^n\frac{1\cdot 3\cdot 5\cdots(2n-3)}{n!2^n}x^{2n} = 1 - \sum_{n=1}^{\infty}(-1)^n\frac{(2n)!}{4^n(2n-1)(n!)^2}x^{2n}.\end{aligned}$$

Integrating term by term, we get

$$\log(x+\sqrt{1+x^2}) = x - \sum_{n=1}^{\infty}\frac{(-1)^n(2n)!}{4^n(4n^2-1)(n!)^2}x^{2n+1}, \quad |x|<1.$$

The equality actually holds at ± 1.