Math 1024 Midterm

Math 1024 Midterm, Spring 2009

(1) Sketch the graph, indicating the critical data (axis of symmetry, distance of extreme point, etc).

1. $r = 1 - \cos\theta - \sin\theta$.
2. $r = \cos 4\theta - \sin 4\theta$.
3. $r = \sin^2\theta$.

(2) Express the length of one cycle of the limaçon $r = a + \cos\theta$ $(a > 0)$ as the length of suitable ellipse.
(3) Find the volume of the solid of revolution obtained by revolving the region between the two leaves of the limaçon $r = a + \cos\theta$ $(1 > a > 0)$ around the x-axis.
(4) Determine the convergence of improper integral.

1. $\displaystyle\int_0^{\frac{\pi}{2}} \tan^p x dx$.
2. $\displaystyle\int_0^{+\infty} \sin x^p dx$, $p > 0$.

(5) The great pyramid has square base. Assume the pyramid is made of stones of uniform density and the whole pyramid is solid. Where is the center of mass of the pyramid?

Answer to Math 1024 Midtrm, Spring 2009

(1.1) The equation is $r = 1 - \sqrt{2}\cos\left(\theta - \frac{\pi}{4}\right) = 1 + \sqrt{2}\cos\left(\theta + \frac{3\pi}{4}\right)$. This is obtained by rotating the limaçon $r = \frac{1}{\sqrt{2}} + \cos\theta$ by angle $-\frac{3\pi}{4}$ and then expand by a factor of $\sqrt{2}$. The graph is symmetric with respect to the angle $-\frac{3\pi}{4}$, which is the same as symmetric with respect to the angle $\frac{\pi}{4}$.
(1.2) The equation is $r = \sqrt{2}\cos\left(4\theta + \frac{\pi}{4}\right)$. This is obtained by rotating the 8-leaved rose $r = \cos 4\theta$ by angle $-\frac{\pi}{16}$ and then expand by a factor of $\sqrt{2}$. The graph is symmetric with respect to the angles $n\frac{\pi}{8} - \frac{\pi}{16}$.

(1.3) The graph is symmetric with respect to the x-axis and y-axis.
(2) The length of one cycle of limaçon is

$$\int_0^{2\pi} \sqrt{(a+\cos\theta)^2+(-\sin\theta)^2}d\theta = \int_0^{2\pi} \sqrt{a^2+2a\cos\theta+1}d\theta.$$

The length of the ellipse $\frac{x^2}{A^2}+\frac{y^2}{B^2}=1$ is

$$\begin{aligned}\int_0^{2\pi} \sqrt{A^2\sin^2 t+B^2\cos^2 t}dt &= \int_0^{2\pi} \sqrt{A^2+(B^2-A^2)\frac{1+\cos 2t}{2}}dt \\ &= \int_0^{4\pi} \frac{1}{2}\sqrt{\frac{A^2+B^2}{2}+\frac{B^2-A^2}{2}\cos u}du \\ &= \int_0^{2\pi} \sqrt{\frac{A^2+B^2}{2}+\frac{B^2-A^2}{2}\cos u}du.\end{aligned}$$

The two lengths are the same when $\frac{A^2+B^2}{2}=a^2+1$ and $\frac{B^2-A^2}{2}=2a$. Solving the system, we get $A=a+1$, $B=|a-1|$.
(3) The revolution is given by $x=(a+\cos\theta)\cos\theta$, $y=(a+\cos\theta)\sin\theta$, $0\le\theta\le\pi$. The volume is

$$\begin{aligned}-\int_{\theta=0}^{\theta=\pi}\pi y^2dx &= -\int_0^{\pi}\pi(a+\cos\theta)^2\sin^2\theta d[(a+\cos\theta)\cos\theta] \\ &= -\int_0^{\pi}\pi(a+\cos\theta)^2(1-\cos^2\theta)(a+2\cos\theta)d(\cos\theta) \\ &= -\int_1^{-1}\pi(a+t)^2(1-t^2)(a+2t)dt \\ &= \int_{-1}^{1}\pi(a^2+2at+t^2)(a+2t)(1-t^2)dt \\ &= \int_{-1}^{1}\pi(a(a^2+t^2)+4at^2)(1-t^2)dt \ \text{(no need to integrate odd function)} \\ &=2\pi a\left(a^2+\frac{1}{3}(5-a^2)-1\right)=\frac{4}{3}\pi a(a^2+1).\end{aligned}$$

The formula actually also compute the solid of revolution when $a\ge 1$.
(4.1) If $p>0$, then the integral is improper at $\frac{\pi}{2}$, by

$$\lim_{x\to\frac{\pi}{2}^-}\frac{\tan x}{\left(x-\frac{\pi}{2}\right)^{-1}}=1,$$

the integral converges if and only if $\int_0^{\frac{\pi}{2}}\frac{dx}{\left(x-\frac{\pi}{2}\right)^p}$ converges. The condition is $p<1$.

If $p < 0$, then the integral is improper at 0, by

$$\lim_{x\to 0^+} \frac{\tan x}{x} = 1,$$

the integral converges if and only if $\int_0^{\frac{\pi}{2}} x^p dx$ converges. The condition is $p > -1$.

The integral is proper when $p = 0$. We conclude that the integral converges if and only if $|p| < 1$.

(4.2) Let $t = x^p$, we have $\int_0^{+\infty} \sin x^p dx = \int_0^{+\infty} t^{\frac{1}{p}-1} \sin t dt$. By the Dirichlet test, if $p > 1$, then the improper integral converges.

Now assume $0 < p \le 1$. Then consider the range of x satisfying $x^p \in [a_n, b_n] = \left[2\pi n + \frac{\pi}{4}, 2\pi n + \frac{3\pi}{4}\right]$. This corresponds to $x \in [\sqrt[p]{a_n}, \sqrt[p]{b_n}]$. We have $\sin x^p \ge \frac{1}{\sqrt{2}}$ on the interval, and $0 < p \le 1$ implies (by mean value theorem, for example) that

$$\sqrt[p]{b_n} - \sqrt[p]{a_n} \ge b_n - a_n = \frac{\pi}{2}.$$

Therefore

$$\int_{\sqrt[p]{b_n}}^{\sqrt[p]{a_n}} \sin x^p dx \ge \frac{1}{\sqrt{2}}(\sqrt[p]{b_n} - \sqrt[p]{a_n}) \ge \frac{\pi}{2\sqrt{2}}.$$

By Cauchy criterion, the improper integral $\int_0^{+\infty} \sin x^p dx$ diverges.

(5) By the symmetry, the center of mass lies in the central vertical line. The problem is to find the height of the center of mass.

Let a be the side length of the square base. Let h be the height of the pyramid. Let t be the height variable. The pyramid is a square at height t, with the side length $a(t)$ satisfying

$$\frac{a(t)}{a} = \frac{h-t}{h}.$$

The height of the center of mass is ($u = a(t)$)

$$\bar{h} = \frac{\int_0^h \rho t a(t)^2 dt}{\int_0^h \rho a(t)^2 dt} = \frac{\int_0^1 \rho h^2 (1-u) u^2 du}{\int_0^1 \rho h^2 u^2 du} = \frac{\rho h^2 \left(\frac{1}{3} - \frac{1}{4}\right)}{\rho h \frac{1}{3}} = \frac{1}{4} h.$$

The center of mass is at $\frac{1}{4}$ of the height.

Math 1024 Midterm, Spring 2011

(1) Compute integral.

1. $\int x^5 (a + bx^2)^{20} dx$, $b \ne 0$.

2. $\int \tan x \sqrt{\sec x} dx$.

(2) Use integral to compute the limit $\lim \sqrt[n]{\dfrac{(2n)!}{n!n^n}}$.

Hint: Consider the limit of the log.

(3) Use the change of variable $x+1 = A\left(u+\dfrac{1}{u}\right)^2$, $x = A\left(u-\dfrac{1}{u}\right)^2$ for suitable constant A to compute the integral $\displaystyle\int \frac{dx}{1+\sqrt{x}+\sqrt{x+1}}$.

(4) Find the recursive relation for $\int \sec^n x dx$. Then use the relation to compute $\displaystyle\int \frac{dx}{(2\cos 3x - \sin 3x)^4}$.

Hint: $2\cos 3x - \sin 3x = A\cos(3x+\theta)$ for suitable A and θ, and you can also find $\sin(3x+\theta)$.

(5) Suppose $f(x)$ is a differentiable function on $(0, +\infty)$ satisfying

$$f(2) = 0, \quad f'(e^x) = \begin{cases} x, & \text{if } x < 0, \\ e^{2x} - 1, & \text{if } x \ge 0. \end{cases}$$

Find $f(x)$.

Answer to Math 1024 Midterm, Spring 2011

(1.1)

$$\begin{aligned}
\int x^5(a+bx^2)^{20}dx &= \frac{1}{2}\int (x^2)^2(a+bx^2)^{20}dx^2 =_{y=a+bx^2} \frac{1}{2b}\int \left(\frac{y-a}{b}\right)^2 y^{20}dy \\
&= \frac{1}{2b^3}\int (y^{22} - 2ay^{21} + a^2y^{20})dy = \frac{1}{2b^3}\left(\frac{1}{23}y^{23} - \frac{a}{11}y^{22} + \frac{a^2}{21}y^{21}\right) + C \\
&= \frac{(a+bx^2)^{21}}{2b^3}\left(\frac{1}{23}(a+bx^2)^2 - \frac{a}{11}(a+bx^2) + \frac{a^2}{21}\right) + C \\
&= \frac{(a+bx^2)^{21}}{2b^3}\left(\frac{1}{23}b^2x^4 + \left(\frac{2}{23} - \frac{1}{11}\right)abx^2 + \left(\frac{1}{23} - \frac{1}{11} + \frac{1}{21}\right)a^2\right) + C \\
&= \frac{(a+bx^2)^{21}}{2b^3}\left(\frac{1}{23}b^2x^4 - \frac{1}{23\cdot 11}abx^2 + \frac{1}{23\cdot 21\cdot 11}a^2\right) + C.
\end{aligned}$$

(1.2) $\displaystyle\int \tan x\sqrt{\sec x}dx = \int \frac{\tan x \sec x dx}{\sqrt{\sec x}} = \int \frac{d\sec x}{\sqrt{\sec x}} = 2\sqrt{\sec x} + C.$

(2) We have

$$\log \sqrt[n]{\frac{(2n)!}{n!n^n}} = \frac{1}{n}\log\frac{2n(2n-1)\cdots(n+1)}{n^n} = \frac{1}{n}\left(\log\frac{2n}{n} + \log\frac{2n-1}{n} + \cdots + \log\frac{n+1}{n}\right).$$

This is the Riemann sum of the function $\log x$ on $[1,2]$ with respect to the partition $x_i = \dfrac{n+i}{n}$ and sample points $x_i^* = \dfrac{n+i}{n}$. Therefore

$$\lim \log \sqrt[n]{\frac{(2n)!}{n!n^n}} = \int_1^2 \log x dx = (x\log x - x)_1^2 = 2\log 2 - 1.$$

Taking the exponential, we get $\lim \sqrt[n]{\frac{(2n)!}{n!n^n}} = e^{2\log 2 - 1} = \frac{4}{e}$.

(3) The change of variable is

$$x + 1 = A\left(u^2 + 2 + \frac{1}{u^2}\right), \quad x = A\left(u^2 - 2 + \frac{1}{u^2}\right).$$

This implies $1 = 4A$. In other words, if

$$x = \frac{1}{4}\left(u^2 + 2 + \frac{1}{u^2}\right),$$

then

$$x + 1 = \frac{1}{4}\left(u + \frac{1}{u}\right)^2, \quad x = \frac{1}{4}\left(u - \frac{1}{u}\right)^2, \quad dx = \left(u - \frac{1}{u^3}\right) du.$$

We also note that the original integral makes sense for $x > 0$. This translates into $u > 1$.

Under the change of variable, we have ($u > 1$ is used for taking square root)

$$\begin{aligned}
\int \frac{dx}{1 + \sqrt{x} + \sqrt{x+1}} &= \int \frac{\left(u - \frac{1}{u^3}\right) du}{1 + \frac{1}{2}\left(u - \frac{1}{u}\right) + \frac{1}{2}\left(u + \frac{1}{u}\right)} \\
&= \int \frac{u^4 - 1}{(u+1)u^3} du = \int \frac{(u-1)(u+1)(u^2+1)}{(u+1)u^3} du \\
&= \int \left(1 - \frac{1}{u} + \frac{1}{u^2} - \frac{1}{u^3}\right) du. \\
&= u - \log|u| - \frac{1}{u} + \frac{1}{2u^2} + C.
\end{aligned}$$

By

$$\sqrt{x+1} = \frac{1}{2}\left(u + \frac{1}{u}\right), \quad \sqrt{x} = \frac{1}{2}\left(u - \frac{1}{u}\right),$$

we get

$$u = \sqrt{x+1} + \sqrt{x}, \quad \frac{1}{u} = \sqrt{x+1} - \sqrt{x},$$

and

$$\begin{aligned}
\int \frac{dx}{1 + \sqrt{x} + \sqrt{x+1}} &= (\sqrt{x+1} + \sqrt{x}) - (\sqrt{x+1} - \sqrt{x}) + \frac{1}{2}(\sqrt{x+1} - \sqrt{x})^2 - \log(\sqrt{x+1} - \sqrt{x}) + C \\
&= x + 2\sqrt{x} - \sqrt{x(x+1)} - \log(\sqrt{x+1} - \sqrt{x}) + C.
\end{aligned}$$

(4) We have

$$\begin{aligned}
\int \sec^n x dx = \int \sec^{n-2} x d\tan x &= \sec^{n-2} x \tan x - (n-2) \int \tan^2 x \sec^{n-2} x dx \\
&= \sec^{n-2} x \tan x - (n-2) \int (\sec^2 x - 1) \sec^{n-2} x dx \\
&= \sec^{n-2} x \tan x - (n-2) \int \sec^n x dx + (n-2) \int \sec^{n-2} x dx.
\end{aligned}$$

Therefore

$$\int \sec^n x dx = \frac{1}{n-1} \sec^{n-2} x \tan x + \frac{n-2}{n-1} \int \sec^{n-2} x dx.$$

Now we have $0 < \theta < \frac{\pi}{2}$ satisfying

$$\cos\theta = \frac{2}{2^2+1^2} = \frac{2}{5}, \quad \sin\theta = \frac{1}{2^2+1^2} = \frac{1}{5}.$$

Then

$$2\cos 3x - \sin 3x = \sqrt{5}(\cos\theta\cos 3x - \sin\theta\sin 3x) = \sqrt{5}\cos(3x+\theta).$$

Let $y = 3x + \theta$, we have

$$\begin{aligned}
\int \frac{dx}{(2\cos 3x - \sin 3x)^4} = \int \frac{dx}{(\sqrt{5}\cos(3x+\theta))^4} &= \frac{1}{5^2}\int \frac{dy}{3\cos^4 y} = \frac{1}{3\cdot 5^2}\int \sec^4 y dy \\
&= \frac{1}{3\cdot 5^2}\left(\frac{1}{3}\sec^2 y\tan y + \frac{2}{3}\int \sec^2 y dy\right) \\
&= \frac{1}{3\cdot 5^2}\left(\frac{1}{3}\sec^2 y\tan y + \frac{2}{3}\tan y\right) + C = \frac{(1+2\cos^2 y)\sin y}{3^2\cdot 5^2\cos^3 y} + C.
\end{aligned}$$

We already know

$$\cos y = \cos(3x+\theta) = \frac{2\cos 3x - \sin 3x}{\sqrt{5}}.$$

We also know

$$\sin y = \sin(3x+\theta) = \sin\theta\cos 3x + \cos\theta\sin 3x = \frac{\cos 3x + 2\sin 3x}{\sqrt{5}}.$$

Therefore

$$\begin{aligned}
\int \frac{dx}{(2\cos 3x - \sin 3x)^4} &= \frac{\left(1 + 2\left(\frac{2\cos 3x - \sin 3x}{\sqrt{5}}\right)^2\right)\frac{\cos 3x + 2\sin 3x}{\sqrt{5}}}{3^2\cdot 5^2\left(\frac{2\cos 3x - \sin 3x}{\sqrt{5}}\right)^3} + C \\
&= \frac{\left(5 + 2\left(2\cos 3x - \sin 3x\right)^2\right)\left(\cos 3x + 2\sin 3x\right)}{3^2\cdot 5^2(2\cos 3x - \sin 3x)^3} + C.
\end{aligned}$$

(5) Let $t = e^x$. Then we have

$$f(2) = 0, \quad f'(t) = \begin{cases} \log t, & \text{if } 0 < t < 1, \\ t^2 - 1, & \text{if } t \geq 1. \end{cases}$$

For $t \in (0,1)$, we have $f(t) = \int \log t dt = t\log t - t + C_1$. For $t \in [1, +\infty)$, we have $f(t) = \int (t^2-1)dt = \frac{t^3}{3} - t + C_2$. The continuity at $t = 1$ means

$$-1 + C_1 = \frac{1}{3} - 1 + C_2.$$

The condition $f(2) = 0$ means

$$\frac{8}{3} - 2 + C_2 = 0.$$

Therefore $C_2 = -\frac{2}{3}$, $C_1 = -\frac{1}{3}$, and we have

$$f(t) = \begin{cases} t \log t - t - \frac{1}{3}, & \text{if } 0 < t < 1, \\ \frac{t^3}{3} - t - \frac{2}{3}, & \text{if } t \geq 1. \end{cases}$$

Math 1024 Midterm, Spring 2013

(1) Compute area.

1. The region between $y = x^2 e^x$ and $y = e^x$.
2. The bigger of two regions between the circle $x^2 + y^2 = 2$ and the parabola $y = x^2$.

(2) Find the Taylor expansion of $f(x) = \int_x^{x^2} \left(\int_0^u \cos t^2 dt \right) du$ up to the 13-th order.

(3) Compute integral.

1. $\displaystyle\int_0^1 \left(\frac{x-1}{x+1} \right)^4 dx.$
2. $\displaystyle\int \frac{xdx}{\cos^2 x}.$
3. $\displaystyle\int \frac{dx}{2x + \sqrt{x(x+1)}}.$

(4) Find the recursive relation for $I_p = \int (ax^2 + bx + c)^p dx$, where $a \neq 0$ and $b^2 \neq 4ac$. What happens when $b^2 = 4ac$?
(5) For what kind of function does the Trapezoidal rule always compute the exact value of the definite integral? What about Simpson's rule? Explain.

Answer to Math 1024 Midterm, Spring 2013

(1.1) By $x^2 e^x - e^x = (x^2 - 1)e^x$, the two curves intersect at $x = -1$ and $x = 1$. On the interval $[-1, 1]$, we have $x^2 e^x \leq e^x$, and the area of the region is

$$\begin{aligned} \int_{-1}^1 (e^x - x^2 e^x) dx &= \int_{-1}^1 (1 - x^2) de^x = -\int_{-1}^1 e^x (-2x) dx = 2 \int_{-1}^1 x de^x \\ &= 2xe^x|_{-1}^1 - 2 \int_{-1}^1 e^x dx = 2(e + e^{-1}) - 2(e - e^{-1}) = 4e^{-1}. \end{aligned}$$

(1.2) The two curves intersect at $(x, y) = (1, 1)$ and $(-1, 1)$. The smaller of the two regions is over the interval $x \in [-1, 1]$, on which the circle $y = \sqrt{2 - x^2}$ is bigger than the parabola $y = x^2$. So the area of the smaller region is

$$\begin{aligned}\int_{-1}^{1} (\sqrt{2 - x^2} - x^2) dx &= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sqrt{2 - (\sqrt{2}\sin t)^2} d(\sqrt{2}\sin t) - \int_{-1}^{1} x^2 dx \\ &= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} 2\cos^2 t dt - \frac{2}{3} = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} (1 + \cos 2t) dt - \frac{2}{3} = \frac{\pi}{2} + \frac{1}{3}.\end{aligned}$$

The bigger region is obtained by deleting the smaller region from the disk, so that it has area

$$\pi(\sqrt{2})^2 - \left(\frac{\pi}{2} + \frac{1}{3}\right) = \frac{3\pi}{2} - \frac{1}{3}.$$

(2) We have

$$\begin{aligned}
f(0) &= 0, \\
f'(x) &= \frac{d}{dx} \int_{x}^{x^2} \left(\int_{0}^{u} \cos t^2 dt \right) du = 2x \int_{0}^{x^2} \cos t^2 dt - \int_{0}^{x} \cos t^2 dt, \\
f'(0) &= 0, \\
f''(x) &= \frac{d}{dx} \left(2x \int_{0}^{x^2} \cos t^2 dt - \int_{0}^{x} \cos t^2 dt \right) = 2 \int_{0}^{x^2} \cos t^2 dt + 2x \cdot 2x \cos x^4 - \cos x^2, \\
f''(0) &= -1, \\
f'''(x) &= \frac{d}{dx} \left(2 \int_{0}^{x^2} \cos t^2 dt + 4x^2 \cos x^4 - \cos x^2 \right) \\
&= 4x \cos x^4 + 8x \cos x^4 - 16x^5 \sin x^4 + 2x \sin x^2 \\
&= 12x \left(1 - \frac{1}{2}x^8 + o(x^{15}) \right) - 16x^5(x^4 + o(x^{11})) + 2x \left(x^2 - \frac{x^6}{6} + o(x^9) \right) \\
&= 12x + 2x^3 - \frac{1}{3}x^7 - 22x^9 + o(x^{10}).
\end{aligned}$$

By the derivatives up to second order, we have

$$f(x) = 0 + 0x - \frac{1}{2}x^2 + a_3 x^3 + a_4 x^4 + \cdots + a_{13} x^{13} + o(x^{13}).$$

Then

$$f'''(x) = 3 \cdot 2 \cdot 1 a_3 + 4 \cdot 3 \cdot 2 a_4 x + \cdots + 13 \cdot 12 \cdot 11 a_{13} x^{10} + o(x^{10}).$$

Compared with the Taylor expansion of $f'''(x)$ up to the 10-th power, we get

$$a_3 = 0, \ a_4 = \frac{12}{4 \cdot 3 \cdot 2} = \frac{1}{2}, \ a_5 = 0, \ a_6 = \frac{2}{6 \cdot 5 \cdot 4} = \frac{1}{60}, \ a_7 = a_8 = a_9 = 0,$$

$$a_{10} = -\frac{1}{3(10 \cdot 9 \cdot 8)} = -\frac{1}{2160}, \ a_{11} = 0, \ a_{12} = -\frac{22}{12 \cdot 11 \cdot 10} = -\frac{1}{60}, \ a_{13} = 0.$$

Therefore

$$f(x) = -\frac{1}{2}x^2 + \frac{1}{2}x^4 + \frac{1}{60}x^6 - \frac{1}{2160}x^{10} - \frac{1}{60}x^{12} + o(x^{10}).$$

(3.1) Let $y = x + 1$. We have

$$\begin{aligned}\int_0^1 \left(\frac{x-1}{x+1}\right)^4 dx &= \int_1^2 \frac{(y-2)^4}{y^4} dy = \int_1^2 \frac{y^4 - 4\cdot 2y^3 + 6\cdot 4y^2 - 4\cdot 8y + 16}{y^4} dy \\ &= \left(y - 8\log y - 24y^{-1} + 16y^{-2} - \frac{16}{3}y^{-3}\right)_1^2 \\ &= 1 - 8\log 2 - 24\left(\frac{1}{2} - 1\right) + 16\left(\frac{1}{4} - 1\right) - \frac{16}{3}\left(\frac{1}{8} - 1\right) = \frac{17}{3} - 8\log 2.\end{aligned}$$

(3.2) $\displaystyle\int \frac{xdx}{\cos^2 x} = \int xd\tan x = x\tan x - \int \tan xdx = x\tan x - \log|\cos x| + C$

(3.3) The integral makes sense when $x > 0$ or $x < -1$. Let $y = \sqrt{\frac{x+1}{x}}$ for $x > 0$ and $y = -\sqrt{\frac{x+1}{x}}$ for $x < -1$. Then $\sqrt{x(x+1)} = xy$, $x = \frac{1}{y^2-1}$, $dx = -\frac{2ydy}{(y^2-1)^2}$ and

$$\begin{aligned}\int \frac{dx}{2x + \sqrt{x(x+1)}} &= \int \frac{dx}{x(2+y)} = \int \frac{-\frac{2ydy}{(y^2-1)^2}}{\frac{1}{y^2-1}(y+2)} = -\int \frac{2ydy}{(y+2)(y+1)(y-1)} \\ &= \int \left(\frac{4}{3(y+2)} - \frac{1}{y+1} - \frac{1}{3(y-1)}\right) dy \\ &= \frac{1}{3}\log\left|\frac{(y+2)^4}{(y+1)^3(y-1)}\right| + C = \frac{1}{3}\log\left|\frac{(xy+2x)^4}{(xy+x)^3(xy-x)}\right| + C \\ &= \frac{1}{3}\log\left|\frac{(2x+\sqrt{x(x+1)})^4}{(x+\sqrt{x(x+1)})^3(x-\sqrt{x(x+1)})}\right| + C.\end{aligned}$$

[*Alternative*]

For $x > 0$, let $y = \sqrt{x+1} + \sqrt{x}$. Then $\frac{1}{y} = \sqrt{x+1} - \sqrt{x}$, $y + \frac{1}{y} = 2\sqrt{x+1}$,

$y - \frac{1}{y} = 2\sqrt{x}$, $x = \frac{1}{4}\left(y - \frac{1}{y}\right)^2$, $dx = \frac{1}{2}\left(1 - \frac{1}{y^4}\right) ydy$. We have

$$\int \frac{dx}{2x + \sqrt{x(x+1)}} = \int \frac{dx}{\sqrt{x}(2\sqrt{x} + \sqrt{x+1})} = \int \frac{\frac{1}{2}\left(1 - \frac{1}{y^4}\right) ydy}{\frac{1}{2}\left(y - \frac{1}{y}\right)\left[\left(y - \frac{1}{y}\right) + \frac{1}{2}\left(y + \frac{1}{y}\right)\right]}$$
$$= 2\int \frac{(y^2+1)dy}{y(3y^2-1)} = 2\int \left(-\frac{1}{y} + \frac{2}{3y+\sqrt{3}} + \frac{2}{3y-\sqrt{3}}\right) dy$$
$$= -2\log|y| + \frac{4}{3}\log|(3y+\sqrt{3})(3y-\sqrt{3})| + C$$
$$= -2\log|y| + \frac{4}{3}\log\left|y^2 - \frac{1}{3}\right| + C$$
$$= -2\log|\sqrt{x+1} + \sqrt{x}| + \frac{4}{3}\log\left|\sqrt{x(x+1)} + x + \frac{1}{3}\right| + C.$$

(4) We have

$$\begin{aligned} I_p &= x(ax^2+bx+c)^p - \int xd(ax^2+bx+c)^p \\ &= x(ax^2+bx+c)^p - \int x(ax^2+bx+c)^{p-1}p(2ax+b)dx \\ &= x(ax^2+bx+c)^p \\ &\quad - \int \left(2p(ax^2+bx+c) - \frac{pb}{2a}(ax^2+bx+c)' + \frac{pb^2}{2a} - 2pc\right)(ax^2+bx+c)^{p-1}dx \\ &= \left(x + \frac{b}{2a}\right)(ax^2+bx+c)^p - 2pI_p + \left(-\frac{pb^2}{2a} + 2pc\right) I_{p-1}. \end{aligned}$$

Therefore

$$I_p = \frac{1}{(2p+1)2a}(2ax+b)(ax^2+bx+c)^p - \frac{p(b^2-4ac)}{(2p+1)2a} I_{p-1}, \quad p \neq -\frac{1}{2}.$$

On the other hand, we may also express I_{p-1} in terms of I_p. After substituting p by $p+1$, we get

$$I_p = \frac{1}{(p+1)(b^2-4ac)}(2ax+b)(ax^2+bx+c)^{p+1} + \frac{(2p+3)2a}{(p+1)(b^2-4ac)} I_{p+1}, \quad p \neq -1.$$

If $b^2 = 4ac$, then

$$ax^2 + bx + c = a(x-r)^2, \quad r = \frac{b}{2a}$$

has a double real root, and

$$I_p = \int a^p(x-r)^{2p}dx = \begin{cases} \dfrac{a^p(x-r)^{2p+1}}{2p+1} + C, & \text{if } p \neq -\dfrac{1}{2}, \\ \dfrac{\log|x-r|}{\sqrt{a}} + C, & \text{if } p = -\dfrac{1}{2}. \end{cases}$$

(5) The trapezoidal rule approximates a function by straight lines connecting end points of partition intervals. Therefore the rule gives the exact value of the definite integral when the function is linear.

The Simpson's rule approximates a function by quadratic functions passing three consecutive partition points of the function. Therefore the rule gives the exact value of the definite integral when the function is quadratic.

Math 1024 Midterm, Spring 2015

(1) Find the second order Taylor expansion of $G(x) = \displaystyle\int_0^x \frac{\sin t^2}{t} dt$ at 0. Then find $\lim_{x\to 0} \dfrac{\left(\displaystyle\int_0^x \frac{\sin t}{t} dt\right)^2}{\displaystyle\int_0^x \frac{\sin t^2}{t} dt}$.

(2) Find a recursive relation for $I_p = \displaystyle\int (\arcsin x)^p dx$. Then find $\displaystyle\int (\arcsin x)^5 dx$ and $\displaystyle\int (\arcsin x)^6 dx$.

(3) Let

$$f(x) = \begin{cases} Ae^x + B, & \text{if } x < 0, \\ Cx \sin \dfrac{1}{x}, & \text{if } x > 0, \\ D, & \text{if } x = 0, \end{cases} \quad F(x) = \int_0^x f(t) dt.$$

Find the conditions on A, B, C, D such that $F(x)$ is continuous. Find the condition such that (x) is differentiable.

(4) Compute the integrals.

1. $\displaystyle\int \frac{\sin x dx}{\cos x \sqrt{1 + \sin^2 x}}$.
2. $\displaystyle\int \frac{dx}{x^2 \sqrt{a^2 - x^2}}$, $a > 0$.
3. $\displaystyle\int \arctan x^2 dx$.

Answer to Math 1024 Midterm, Spring 2015

(1) The function

$$g(t) = \begin{cases} \dfrac{\sin t^2}{t}, & \text{if } t \neq 0 \\ 0, & \text{if } t = 0 \end{cases}$$

is continuos at 0. Then by the fundamental theorem of calculus, $G(x) = \int_0^x \frac{\sin t^2}{t} dt = \int_0^x g(x) dx$ has derivative

$$G'(x) = g(x) = \begin{cases} \frac{\sin x^2}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}.$$

By the definition, the second order derivative at 0 is

$$G''(0) = \lim_{x \to 0} \frac{G(x) - G(0)}{x} = \lim_{x \to 0} \frac{1}{x} \left(\frac{\sin x^2}{x} \right) = 1.$$

This gives the Taylor expansion

$$\int_0^x \frac{\sin t^2}{t} dt = 0 + 0x + \frac{1}{2}x^2 + o(x^2).$$

Alternative method: By $\sin t^2 = t^2 + o(t^5)$, we have $\frac{\sin t^2}{t} = t + o(t^4)$. Then

$$\int_0^x \frac{\sin t^2}{t} dt = \int_0^x (t + o(t^4)) dt = \frac{1}{2}x^2 + o(x^5).$$

By the same argument, we have

$$\int_0^x \frac{\sin t}{t} dt = x + o(x).$$

Therefore

$$\lim_{x \to 0} \frac{\left(\int_0^x \frac{\sin t}{t} dt \right)^2}{\int_0^x \frac{\sin t^2}{t} dt} = \lim_{x \to 0} \frac{(x + o(x))^2}{\frac{1}{2}x^2 + o(x^2)} = \lim_{x \to 0} \frac{x^2 + o(x^2)}{\frac{1}{2}x^2 + o(x^2)} = 2.$$

(2) For $p \neq 1, 2$, we have

$$\begin{aligned}
I_p &= \int (\arcsin x)^p dx = x(\arcsin x)^p - \int x d(\arcsin x)^p \\
&= x(\arcsin x)^p - \int xp(\arcsin x)^{p-1} \frac{1}{\sqrt{1-x^2}} dx \\
&= x(\arcsin x)^p + p \int (\arcsin x)^{p-1} d\sqrt{1-x^2} \\
&= x(\arcsin x)^p + p\sqrt{1-x^2}(\arcsin x)^{p-1} - p \int \sqrt{1-x^2} d(\arcsin x)^{p-1} \\
&= x(\arcsin x)^p + p\sqrt{1-x^2}(\arcsin x)^{p-1} - p \int \sqrt{1-x^2}(p-1)(\arcsin x)^{p-2} \frac{1}{\sqrt{1-x^2}} dx \\
&= x(\arcsin x)^p + p\sqrt{1-x^2}(\arcsin x)^{p-1} - p(p-1)I_{p-2}.
\end{aligned}$$

We further have

$$\int \arcsin x dx = x \arcsin x - \int \frac{x dx}{\sqrt{1-x^2}} = x \arcsin x + \sqrt{1-x^2} + C.$$

$$\int (\arcsin x)^0 dx = \int dx + C,$$

$$\begin{aligned}
\int (\arcsin x)^5 dx &= x(\arcsin x)^5 + 5\sqrt{1-x^2}(\arcsin x)^4 - 5\cdot 4 \int (\arcsin x)^3 dx \\
&= x\left((\arcsin x)^5 - 5\cdot 4(\arcsin x)^3\right) + \sqrt{1-x^2}\left(5(\arcsin x)^4 - 5\cdot 4\cdot 3(\arcsin x)^2)\right) \\
&\quad + 5\cdot 4\cdot 3\cdot 2 \int \arcsin x dx \\
&= x\left((\arcsin x)^5 - 5\cdot 4(\arcsin x)^3 + 5\cdot 4\cdot 3\cdot 2 \arcsin x\right) \\
&\quad + \sqrt{1-x^2}\left(5(\arcsin x)^4 - 5\cdot 4\cdot 3(\arcsin x)^2 + 5\cdot 4\cdot 3\cdot 2\right) + C \\
&= x(\arcsin x)^5 + 5\sqrt{1-x^2}(\arcsin x)^4 - 20x(\arcsin x)^3 - 60\sqrt{1-x^2}(\arcsin x)^2 \\
&\quad + 120x \arcsin x + 120\sqrt{1-x^2} + C.
\end{aligned}$$

$$\begin{aligned}
\int (\arcsin x)^6 dx &= x\left((\arcsin x)^6 - 6\cdot 5(\arcsin x)^4 + 6\cdot 5\cdot 4\cdot 3(\arcsin x)^2 - 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1\right) \\
&\quad + \sqrt{1-x^2}\left(6(\arcsin x)^5 - 6\cdot 5\cdot 4(\arcsin x)^3 + 6\cdot 5\cdot 4\cdot 3\cdot 2 \arcsin x\right) + C \\
&= x(\arcsin x)^6 + 6\sqrt{1-x^2}(\arcsin x)^5 - 30x(\arcsin x)^4 - 120\sqrt{1-x^2}(\arcsin x)^3 \\
&\quad + 360x(\arcsin x)^2 + 720\sqrt{1-x^2} \arcsin x - 720x + C.
\end{aligned}$$

(3) The integral $F(x)$ is always continuous. Since $f(x)$ is continuous on $\mathbb{R} = 0$, $F(x)$ is also differentiable on $\mathbb{R} = 0$. It remains to discuss the differentiability of $F(x)$ at 0.

For $x \le 0$, we have $F(x) = \int_0^x (Ae^x + B)dx$. This implies $F'_-(0) = (Ae^x + B)_{x=0} = A + B$. For $x \ge 0$, we have $F(x) = \int_0^x Cx \sin \frac{1}{x} dx$. Extend $Cx \sin \frac{1}{x}$ to be continuous at 0 by assiging value $\lim_{x\to 0} Cx \sin \frac{1}{x} = 0$ at 0, we get $F'_+(0) = 0$. So F is differentiable at 0 if and only if$A + B = 0$.

(4.1)

$$\begin{aligned}
\int \frac{\sin x dx}{\cos x\sqrt{1+\sin^2 x}} &=_{y=\cos x} \int \frac{-dy}{y\sqrt{2-y^2}} = \int \frac{d(2-y^2)}{2y^2\sqrt{2-y^2}} \\
&=_{z=\sqrt{2-y^2}} \int \frac{dz^2}{2(2-z^2)z} = \int \frac{dz}{2-z^2} = \frac{1}{2\sqrt{2}} \int \left(\frac{1}{z+\sqrt{2}} - \frac{1}{z-\sqrt{2}}\right) dz \\
&= \frac{1}{2\sqrt{2}} \log\left|\frac{z+\sqrt{2}}{z-\sqrt{2}}\right| + C = \frac{1}{2\sqrt{2}} \log\left|\frac{(z+\sqrt{2})^2}{z^2-2}\right| + C \\
&= \frac{1}{\sqrt{2}} \log\left|\frac{\sqrt{1+\sin^2 x}+\sqrt{2}}{\cos x}\right| + C.
\end{aligned}$$

Alternative 1.

$$\int \frac{\sin x dx}{\cos x\sqrt{1+\sin^2 x}} = \int \frac{\sec x \tan x dx}{\sqrt{\sec^2 x + \tan^2 x}} = \int \frac{d \sec x}{\sqrt{2\sec^2 x - 1}} =_{y=\sqrt{2}\sec x} \int \frac{dy}{\sqrt{2}\sqrt{y^2-1}}$$
$$= \frac{1}{\sqrt{2}} \log|y+\sqrt{y^2-1}| + C = \frac{1}{\sqrt{2}} \log|\sqrt{2}\sec x + \sqrt{2\sec^2 x - 1}| + C$$
$$= \frac{1}{\sqrt{2}} \log\left|\frac{\sqrt{2}+\sqrt{2-\cos^2 x}}{\cos x}\right| + C.$$

(4.2) Let $x = a\sin y$, $x \in (-a, a)$, $y \in (-\frac{\pi}{2}, \frac{\pi}{2})$, $\sqrt{a^2-x^2} = a\cos y$. Then $\dfrac{dx}{\sqrt{a^2-x^2}} = dy$ and

$$\int \frac{dx}{x^2\sqrt{a^2-x^2}} = \int \frac{dy}{a^2\sin^2 y} = \frac{1}{a^2}\int \csc^2 y dy$$
$$= -\frac{1}{a^2}\cot y + C = -\frac{(a\cos y)}{a^2(a\sin y)} + C = -\frac{\sqrt{a^2-x^2}}{a^2 x} + C.$$

(4.3) By integration by parts, $\int \arctan x^2 dx = x\arctan x^2 - \int \frac{2x^2 dx}{1+x^4}$. By

$$1+x^4 = 1+2x^2+x^4-2x^2 = (x^2+1)^2 - (\sqrt{2}x)^2 = (x^2+\sqrt{2}x+1)(x^2-\sqrt{2}x+1),$$

we have
$$\frac{2x^2}{1+x^4} = \frac{Ax+B}{x^2+\sqrt{2}x+1} + \frac{Cx+D}{x^2-\sqrt{2}x+1}.$$
Then
$$2x^2 = (Ax+B)(x^2-\sqrt{2}x+1) + (Cx+D)(x^2+\sqrt{2}x+1).$$
Changing x to $-x$, we get

$$2x^2 = (-Ax+B)(x^2+\sqrt{2}x+1) + (-Cx+D)(x^2-\sqrt{2}x+1).$$

Therefore $Ax+B = -Cx+D$, $Cx+D = -Ax+B$, and

$$2x^2 = (Ax+B)(x^2-\sqrt{2}x+1) + (-Ax+B)(x^2+\sqrt{2}x+1).$$

Taking $x=0$, we get $B=0$. Substituting in $B=0$, we get

$$2x^2 = Ax(x^2-\sqrt{2}x+1) - Ax(x^2+\sqrt{2}x+1).$$

This tells us $A = -\dfrac{1}{\sqrt{2}}$, and we get

$$\frac{2x^2}{1+x^4} = -\frac{x}{\sqrt{2}(x^2+\sqrt{2}x+1)} + \frac{x}{\sqrt{2}(x^2-\sqrt{2}x+1)}.$$

We have

$$\begin{aligned}
\int \frac{xdx}{x^2+\sqrt{2}x+1} &= \int \frac{d(x^2+\sqrt{2}x+1)}{2(x^2+\sqrt{2}x+1)} - \int \frac{\sqrt{2}dx}{2(x^2+\sqrt{2}x+1)} \\
&= \frac{1}{2}\log(x^2+\sqrt{2}x+1) - \int \frac{dx}{\sqrt{2}\left(\left(x+\frac{1}{\sqrt{2}}\right)^2+\left(\frac{1}{\sqrt{2}}\right)^2\right)} \\
&= \frac{1}{2}\log(x^2+\sqrt{2}x+1) - \arctan(\sqrt{2}x+1) + C.
\end{aligned}$$

Similarly, we have

$$\int \frac{xdx}{x^2-\sqrt{2}x+1} = \frac{1}{2}\log(x^2-\sqrt{2}x+1) + \arctan(\sqrt{2}x-1) + C.$$

Therefore

$$\begin{aligned}
\int \arctan x^2 dx = x\arctan x^2 &+ \frac{1}{\sqrt{2}}\left(\frac{1}{2}\log(x^2+\sqrt{2}x+1) - \arctan(\sqrt{2}x+1)\right) \\
&- \frac{1}{\sqrt{2}}\left(\frac{1}{2}\log(x^2-\sqrt{2}x+1) + \arctan(\sqrt{2}x-1)\right) + C \\
= x\arctan x^2 &- \frac{1}{\sqrt{2}}\left(\arctan(\sqrt{2}x+1) + \arctan(\sqrt{2}x-1)\right) \\
&+ \frac{1}{2\sqrt{2}}\log\frac{x^2+\sqrt{2}x+1}{x^2-\sqrt{2}x+1} + C.
\end{aligned}$$

Alternative.

$$\begin{aligned}
\int \frac{2x^2}{1+x^4}dx &= \int \frac{x^2-1}{1+x^4}dx + \int \frac{x^2+1}{1+x^4}dx \\
&= \int \frac{1-\frac{1}{x^2}}{\left(x+\frac{1}{x}\right)^2-2}dx + \int \frac{1+\frac{1}{x^2}}{\left(x-\frac{1}{x}\right)^2+2}dx \\
&= \int \frac{d\left(x+\frac{1}{x}\right)}{\left(x+\frac{1}{x}\right)^2-2} + \int \frac{d\left(x-\frac{1}{x}\right)}{\left(x-\frac{1}{x}\right)^2+2} \\
&= \frac{1}{2\sqrt{2}}\log\left|\frac{x+\frac{1}{x}-\sqrt{2}}{x+\frac{1}{x}+\sqrt{2}}\right| + \frac{1}{\sqrt{2}}\arctan\frac{1}{\sqrt{2}}\left(x-\frac{1}{x}\right) + C \\
&= \frac{1}{2\sqrt{2}}\log\frac{x^2-\sqrt{2}x+1}{x^2+\sqrt{2}x+1} + \frac{1}{\sqrt{2}}\arctan\frac{1}{\sqrt{2}}\left(x-\frac{1}{x}\right) + C.
\end{aligned}$$

This gives

$$\int \arctan x^2 dx = x\arctan x^2 - \frac{1}{2\sqrt{2}}\log\frac{x^2-\sqrt{2}x+1}{x^2+\sqrt{2}x+1} - \frac{1}{\sqrt{2}}\arctan\frac{1}{\sqrt{2}}\left(x-\frac{1}{x}\right) + C.$$