Exercise 3.1.1

(1) $S(P_n,x)=\sum_{i=1}^{n}\frac{i-1}{n}\frac{1}{n}=\frac{1}{n^2}\frac{n(n-1)}{2}=\frac{n-1}{2n}$.

(2) $S(P_n,x^2)=\sum_{i=1}^{n}\frac{i^2}{n^2}\frac{1}{n}=\frac{1}{4n^3}\left(4\frac{n(n+1)(2n+1)}{6}-4\frac{n(n+1)}{2}+n\right)=\frac{(n+1)(2n+1)}{6n^2}$.

(3) $S(P_n,x^2)=\sum_{i=1}^{n}\frac{(2i-1)^2}{(2n)^2}\frac{1}{n}=\frac{1}{n^3}\frac{n(n+1)(2n+1)}{6}=\frac{4n^2+n-4}{12n^2}$.

(4) $S(P_n,\alpha^x)=\sum_{i=1}^{n}\alpha^{\frac{i-1}{n}}\frac{1}{n}=\frac{1}{n}\left(1+\sqrt[n]{\alpha}+\sqrt[n]{\alpha}^2+\cdots+\sqrt[n]{\alpha}^{n-1}\right)=\frac{1-\alpha}{n(1-\sqrt[n]{\alpha})}$.

Exercise 3.1.2

(1) For any partition P of $[0,2]$, we have only one index i satisfying $x_{i-1}\le 1<x_i$. Then $S(P,f)=f(x_i^*)(x_i-x_{i-1})+1(x_n-x_i)=f(x_i^*)(x_i-x_{i-1})+1-x_i$. Since $f(x_i^*)=0$ or 1 and $0<x_i\le x_i-x_{i-1}\le\|P\|$, we have $|S(P,f)-1|\le 2\|P\|$. Therefore $f(x)$ is integrable with $\int_0^2 fdx=1$.

(2) For any partition P, by taking all x_i^* to be irrational, we get $S(P,f)=0$. By taking all x_i^* rational, we get $S(P,f)=\sum x_i^*(x_i-x_{i-1})\ge\frac{1}{2}\left(\frac{1}{2}-\|P\|\right)$, where is inequality is obtained by considering only those $x_i^*\ge\frac{1}{2}$. This implies that $S(P,f)$ does not converge as $\|P\|\to 0$. Therefore the function is not integrable.

(3) For any natural number N, assume $\|P\|<\frac{1}{2N}$. Then any interval in the partition is either contained in $\left[-1,-\frac{1}{2N}\right]\cup\left[\frac{1}{2N},1\right]$ or contained in $\left[-\frac{1}{N},\frac{1}{N}\right]$. We may divide the Riemann sum into two parts $S(P,f)=\sum'+\sum''$, where the intervals in $\sum'$ are contained in $\left[-1,-\frac{1}{N}\right]\cup\left[\frac{1}{N},1\right]$ and the intervals in $\sum''$ are contained in $\left[-\frac{1}{N},\frac{1}{N}\right]$. Then there are at most $8N$ intervals in $\sum'$, such that the corresponding $f(x_i^*)=1$, and we also have $f(x_i^*)=0$ for the other intervals in $\sum'$. Therefore we get $|\sum'|\le 4N\|P\|$. On the other hand, the total length of the intervals in $\sum''$ is $\le\frac{2}{N}$, and we always have $f(x_i^*)=0$ or 1. Therefore $|\sum''|\le\frac{2}{N}$. We conclude that $|S(P,f)|\le|\sum'|+|\sum''|\le 8N\|P\|+\frac{2}{N}$. This implies that

$$\|P\|<\frac{1}{2N^2}\implies|S(P,f)|\le\frac{6}{N},$$

which further implies that the function is integrable, with $\int_{-1}^{1}f(x)dx=0$.

(4) For any natural number N, the Riemann sum $S(P,f)$ can be divided into two parts according to $x_i^*=\frac{1}{n}$ for some $n<N$ or otherwise. The number of terms in the first part is $<N$, so that the sum is $<N(\max|f|)\|P\|=N\|P\|$. For the second part we have $f(x_i^*)\le\frac{1}{N}$, so that the sum of the second part is $\le\frac{1}{N}(1-0)=\frac{1}{N}$. Thus $0\le S(P,f)\le N\|P\|+\frac{1}{N}$. Then for any $\epsilon>0$, choosing $N=\frac{2}{\epsilon}$ and $\|P\|<\frac{\epsilon^2}{4}$ would imply that $|S(P,f)|<\epsilon$. Therefore the function is integrable, with $\int_0^1 fdx=0$.

EXERCISE 3.1.3

By taking middle points, we get $S_m(P,x) = \sum \frac{x_i + x_{i-1}}{2}(x_i - x_{i-1}) = \frac{1}{2}\sum(x_i^2 - x_{i-1}^2) = \frac{1}{2}(x_n^2 - x_0^2) = \frac{1}{2}$.

Then for any choice of x_i^*, we have $\left|x_i^* - \frac{x_i + x_{i-1}}{2}\right| \leq \frac{1}{2}\|P\|$. Therefore $\left|S(P,x) - \frac{1}{2}\right| = |S(P,x) - S_m(P,x)| \leq \sum \left|x_i^* - \frac{x_i + x_{i-1}}{2}\right| \Delta x_i \leq \frac{1}{2}\|P\| \sum \Delta x_i = \frac{1}{2}\|P\|$. This implies that x is integrable, with $\int_0^1 x dx = \frac{1}{2}$.

Exercise 3.1.4

For $b > a \geq 0$, we have $\omega_{[a,b]}(x^2) = b^2 - a^2$. Then

$$\sum \omega_{[x_{i-1},x_i]}(x^2)\Delta x_i = \sum (x_i^2 - x_{i-1}^2)\Delta x_i \leq \sum (x_i^2 - x_{i-1}^2)\|P\| = (x_n^2 - x_0^2)\|P\| = \|P\|.$$

Thus for any $\epsilon > 0$, $\|P\| < \epsilon \implies \sum \omega_{[x_{i-1},x_i]}(x^2)\Delta x_i < \epsilon$.

Exercise 3.1.5

(1) For any partition P, we have only one index j satisfying $x_{j-1} \leq 0 < x_j$. Then $\omega_{[x_{i-1},x_i]}(f) = 0$ for $j \neq i$ and $\omega_{[x_{j-1},x_j]}(f) = 1$. Therefore $\sum \omega_{[x_{i-1},x_i]}(f)\Delta x_i = x_j - x_{j-1} \leq \|P\|$. Thus for any $\epsilon > 0$, $\|P\| < \epsilon \implies \sum \omega_{[x_{i-1},x_i]}(f)\Delta x_i < \epsilon$.

(2) We have $\omega_{[x_{i-1},x_i]}(f) = x_i$. If $\|P\| < \dfrac{1}{3}$, then the total length of the intervals with $x_i > \dfrac{2}{3}$ is at least $\dfrac{1}{3}$. Therefore $\sum \omega_{[x_{i-1},x_i]}(f)\Delta x_i = \sum x_i \Delta x_i \geq \dfrac{2}{3}\dfrac{1}{3}$, and the function is not integrable.

(3) For any natural number N, assume $\|P\| < \dfrac{1}{2N}$. Then any interval in the partition is either contained in $\left[-1, -\dfrac{1}{2N}\right] \cup \left[\dfrac{1}{2N}, 1\right]$ or contained in $\left[-\dfrac{1}{N}, \dfrac{1}{N}\right]$. The number of intervals of the first type with $\omega_{[x_{i-1},x_i]}(f) = 1$ is $\leq 8N$. The total length of the second type of intervals is $\leq \dfrac{2}{N}$. Moreover, each $\omega_{[x_{i-1},x_i]}(f)$ must be either 0 or 1. Therefore $\sum \omega_{[x_{i-1},x_i]}(f) \leq 8N\|P\| + \dfrac{2}{N}$. In particular, we have $\sum \omega_{[x_{i-1},x_i]}(f) \leq \dfrac{6}{N}$ when $\|P\| < \dfrac{1}{2N^2}$.

(4) For any natural number N, the number of intervals with $\omega_{[x_{i-1},x_i]}(f) \geq \dfrac{1}{N}$ is $\leq 2N$. Since we always have $\omega_{[x_{i-1},x_i]}(f) \leq 1$, we get

$$\sum \omega_{[x_{i-1},x_i]}(f)\Delta x_i = \sum_{\omega \geq \frac{1}{N}} + \sum_{\omega < \frac{1}{N}} < 2N\|P\| + \frac{1}{N}(1-0).$$

Thus $\|P\| < \dfrac{1}{N^2}$ implies $\sum \omega_{[x_{i-1},x_i]}(f)\Delta x_i < \dfrac{3}{N}$.

Exercise 3.1.6

Since $||f(x)| - |f(y)|| \leq |f(x) - f(y)|$, we have $\omega(|f|) = \sup ||f(x)| - |f(y)|| \leq \sup |f(x) - f(y)| = \omega(f)$. As a result, $\sum \omega_{[x_{i-1},x_i]}(|f|)\Delta x_i \leq \sum \omega_{[x_{i-1},x_i]}(f)\Delta x_i$. Then by Theorem 3.1.3, it is easy to see that the integrability of f implies the integrability of $|f|$.

Exercise 3.1.7

Suppose f is integrable, then $|f(x)| < M$ for a constant M by Proposition 3.1.2. Then $|f(x)^2 - f(y)^2| = |f(x)+f(y)||f(x)-f(y)| \leq 2M|f(x)-f(y)|$ and $\omega(f^2) = \sup |f(x)^2 - f(y)^2| \leq \sup 2M|f(x)-f(y)| = 2M\omega(f)$. As a result, $\sum \omega_{[x_{i-1},x_i]}(f^2)\Delta x_i \leq 2M \sum \omega_{[x_{i-1},x_i]}(f)\Delta x_i$. Then by Theorem 3.1.3, it is easy to see that the integrability of f implies the integrability of f^2.

Exercise 3.1.8

Suppose $|f(x)| < M$ for a constant M. Then the oscillation of f on any interval is at most $2M$. For any $\epsilon > 0$, by the integrability of f on $[a+\epsilon, b]$, there is $\delta > 0$, such that any partition P of $[a+\epsilon, b]$ satisfying $\|P\| < \delta$ would imply $\sum \omega_{[x_{i-1},x_i]}(f)\Delta x_i \leq \epsilon$.

Now let P be a partition of $[a, b]$ satisfying $\|P\| < \min\{\epsilon, \delta\}$. Let $x_{j-1} \leq a + \epsilon < x_j$. Let P_ϵ be the partition $a + \epsilon < x_j < x_{j+1} < \cdots < x_n = b$ of $[a+\epsilon, b]$. Since the partition P_ϵ satisfies

$\|P_\epsilon\| \leq \|P\| < \delta$, the Riemann sum of the oscillation with respect to it is

$$\omega_{[a+\epsilon, x_j]}(f)(x_j - (a+\epsilon)) + \sum_{i=j}^{n} \omega_{[x_{i-1}, x_i]}(f)\Delta x_i < \epsilon,$$

which implies

$$\sum_{i=j}^{n} \omega_{[x_{i-1}, x_i]}(f)\Delta x_i < \epsilon.$$

On the other hand, $x_j - a = (x_j - (a+\epsilon)) + \epsilon \leq (x_j - x_{j-1}) + \epsilon \leq \|P\| + \epsilon < 2\epsilon$. Therefore

$$\sum_{i=1}^{j} \omega_{[x_{i-1}, x_i]}(f)\Delta x_i \leq 2M \sum_{i=1}^{j} \Delta x_i = 2M(x_j - a) < 4M\epsilon.$$

Thus we conclude that the Riemann sum of the oscillation with respect to P is

$$\sum_{i=j}^{n} \omega_{[x_{i-1}, x_i]}(f)\Delta x_i + \sum_{i=1}^{j} \omega_{[x_{i-1}, x_i]}(f)\Delta x_i \leq (4M+1)\epsilon.$$

By Theorem 3.1.3, the function is integrable on $[a, b]$.

Exercise 3.1.9

Since f is convex, $f'_-(x)$ and $f'_+(x)$ exists and are increasing. Therefore the one side derivatives are integrable. By choosing $x_i^* = x_i$, we have

$$S(P, f'_-(x)) = \sum f'_-(x_i)(x_i - x_{i-1}) \geq \sum(f(x_i) - f(x_{i-1})) = f(b) - f(a).$$

This implies $\int_a^b f'_-(x)dx \geq f(b) - f(a)$. On the other hand, by choosing $x_i^* = x_{i-1}$, we have

$$S(P, f'_-(x)) = \sum f'_-(x_{i-1})(x_i - x_{i-1}) \leq \sum f'_+(x_{i-1})(x_i - x_{i-1}) \leq \sum(f(x_i) - f(x_{i-1})) = f(b) - f(a).$$

This implies $\int_a^b f'_-(x)dx \leq f(b) - f(a)$. Thus we conclude $\int_a^b f'_-(x)dx = f(b) - f(a)$. The proof for $\int_a^b f'_+(x)dx = f(b) - f(a)$ is similar.

Exercise 3.1.10

The integrability of $f(x)^2$ or $|f(x)|$ does not imply the integrability of $f(x)$. A counterexample is given by $\begin{cases} 1 & \text{if } x \text{ rational} \\ -1 & \text{if } x \text{ irrational} \end{cases}$. The integrability of $f(x)^3$ implies the integrability of $f(x)$ because $f(x)$ is the composition of $\sqrt[3]{y}$ with $f(x)^3$, and $\sqrt[3]{y}$ is continuous.

Exercise 3.1.11

For $x_2 > x_1$, we have $\phi(x_2) - \phi(x_1) > A(x_2 - x_1) > 0$, so that ϕ is strictly increasing. The inequality $0 < \phi(x_2) - \phi(x_1) < B(x_2 - x_1)$ also implies ϕ is continuous. Therefore $y = \phi(x)$ is a change of variable between $x_1 \leq x \leq x_2$ and $\phi(x_1) \leq y \leq \phi(x_2)$, and

$$\sup_{x_1 \leq x \leq x_2} f(\phi(x)) = \sup_{\phi(x_1) \leq y \leq \phi(x_2)} f(y), \qquad \inf_{x_1 \leq x \leq x_2} f(\phi(x)) = \inf_{\phi(x_1) \leq y \leq \phi(x_2)} f(y).$$

As the difference between the supremum and infimum, the oscillations are equal:

$$\omega_{[x_1,x_2]}(f \circ \phi) = \omega_{[\phi(x_1),\phi(x_2)]}(f).$$

Let P be a partition of $[a, b]$. We have $\omega_{[x_{i-1},x_i]}(f \circ \phi) = \omega_{[\phi(x_{i-1}),\phi(x_i)]}(f)$ from the second part. Then

$$\sum \omega_{[x_{i-1},x_i]}(f \circ \phi)(x_i - x_{i-1}) \leq \frac{1}{A} \sum \omega_{[\phi(x_{i-1}),\phi(x_i)]}(f)(\phi(x_i) - \phi(x_{i-1}))$$

Now $\phi(P)$ is a partition of $[\phi(a), \phi(b)]$. By $\phi(x_2) - \phi(x_1) < B(x_2 - x_1)$, we also know $\|P\| < B\|P\|$.

Since $f(y)$ is integrable on $[\phi(a), \phi(b)]$, for any $\epsilon > 0$, there is $\delta > 0$, such that $\|\phi(P)\| < \delta$ implies $\sum \omega_{[\phi(x_{i-1}),\phi(x_i)]}(f)(\phi(x_i) - \phi(x_{i-1})) \leq A\epsilon$. Thus

$$\|P\| < \frac{\delta}{B} \implies \|\phi(P)\| < B\|P\| < \delta \implies \sum \omega_{[x_{i-1},x_i]}(f \circ \phi)(x_i - x_{i-1}) < \frac{1}{A} A\epsilon = \epsilon.$$

This completes the proof of the integrability of $f(\phi(x))$.

Finally, assume ϕ is differentiable, with $A < \phi'(x) < B$. Then for $x_1 < x_2$, by the mean value theorem, we have $\phi(x_2) - \phi(x_1) = \phi'(c)(x_2 - x_1)$. Then $A < \phi'(c) < B$ implies $A(x_2 - x_1) < \phi(x_2) - \phi(x_1) < B(x_2 - x_1)$.

EXERCISE 3.1.12

We have $\max\{f,g\} = \frac{1}{2}(|f+g|+f+g)$. The integrability of f and g implies the integrability of $f+g$. Then by the continuity of the absolute value, $|f+g|$ is integrable. Then as a linear combination of $|f+g|$, f, g, $\max\{f,g\}$ is also integrable.

EXERCISE 3.1.13

The function $f(x) = \begin{cases} x & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$ is integrable on $[0,1]$. However, $\frac{1}{f(x)}$ is not bounded on $[0,1]$, so that it is not integrable on $[0,1]$.

EXERCISE 3.1.14

We have $\inf_{[a,b]} f \le f(x) \le \sup_{[a,b]} f$ for any $a \le x \le b$. Then by Proposition 3.1.8, we have

$$(b-a)\inf_{[a,b]} f = \int_a^b \inf_{[a,b]} f dx \le \int_a^b f(x)dx \le \int_a^b \sup_{[a,b]} f dx = (b-a)\sup_{[a,b]} f.$$

EXERCISE 3.1.15

By Exercise 3.1.14, we have $\min_{[a,b]} f \le \frac{1}{b-a}\int_a^b f(x)dx \le \max_{[a,b]} f$. Then by the continuity of $f(x)$ and the intermediate value theorem, we have $\frac{1}{b-a}\int_a^b f(x)dx = f(c)$ for some $a < c < b$.

More generally, for $g(x) \ge 0$, we have $(\min_{[a,b]} f)g(x) \le f(x)g(x) \le (\max_{[a,b]} f)g(x)$. Therefore $(\min_{[a,b]} f)\int_a^b g(x)dx \le \int_a^b f(x)g(x)dx \le (\max_{[a,b]} f)\int_a^b g(x)dx$. This is the same as $\min_{[a,b]} f \le \frac{\int_a^b f(x)g(x)dx}{\int_a^b g(x)dx} \le \max_{[a,b]} f$. By the continuity of f and the intermediate value theorem, we get $\frac{\int_a^b f(x)g(x)dx}{\int_a^b g(x)dx} = f(c)$ for some $a < c < b$.

EXERCISE 3.1.16

For any fixed $a \le c \le b$, we have $|f(c) - f(x)| \le \omega_{[a,b]}(f)$ for any x in $[a,b]$. Therefore

$$\begin{aligned}\left|f(c)(b-a) - \int_a^b f(x)dx\right| &= \left|\int_a^b f(c)dx - \int_a^b f(x)dx\right| = \left|\int_a^b (f(c) - f(x))dx\right| \\ &\le \int_a^b |f(c) - f(x)|dx \le \int_a^b \omega_{[a,b]}(f)dx = \omega_{[a,b]}(f)(b-a).\end{aligned}$$

The estimation further implies

$$\begin{aligned}\left|S(P,f) - \int_a^b f(x)dx\right| &= \left|\sum \left(f(x_i^*)\Delta x_i - \int_{x_{i-1}}^{x_i} f(x)dx\right)\right| \\ &\le \sum \left|f(x_i^*)(x_i - x_{i-1}) - \int_{x_{i-1}}^{x_i} f(x)dx\right| \le \sum \omega_{[x_{i-1},x_i]}(f)\Delta x_i.\end{aligned}$$

EXERCISE 3.1.17

The continuous function is uniformly continuous on $[a-h, b+h]$. Thus for any $\epsilon > 0$, there is $\delta > 0$, such that $|x-y| < \delta$ and $a-h \le x, y \le b+h$ imply $|f(x)-f(y)| < \epsilon$. Now if $|h| < \delta$, then

$$\int_a^b |f(x+h)-f(x)|dx \le (b-a)\epsilon.$$

This implies $\lim_{h\to 0} \int_a^b |f(x+h)-f(x)|dx = 0$.

EXERCISE 3.1.18

It is clear that the first implies the second. The second implies the third because

$$\int_a^b |f(x)|dx = \int_a^c |f(x)|dx + \int_c^d |f(x)|dx + \int_d^b |f(x)|dx \ge \int_c^d |f(x)|dx \ge \left|\int_c^d f(x)dx\right|.$$

The second also implies the fourth because for $M = \max_{[a,b]} |g|$, we have

$$M\int_a^b |f(x)|dx \ge \int_a^b |f(x)g(x)|dx \ge \left|\int_a^b f(x)g(x)dx\right|.$$

It remains to prove that either third or the fourth implies $f = 0$. Equivalently, we will assume $f \ne 0$ and prove that the third and the fourth are broken.

Assume $f(c) \ne 0$ for some $c \in [a, b]$. If $f(c) > 0$, then by the continuity of $f(x)$ at c, there is $\delta > 0$, such that $f(x) > \dfrac{f(c)}{2}$ for $x \in [c-\delta, c+\delta]$ (the interval is $[a, a+\delta]$ if $c = a$ and is $[b-\delta, b]$ if $c = b$). Therefore $\displaystyle\int_{c-\delta}^{c+\delta} f(x)dx \ge 2\delta\frac{f(c)}{2} > 0$, and the third is broken. Moreover, we construct a continuous function $g(x)$ satisfying

$$g(x) \ge 0, \quad g(x) = 1 \text{ on } \left[c-\frac{\delta}{2}, c+\frac{\delta}{2}\right], \quad g(x) = 0 \text{ on } [a, c-\delta] \cup [c+\delta, b].$$

Then

$$\int_a^b f(x)g(x)dx = \int_{c-\delta}^{c+\delta} f(x)g(x)dx \ge \int_{c-\frac{\delta}{2}}^{c+\frac{\delta}{2}} f(x)g(x)dx \ge \delta\frac{f(c)}{2} > 0.,$$

and the fourth is broken. The argument for the case $f(c) < 0$ is similar.

EXERCISE 3.1.19

If $f(x)$ changes sign, then both $|f(x)| - f(x)$ and $|f(x)| + f(x)$ are continuous non-negative functions that are not constantly 0. By Exercise 3.1.18, we have

$$\int_a^b (|f(x)| - f(x))dx > 0, \quad \int_a^b (|f(x)| + f(x))dx > 0.$$

Combining the two, we get $\displaystyle\left|\int_a^b f(x)dx\right| < \int_a^b |f(x)|dx$.

EXERCISE 3.1.20

If $f(x)$ is never zero on (a, b), then by the continuity, we have either $f(x) > 0$ on (a, b) or $f(x) < 0$ on (a, b). Then it follows from Exercise 3.1.18 that $\displaystyle\int_a^b f(x)dx = \pm\int_a^b |f(x)|dx \ne 0$.

If $f(x)$ has only one zero at $c \in (a,b)$, then by continuity, we have either $f(x) \geq 0$ on (a,b), or $f(x) \leq 0$ on (a,b), or $f(x) < 0$ on (a,c) and $f(x) > 0$ on (c,b), or $f(x) > 0$ on (a,c) and $f(x) < 0$ on (c,b). In the first two cases, since f is not constantly zero, it follows again from Exercise 3.1.18 that $\int_a^b f(x)dx \neq 0$. In the last two cases, we have $(x-c)f(x) \geq 0$ on (a,b) or $(x-c)f(x) < 0$ on (a,b). Then

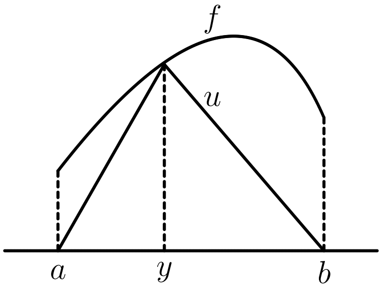
$$\int_a^b xf(x)dx - c\int_a^b f(x)dx = \int_a^b (x-c)f(x)dx \neq 0$$

by the same reason.

In all the cases, we find contradiction to the assumptions. Therefore $f(x)$ must be zero at two places in (a,b).

EXERCISE 3.1.21

Let $u(x)$ be the straight line connecting $(a,0)$ to $(y,f(y))$, followed by straight line connecting $(y,f(y))$ to $(b,0)$. By $f(x) \geq 0$ and the concavity, we have $f \geq u$ on $[a,b]$. Then $\int_a^b f(x)dx \geq \int_a^b u(x)dx = \frac{1}{2}f(y)(b-a)$.



EXERCISE 3.1.22

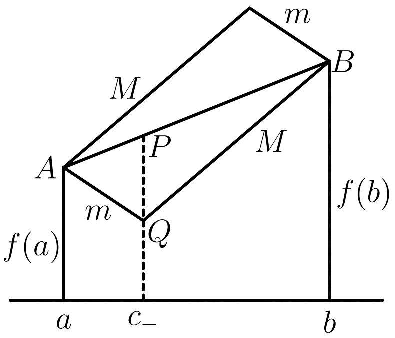
Let L_- be the straight line $f(a)+m(x-a)$ followed by the straight line $f(b)+M(x-b)$. Then

$$L_-(x) = \begin{cases} f(a)+m(x-a) & \text{if } a \leq x \leq c_- \\ f(b)+M(b-x) & \text{if } c_- \leq x \leq b \end{cases},$$

where $c_- = \dfrac{f(a)-f(b)+Mb-ma}{M-m}$ is where the two lines meet. We clearly have $L_-(x) \leq f(x)$ on $[a,b]$. Similarly, we have $L_+(x) \geq f(x)$ on $[a,b]$, where

$$L_+(x) = \begin{cases} f(a)+M(x-a) & \text{if } a \leq x \leq c_+ \\ f(b)+m(b-x) & \text{if } c_+ \leq x \leq b \end{cases},$$

and $c_+ = \dfrac{f(a)-f(b)+Mb-ma}{M-m}$. As illustrated in the picture, the graph of f lies in the parallelogram. The two lower edges of the parallelogram form the graph of L_-, and the two upper edges form the graph of L_+.



The difference $\left|\int_a^b f(x)dx - \frac{f(a)+f(b)}{2}(b-a)\right|$ is no more than the area of the triangle ABQ. The area of the triangle is half of $(b-a)$ multiplied to the length of PQ. The length of PQ is

$$(c_- - a)(\text{slope of } AP - \text{slope of } AQ) = \frac{f(a)-f(b)+M(b-a)}{M-m}\left(\frac{f(b)-f(a)}{b-a} - m\right)$$

Thus the area of the triangle is $\dfrac{(f(a)-f(b)+M(b-a))(f(b)-f(a)-m(b-a))}{2(M-m)}$ and the inequality follows.

Exercise 3.1.23

Suppose $f(b) \le 1$. For any $b-a > \epsilon > 0$, we have

$$\begin{aligned} 0 \le \int_a^b f(x)^n dx &= \int_a^{b-\epsilon} f(x)^n dx + \int_{b-\epsilon}^b f(x)^n dx \le \int_a^{b-\epsilon} dx + \int_{b-\epsilon}^b f(1)^n dx \\ &\le f(b-\epsilon)^n(b-\epsilon-a) + \epsilon f(1)^n \le f(b-\epsilon)^n(b-a) + \epsilon. \end{aligned}$$

By $0 \le f(b-\epsilon) < f(1) \le 1$, there is N, such that $n > N$ implies $f(b-\epsilon)^n < \epsilon$. This further implies $0 \le \int_a^b f(x)^n dx < (b-a+1)\epsilon$. Therefore $\lim_{n\to\infty} \int_a^b f(x)^n dx = 0$.

Suppose $f(b) > 1$ and f is continuous at b. Then there is $a < c < b$, such that $f(c) > 1$. By non-negativity and the increasing property, have

$$\int_a^b f(x)^n dx = \int_a^c f(x)^n dx + \int_c^b f(x)^n dx \ge \int_c^b f(x)^n dx \ge (b-c)f(c)^n.$$

By $f(c) > 1$, we have $\lim_{n\to\infty} f(c)^n = +\infty$. Therefore $\lim_{n\to\infty} \int_a^b f(x)^n dx = +\infty$.

Exercise 3.1.24

First assume $f(a) \ne 0$, $f(b) \ne 0$. Then the condition tells us $0 < m \le f(x) \le M$ for some constants m, M and all $x \in [a,b]$. Since $\lim_{n\to\infty} \sqrt[n]{m} = \lim_{n\to\infty} \sqrt[n]{M} = 1$, for any $\epsilon > 0$, there is N, such that $n > N$ implies $|\sqrt[n]{m} - 1| < \epsilon$ and $|\sqrt[n]{M} - 1| < \epsilon$. Since $\sqrt[n]{f(x)}$ is sandwiched between $\sqrt[n]{m}$ and $\sqrt[n]{M}$, this further implies $|\sqrt[n]{f(x)} - 1| < \epsilon$ for all $x \in [a,b]$, and

$$\left|\int_a^b g(x)\sqrt[n]{f(x)}dx - \int_a^b g(x)dx\right| \le \int_a^b |g(x)||\sqrt[n]{f(x)} - 1|dx \le \epsilon\int_a^b |g(x)|dx.$$

This completes the proof that $\lim_{n\to\infty} \int_a^b g(x)\sqrt[n]{f(x)}dx = \int_a^b g(x)dx$.

Now assume $f(a)=0$, $f(b)\neq 0$. There is $\delta>0$, such that $f(x)<1$ on $[a,a+\delta]$. Now for any fixed $a<c\le a+\delta$, we have $\lim_{n\to\infty}\int_c^b g(x)\sqrt[n]{f(x)}dx=\int_c^b g(x)dx$. Thus for any $\epsilon>0$, there is N, such that $n>N$ implies $\left|\int_c^b g(x)\sqrt[n]{f(x)}dx-\int_c^b g(x)dx\right|<\epsilon$. Then by $0\le f(x)<1$ on $[a,c]$, $n>N$ implies

$$\begin{aligned}&\left|\int_a^b g(x)\sqrt[n]{f(x)}dx-\int_a^b g(x)dx\right|\\ \le&\left|\int_c^b g(x)\sqrt[n]{f(x)}dx-\int_c^b g(x)dx\right|+\left|\int_a^c g(x)\sqrt[n]{f(x)}dx\right|+\left|\int_a^c g(x)dx\right|\\ \le&\epsilon+2(c-a)\int_a^c|g(x)|dx.\end{aligned}$$

By fixing c very close to a at first, the right side can be made very small ($<2\epsilon$ for example). This proves that $\lim_{n\to\infty}\int_a^b g(x)\sqrt[n]{f(x)}dx=\int_a^b g(x)dx$.

The proof for the other cases is similar.

Exercise 3.1.25

Since $f(x)$ is continuous, for any $\epsilon>0$, there are $a<c<d<b$, such that $f(x)>\max f-\epsilon$ on $[c,d]$. Then

$$\int_a^b f(x)^p dx=\int_a^c f(x)^p dx+\int_c^d f(x)^p dx+\int_d^b f(x)^p dx\ge\int_c^d f(x)^p dx\ge(d-c)(\max f-\epsilon)^p,$$

and

$$\int_a^b f(x)^p dx\le(b-a)(\max f)^p.$$

Therefore

$$(b-a)^{\frac{1}{p}}\max f\ge\left(\int_a^b f(x)^p dx\right)^{\frac{1}{p}}\ge(d-c)^{\frac{1}{p}}(\max f-\epsilon).$$

As $p\to\infty$, the left side has limit $\max f$ and the right side has limit $\max f-\epsilon$. Therefore for sufficiently big p, we get

$$\max f+\epsilon>\left(\int_a^b f(x)^p dx\right)^{\frac{1}{p}}>(\max f-\epsilon)-\epsilon=\max f-2\epsilon.$$

This proves that $\lim_{p\to+\infty}\left(\int_a^b f(x)^p dx\right)^{\frac{1}{p}}=\max_{[a,b]}f(x)$.

Exercise 3.1.26

For any partition P, we have

$$\begin{aligned}\left|f(x_i^*)\Delta x_i-\int_{x_{i-1}}^{x_i}f(x)dx\right|&\le\int_{x_{i-1}}^{x_i}|f(x_i^*)-f(x)|dx\le L\int_{x_{i-1}}^{x_i}|x_i^*-x|dx\\ &=L\int_{x_{i-1}}^{x_i}(x_i^*-x)dx+L\int_{x_{i-1}}^{x_i}(x-x_i^*)dx=L\frac{(x_i^*-x_{i-1})^2+L(x_i-x_i^*)^2}{2}\\ &=L\frac{(x_i-x_{i-1})^2}{2}-L(x_i^*-x_{i-1})(x_i-x_i^*)\le\frac{L}{2}\Delta x_i^2.\end{aligned}$$

Then adding together, we get

$$\begin{aligned}\left|S(P,f)-\int_a^b f(x)dx\right| &= \left|\sum f(x_i^*)\Delta x_i-\sum\int_{x_{i-1}}^{x_i} f(x)dx\right| \\ &\le \sum\left|f(x_i^*)\Delta x_i-\int_{x_{i-1}}^{x_i} f(x)dx\right| \le \frac{L}{2}\sum\Delta x_i^2.\end{aligned}$$

Exercise 3.1.27

$$\left|\sum f(x_i^*)\int_{x_{i-1}}^{x_i} g(x)dx-\int_a^b f(x)g(x)dx\right| = \left|\sum\left(f(x_i^*)\int_{x_{i-1}}^{x_i} g(x)dx-\int_{x_{i-1}}^{x_i} f(x)g(x)dx\right)\right|$$

$$\le\sum\left|\int_{x_{i-1}}^{x_i}(f(x_i^*)-f(x))g(x)dx\right| \le \sum\int_{x_{i-1}}^{x_i}\omega_{[x_{i-1},x_i]}(f)\sup_{[x_{i-1},x_i]}|g|dx$$

$$\le\sum\omega_{[x_{i-1},x_i]}(f)\sup_{[x_{i-1},x_i]}|g|\Delta x_i \le \sup_{[a,b]}|g|\sum\omega_{[x_{i-1},x_i]}(f)\Delta x_i.$$

Exercise 3.1.28

The condition on ϕ tells us that for any $\epsilon > 0$, there is $\delta > 0$, such that

$$|t| < \delta \implies |\phi(t) - t| \le \epsilon |t|.$$

Since $f(x)$ is integrable, we have $|f(x)| \le M$ for some constant M. Then

$$\|P\| < \frac{\delta}{M} \implies |f(x_i^*)\Delta x_i| < M\|P\| < \delta \implies |\phi(f(x_i^*)\Delta x_i) - f(x_i^*)\Delta x_i| \le \epsilon |f(x_i^*)\Delta x_i|.$$

Therefore

$$|S_\phi(P,f) - S(P,f)| \le \sum_{i=1}^n |\phi(f(x_i^*)\Delta x_i) - f(x_i^*)\Delta x_i| \le \epsilon \sum_{i=1}^n |f(x_i^*)\Delta x_i| \le \epsilon M(b-a).$$

This shows that $\lim_{\|P\|\to 0} S_\phi(P,f)$ converges if and only if $\lim_{\|P\|\to 0} S(P,f)$ converges, and the two limits are the same.

Exercise 3.1.29

We have $\log \Pi(P,f) = S_\phi(P,f)$, the modified Riemann sum with $\phi(t) = \log(1+t)$. Then the conclusion follows from Exercise 3.1.28.

Exercise 3.1.30

By Theorem 3.1.3, for any $\epsilon > 0$, there is $\delta > 0$, such that

$$\|P\| < \delta \implies \sum \omega_{[x_{i-1},x_i]}(f)\Delta x_i < \epsilon(b-a).$$

Given the right side, we must have $\omega_{[x_{i-1},x_i]}(f) < \epsilon$ for some i, because otherwise, we would have

$$\omega_{[x_{i-1},x_i]}(f) \ge \epsilon \implies \sum \omega_{[x_{i-1},x_i]}(f)\Delta x_i \ge \epsilon \sum \Delta x_i = \epsilon(b-a),$$

contradicting to the right side.

Exercise 3.1.31

Since $\lim_{n\to\infty} \omega_{[a_n,b_n]}(f) = 0$, for any $\epsilon > 0$, there is n, such that $\omega_{[a_n,b_n]}(f) < \epsilon$. This means

$$a_n \le x, y \le b_n \implies |f(x) - f(y)| < \epsilon.$$

Since $a_n < c < b_n$, for $\delta = \min\{c - a_n, b_n - c\} > 0$, we have

$$|x - c| < \delta \implies a_n \le x \le b_n \implies |f(x) - f(c)| < \epsilon.$$

This proves that f is continuous at c.

Exercise 3.1.32

Suppose f is integrable on $[a,b]$. By Exercise 3.1.30, for $\epsilon = 1$, we can find some $[a_1,b_1] = [x_{i-1},x_i] \subset [a,b]$, such that $\omega_{[a_1,b_1]}(f) < 1$. In fact, since f is also integrable on $[a+\delta, b-\delta]$, by carrying out the argument on $[a+\delta, b-\delta]$ instead of $[a,b]$, we may further assume $[a_1,b_1] = [x_{i-1},x_i] \subset (a,b)$. Next, by the integrability of f on $[a_1.b_1]$, for $\epsilon = \frac{1}{2}$, we can find some $[a_2,b_2] \subset (a_1,b_1)$, such that $\omega_{[a_2,b_2]}(f) < \frac{1}{2}$. Keep going, we find a sequence of intervals $[a_n,b_n] \subset [a,b]$, such that $[a_{n+1},b_{n+1}] \subset (a_n,b_n)$ and $\omega_{[a_n,b_n]}(f) < \frac{1}{n}$. The inclusion between the intervals tells us there is c such that $a_n < c < b_n$ for all n. Then we may apply Exercise 3.1.31 to conclude that f is continuous at c.

The existence of a continuous point in any interval (c, d) contained in $[a, b]$ follows from the fact that f is also integrable on $[c, d]$.

EXERCISE 3.1.33

The limit that defines $\omega(x)$ converges because $\omega_{[x-\delta,x+\delta]}(f)$ is increasing in δ. The limit as $\delta \to 0^+$ is then the limit of a monotone function, and we have $\omega(x) \le \omega_{[x-\delta,x+\delta]}(f)$ for any δ.

The continuity of $f(x)$ at x_0 means that for any $\epsilon > 0$, there is $\delta > 0$, such that $|x - x_0| < \delta$ implies $|f(x) - f(x_0)| < \epsilon$. By the Cauchy criterion, it is easy to see that this is equivalent to for any $\epsilon > 0$, there is $\delta > 0$, such that $|x - x_0| \le \delta$ and $|y - x_0| \le \delta$ implies $|f(x) - f(y)| < \epsilon$. Since the condition is equivalent to $\omega_{[x-\delta,x+\delta]}(f) < \epsilon$, the whole Cauchy criterion is the same as $\lim_{\delta\to 0^+} \omega_{[x-\delta,x+\delta]}(f) = 0$.

EXERCISE 3.1.34

Now suppose $f(x)$ is integrable. Then for any $\epsilon > 0$ and $\delta > 0$, by Theorem 3.1.3, there is a partition P (one among all with sufficiently small mesh), such that the corresponding Riemann sum of the oscillation

$$\sum \omega_{[x_{i-1},x_i]}(f)\Delta x_i < \delta\epsilon.$$

If (x_{i-1}, x_i) contains x with $\omega(x) \ge \delta$, then $\omega_{[x_{i-1},x_i]}(f) \ge \omega(x) \ge \delta$. Let U be the union of all such intervals and let $l(U)$ be the sum of the length of such intervals, then

$$\sum \omega_{[x_{i-1},x_i]}(f)\Delta x_i = \sum_{[x_{i-1},x_i]\subset U} \omega_{[x_{i-1},x_i]}(f)\Delta x_i \ge \delta \sum_{[x_{i-1},x_i]\subset U} \Delta x_i = \delta l(U).$$

Therefore $\delta\epsilon > \delta l(U)$ and we conclude $l(U) < \epsilon$.

The argument above missed the possibility that some partition points x_i may satisfy $\omega(x_i) \ge \delta$. However, since P is fixed in the argument above, the exception only happens at finitely many points. BY enclosing these finitely many points in very small open intervals, U is enlarged a little bit and we can still keep the length of $U < \epsilon$.

EXERCISE 3.1.35

By Exercise 3.1.34, for any $\epsilon > 0$ and natural number n, there is a finite union U_n of closed intervals such that

1. $\omega(x) \ge \dfrac{1}{n}$ implies $x \in U_n$.
2. The sum of the lengths of the intervals in U is $< \dfrac{\epsilon}{2^n}$.

Then $U = U_1 \cup U_2 \cup \cdots$ is a countable union of closed intervals satisfying

1. $\omega(x) \ge 0$ implies $x \in U$. By Exercise 3.1.33, this means that all the discontinuous points are contains in U.
2. The sum of the lengths of the intervals in U is $< \sum_{n=1}^{+\infty} \dfrac{\epsilon}{2^n} = \epsilon$.

EXERCISE 3.1.36

By Exercise 3.1.32, there is $c \in (a, b)$, such that $f(x)$ is continuous at c. By $f(c) > 0$, there is $\delta > 0$, such that $f(x) > \dfrac{f(c)}{2}$ on $[c-\delta, c+\delta]$. Then

$$\int_a^b f(x)dx = \int_a^{c-\delta} f(x)dx + \int_{c-\delta}^{c+\delta} f(x)dx + \int_{c+\delta}^b f(x)dx \geq \int_{c-\delta}^{c+\delta} f(x)dx \geq 2\delta\frac{f(c)}{2} > 0.$$

Exercise 3.1.37

The second property implies the fourth by

$$M\int_a^b |f(x)|dx \geq \int_a^b |f(x)g(x)|dx \geq \left|\int_a^b f(x)g(x)dx\right|.$$

The fourth implies the first because

$$\int_c^d f(x)dx = \int_a^b f(x)\chi_{[c,d]}(x)dx,$$

where $\chi_{[c,d]}(x) = \begin{cases} 1 & \text{if } x \in [c,d] \\ 0 & \text{if } x \notin [c,d] \end{cases}$ is integrable. The fourth also implies the third because continuous functions are integrable.

Next suppose the fifth property does not hold. The we have $f(x)$ continuous at some $c \in [a, b]$ and $f(c) \neq 0$. Suppose $f(c) > 0$. Then by the continuity at c, we have $\delta > 0$, such that $f(x) > 0$ on $[c-\delta, c+\delta]$. By Exercise 3.1.36, we get $\displaystyle\int_{c-\delta}^{c+\delta} f(x)dx > 0$, so that the first property does not hold. Moreover, we can also find a continuous function $g(x)$ such that $g(x) > 0$ on $\left[c-\dfrac{\delta}{2}, c+\dfrac{\delta}{2}\right]$, $g(x) = 0$ on $[a, b] - [c-\delta, c+\delta]$, and $g(x) \geq 0$ otherwise. Then $f(x)g(x) \geq 0$ and $f(x)g(x) > 0$ on $\left[c-\dfrac{\delta}{2}, c+\dfrac{\delta}{2}\right]$. By applying Exercise 3.1.36 to $f(x)g(x)$ on $\left[c-\dfrac{\delta}{2}, c+\dfrac{\delta}{2}\right]$, we get

$$\int_a^b f(x)g(x)dx \geq \int_{c-\frac{\delta}{2}}^{c+\frac{\delta}{2}} f(x)g(x)dx > 0.$$

Again the third property does not hold. Thus we have shown that both the first and the third property implies the fifth.

It remains to prove the fifth property implies the second. On any nontrivial interval $[c, d] \subset [a, b]$, by Exercise 3.1.32, f is continuous at some $\xi \in (c, d)$. By the assumption, we have $f(\xi) = 0$, which implies

$$|f(x)| = |f(x) - f(\xi)| \leq \omega_{[c,d]}(f) \text{ for any } x \in [c, d].$$

Therefore we get

$$S(P, |f|) = \sum |f(x_i^*)|\Delta x_i \leq \sum \omega_{[x_{i-1},x_i]}(f)\Delta x_i.$$

By the Riemann criterion, the right side has limit 0 as $\|P\| \to 0$. This implies that the left side has limit 0 as $\|P\| \to 0$, which is exactly the second property.

Exercise 3.1.38

Suppose the part of Q on $[x_{i-1}, x_i]$ is $Q_{[x_{i-1},x_i]}\colon x_{i-1} = y_k < y_{k+1} < \cdots < y_{l-1} < y_l = x_i$. Then for $k < j \le l$, we have $[y_{j-1}, y_j] \subset [x_{i-1}, x_i]$. This implies $\omega_{[y_{j-1},y_j]}(f) \le \omega_{[x_{i-1},x_i]}(f)$ and

$$\sum_{j=l+1}^{k} \omega_{[y_{j-1},y_j]}(f)\Delta y_j \le \omega_{[x_{i-1},x_i]}(f) \sum_{j=l+1}^{k} \Delta y_j = \omega_{[x_{i-1},x_i]}(f)\Delta x_i.$$

Adding these together for all i, we get

$$\sum \omega_{[y_{j-1},y_j]}(f)\Delta y_j \le \sum \omega_{[x_{i-1},x_i]}(f)\Delta x_i.$$

On the other hand, we may divide the sum $\sum \omega_{[x_{i-1},x_i]}(f)\Delta x_i$ into two parts. The first part $\sum'$ consists of the intervals $[x_{i-1}, x_i]$ in which extra points are added in order to obtain Q. The second part $\sum''$ consists of the intervals in which no extra points are added. Since total of k points are added, the first part consists of at most k intervals, and all the oscillations are no bigger than $\omega_{[a,b]}(f)$, we have

$$\sum{}' \omega_{[x_{i-1},x_i]}(f)\Delta x_i \le \omega_{[a,b]}(f) \sum{}' \Delta x_i \le \omega_{[a,b]}(f)k\|P\|.$$

Since P and Q are the same for the second part, we get

$$\sum{}'' \omega_{[x_{i-1},x_i]}(f)\Delta x_i = \sum{}'' \omega_{[y_{j-1},y_j]}(f)\Delta x_i \le \sum \omega_{[y_{j-1},y_j]}(f)\Delta x_i.$$

Combining the two parts, we get

$$\sum \omega_{[x_{i-1},x_i]}(f)\Delta x_i \le \omega_{[a,b]}(f)k\|P\| + \sum \omega_{[y_{j-1},y_j]}(f)\Delta x_i.$$

Exercise 3.1.39

The integrability condition is on one P and is weaker than the one in Theorem 3.1.3. So we only need to prove this condition implies the one in Theorem 3.1.3.

Suppose for a partition P (partition points denoted x_i), we have $\sum \omega_{[x_{i-1},x_i]}(f)\Delta x_i \le \epsilon$. Let Q (partition points denoted y_j) be any partition. Let Q' (partition points denoted z_k) be the partition obtained by combining P and Q together. Then

$$\sum \omega_{[z_{k-1},z_k]}(f)\Delta z_k \le \sum \omega_{[x_{i-1},x_i]}(f)\Delta x_i \le \epsilon$$

by the first inequality in Exercise 3.1.39. Moreover, if P consists of n points, then Q' is obtained from Q by adding no more than n points. Thus by the second inequality in the exercise,

$$\sum \omega_{[y_{j-1},y_j]}(f)\Delta y_j \le \sum \omega_{[z_{k-1},z_k]}(f)\Delta z_k + n\|Q\|\omega_{[a,b]}(f)$$

Combining the two estimations, we get

$$\sum \omega_{[y_{j-1},y_j]}(f)\Delta y_j \le \epsilon + n\|Q\|\omega_{[a,b]}(f).$$

This tells us that

$$\|Q\| < \frac{\epsilon}{n\omega_{[a,b]}(f)} \implies \sum \omega_{[y_{j-1},y_j]}(f)\Delta y_j < 2\epsilon.$$

Since n is a fixed number for the fixed P, the condition in Theorem 3.1.3 is verified.

Exercise 3.1.40

We have $U(P,f) \geq L(P,f)$. Moreover, by

$$U(P,f) = \sum \left(\sup_{[x_{i-1},x_i]} f(x) \right) \Delta x_i \leq \sum \left(\sup_{[a,b]} f(x) \right) \Delta x_i = \left(\sup_{[a,b]} f(x) \right) (b-a),$$

for a refinement Q of P, we have

$$U(Q,f) = \sum U(Q_{[x_{i-1},x_1]}, f) \leq \sum \left(\sup_{[x_{i-1},x_1]} f \right) (x_i - x_{i-1}) = U(P,f).$$

The inequality $L(Q,f) \geq L(P,f)$ can be proved similarly.

Exercise 3.1.41

Denote $\bar{I} = \overline{\int_a^b} f(x)dx$. Then for any $\epsilon > 0$, there is P, such that $U(P,f) < \bar{I} + \epsilon$. Now for any partition Q, let Q' be the partition obtained by combining P and Q together. Suppose P contains n partition points, then Q' is obtained from Q by adding no more than n points. By a proof similar to Exercise 3.1.38, we get

$$U(Q,f) \leq U(Q',f) + n\|Q\| \sup_{[a,b]} f.$$

Since Q' is also a refinement of P, by Exercise 3.1.40, we also have $U(Q',f) \leq U(P,f) < \bar{I} + \epsilon$. Thus we conclude that

$$\|Q\| < \frac{\epsilon}{n \sup_{[a,b]} |f|} \implies U(Q,f) < \bar{I} + 2\epsilon.$$

On the other hand, the definition of $\bar{I}$ tells us $\bar{I} \leq U(Q,f)$. This proves that $\bar{I} = \lim_{\|P\|\to 0} U(P,f)$. The proof for the lower Darboux sum and integral is similar.

Exercise 3.1.42

The inequality follows from Exercise 3.1.40. Moreover,Exercise 3.1.41 tells us that

$$\overline{\int_a^b} f(x)dx - \underline{\int_a^b} f(x)dx = \lim_{\|P\|\to 0} (U(P,f) - L(P,f)).$$

By

$$U(P,f) - L(P,f) = \sum \left(\sup_{[x_{i-1},x_i]} f(x) - \inf_{[x_{i-1},x_i]} f(x) \right) \Delta x_i = \sum \omega_{[x_{i-1},x_1]}(f) \Delta x_i$$

and Theorem 3.1.3, the upper and lower integrals are equal if and only if f is integrable.

Finally, the Riemann integral $\int_a^b f(x)dx$ is the common value by $L(P,f) \leq S(P,f) \leq U(P,f)$ and a proof similar to the sandwich rule.

Exercise 3.1.43

We have $L(P,f)\le\sum\phi_i\Delta x_i\le U(P,f)$. The a proof similar to the sandwich rule shows that

$$\lim_{\|P\|\to 0}L(P,f)=\lim_{\|P\|\to 0}U(P,f)=\int_a^b f(x)dx \implies \lim_{\|P\|\to 0}\sum\phi_i\Delta x_i=\int_a^b f(x)dx.$$

EXERCISE 3.1.44

First we consider Proposition 3.1.9.

For the same partition P and the same choice of x_i^*, we have $S(P,f+g)=S(P,f)+S(P,g)$, $S(P,cf)=cS(P,f)$. Then

$$U(P,f+g)=\sup_{\text{all }x_i^*}(S(P,f)+S(P,g))\le\sup_{\text{all }x_i^*}S(P,f)+\sup_{\text{all }x_i^*}S(P,g)=U(P,f)+U(P,g),$$

and

$$U(P,cf)=\sup_{\text{all }x_i^*}cS(P,f)=\begin{cases}c\sup_{\text{all }x_i^*}S(P,f)=cU(P,f) & \text{if } c\ge 0\\ c\inf_{\text{all }x_i^*}S(P,f)=cL(P,f) & \text{if } c<0\end{cases}.$$

Therefore

$$\overline{\int_a^b}(f(x)+g(x))dx\le\overline{\int_a^b}f(x)dx+\overline{\int_a^b}g(x)dx,\quad \overline{\int_a^b}cf(x)dx=\begin{cases}c\overline{\int_a^b}f(x)dx & \text{if } c\ge 0\\ c\underline{\int_a^b}f(x)dx & \text{if } c<0\end{cases}.$$

There are similar relations for the lower Darboux integrals. Moreover, by

$$\overline{\int_a^b}(f(x)+g(x))dx-\underline{\int_a^b}f(x)dx=\overline{\int_a^b}(f(x)+g(x))dx+\overline{\int_a^b}(-f(x))dx\ge\overline{\int_a^b}g(x)dx,$$

we get

$$\overline{\int_a^b}(f(x)+g(x))dx\ge\underline{\int_a^b}f(x)dx+\overline{\int_a^b}g(x)dx.$$

The inequalities may become strict. For example, take $f(x)=D(x)$, $g(x)=1-D(x)$. Then

$$\overline{\int_a^b}(f(x)+g(x))dx=\overline{\int_a^b}f(x)dx=\overline{\int_a^b}g(x)dx=b-a.$$

Next we consider Proposition 3.1.8.

We have

$$f\le g\implies\sup f\le\sup g\implies U(P,f)\le U(P,g)\implies\overline{\int_a^b}f(x)dx\le\overline{\int_a^b}g(x)dx.$$

Finally we consider Proposition 3.1.9.

For any $\epsilon>0$, there are partitions P' and P'' of $[a,b]$ and $[a,c]$ and choices of x_i^* for both, such that

$$S(P',f)\le\overline{\int_a^b}f(x)dx+\epsilon,\quad S(P'',f)\le\overline{\int_b^c}f(x)dx+\epsilon.$$

By combining the partitions and choices of x_i^* together, we have a partition P of $[a,c]$ and choices, such that

$$\overline{\int_a^c} f(x)dx \le S(P,f) = S(P',f) + S(P'',f) \le \overline{\int_a^b} f(x)dx + \overline{\int_b^c} f(x)dx + 2\epsilon.$$

On the other hand, for any $\epsilon > 0$, there is a partition P of $[a,c]$ and choices of x_i^*, such that

$$\overline{\int_a^c} f(x)dx + \epsilon \ge S(P,f).$$

By the proof of Proposition 3.1.9, we can construct partitions P' and P'' of $[a,b]$ and $[a,c]$ and choices of x_i^* for both, such that

$$|S(P,f) - S(P',f) - S(P'',f)| \le 2(\sup|f|)\|P\|.$$

Therefore

$$\begin{aligned}\overline{\int_a^b} f(x)dx + \overline{\int_b^c} f(x)dx &\le S(P',f) + S(P'',f) \le S(P,f) + 2(\sup|f|)\|P\| \\ &\le \overline{\int_a^c} f(x)dx + \epsilon + 2(\sup|f|)\|P\|.\end{aligned}$$

Combining the inequalities and the fact that ϵ and $\|P\|$ can be arbitrarily small, we conclude that

$$\overline{\int_a^b} f(x)dx + \overline{\int_b^c} f(x)dx = \overline{\int_a^c} f(x)dx.$$

Exercise 3.1.45

Since intervals in P has equal length δ, for $x \in [x_{i-1}, x_i]$ and $|t| < \delta$, we have $x + t \in [x_{i-2}, x_{i-1}] \cup [x_{i-1}, x_i] \cup [x_i, x_{i+1}] = [x_{i-2}, x_{i+1}]$. This implies

$$|f(x+t) - f(x)| \le \omega_{[x_{i-2},x_{i+1}]}(f) \le \omega_{[x_{i-2},x_{i-1}]}(f) + \omega_{[x_{i-1},x_i]}(f) + \omega_{[x_i,x_{i+1}]}(f)$$

and

$$\begin{aligned}\int_{x_{i-1}}^{x_i} |f(x+t) - f(x)|dx \le& \Delta x_i(\omega_{[x_{i-2},x_{i-1}]}(f) + \omega_{[x_{i-1},x_i]}(f) + \omega_{[x_i,x_{i+1}]}(f)) \\ =& \delta(\omega_{[x_{i-2},x_{i-1}]}(f) + \omega_{[x_{i-1},x_i]}(f) + \omega_{[x_i,x_{i+1}]}(f)).\end{aligned}$$

Exercise 3.1.46

For any natural number n, we divide $[a,b]$ evenly into n parts and get the partition P_n, with $x_i = \dfrac{(n-i)a + ib}{n}$. Further denote $\delta_n = \|P_n\| = \dfrac{b-a}{n}$, $x_{-1} = a - \delta$, $x_{n+1} = b + \delta$. Then by

Exercise 3.1.45, for $|t| < \delta$, we have

$$\begin{aligned}\int_a^b |f(x+t)-f(x)|dx =&\sum_{i=1}^n \int_{x_{i-1}}^{x_i} |f(x+t)-f(x)|dx\\ \le&\sum_{i=1}^n \delta(\omega_{[x_{i-2},x_{i-1}]}(f)+\omega_{[x_{i-1},x_i]}(f)+\omega_{[x_i,x_{i+1}]}(f))\\ \le&\delta\left(\sum_{i=0}^{n-1}\omega_{[x_{i-1},x_i]}(f)+\sum_{i=1}^{n}\omega_{[x_{i-1},x_i]}(f)+\sum_{i=2}^{n+1}\omega_{[x_{i-1},x_i]}(f)\right)\\ \le&\delta\left(3\sum_{i=1}^n\omega_{[x_{i-1},x_i]}(f)+\omega_{[a-\delta,a]}(f)+\omega_{[b,b+\delta]}(f)\right).\end{aligned}$$

By the integrability of f, we have $\lim_{n\to\infty}\delta\sum_{i=1}^n\omega_{[x_{i-1},x_i]}(f)=\lim_{n\to\infty}\sum_{i=1}^n\omega_{[x_{i-1},x_i]}(f)\Delta x_i=0$. Moreover, since f is bounded, we have $\lim_{n\to\infty}\delta\left(\omega_{[a-\delta,a]}(f)+\omega_{[b,b+\delta]}(f)\right)=0$. Therefore we conclude that $\lim_{t\to 0}\displaystyle\int_a^b|f(x+t)-f(x)|dx=0$.

The product version $\lim_{t\to 1}\displaystyle\int_a^b|f(tx)-f(x)|dx=0$ of the integral continuity can be proved similarly. Under the set up in Exercise 3.1.45, we have $|tx-x|\le|t-1|\max\{|a|,|b|\}$. Therefore, when $|t-1|<\dfrac{\delta}{\max\{|a|,|b|\}}$, we have $|tx-x|<\delta$, so that the argument in Exercise 3.1.45 still works, and the estimation above still works.

EXERCISE 3.1.47

Let L be the linear function satisfying $L(a)=f(a)$, $L(b)=f(b)$. Then $f(x)\le L(x)$ on $[a,b]$, and we have

$$\int_a^b f(x)dx\le\int_a^b L(x)dx=\frac{L(a)+L(b)}{2}(b-a)=\frac{f(a)+f(b)}{2}(b-a),$$

where the first equality follows from the linearity of L.

Let K be the linear function satisfying $K\left(\dfrac{a+b}{2}\right)=f\left(\dfrac{a+b}{2}\right)$ and $f(x)\ge K(x)$ on $[a,b]$ (see Proposition 2.3.5). Then

$$\int_a^b f(x)dx\ge\int_a^b K(x)dx=\frac{K(a)+K(b)}{2}(b-a)=K\left(\frac{a+b}{2}\right)(b-a)=f\left(\frac{a+b}{2}\right)(b-a).$$

EXERCISE 3.1.48

For any linear function $L(x)=Ax+B$, we have

$$\int_a^b\lambda(x)L(x)dx=A((1-\mu)a+\mu b)(b-a)+B(b-a)=(1-\mu)L(a)+\mu L(b)=L((1-\mu)a+\mu b).$$

Let L be the linear function satisfying $L(a)=f(a)$, $L(b)=f(b)$. Then $f(x)\le L(x)$ on $[a,b]$, and we have

$$\int_a^b\lambda(x)f(x)dx\le\int_a^b\lambda(x)L(x)dx=(1-\mu)L(a)+\mu L(b)=(1-\mu)f(a)+\mu f(b).$$

Let K be the linear function satisfying $K((1-\mu)a+\mu b) = f((1-\mu)a+\mu b)$ and $f(x) \geq K(x)$ on $[a, b]$ (see Proposition 2.3.5). Then

$$\int_a^b \lambda(x)f(x)dx \geq \int_a^b \lambda(x)K(x)dx = K((1-\mu)a+\mu b) = f((1-\mu)a+\mu b).$$

Exercise 3.1.49 [$\lambda(x)$ is a weight function on $[\alpha, \beta]$ instead of on $[a, b]$]

Let $c = \dfrac{1}{\beta-\alpha}\displaystyle\int_\alpha^\beta \lambda(t)\phi(t)dt$. Let $K(x) = Ax+B$ be a linear function satisfying $K(c) = f(c)$ and $f(x) \geq K(x)$ on $[a, b]$ (see Proposition 2.3.5). Then $f(x) \geq L(x)$ on $[a, b]$, and we have

$$\begin{aligned}\frac{1}{\beta-\alpha}\int_\alpha^\beta \lambda(t)f(\phi(t))dt \geq &\frac{1}{\beta-\alpha}\int_\alpha^\beta \lambda(t)K(\phi(t))dt = A\frac{1}{\beta-\alpha}\int_\alpha^\beta \lambda(t)\phi(t)dt + B \\ =&K(c) = f\left(\frac{1}{\beta-\alpha}\int_\alpha^\beta \lambda(t)\phi(t)dt\right).\end{aligned}$$

Exercise 3.1.50

Denote

$$\|f\|_p = \left(\int_a^b |f(x)|^p dx\right)^{\frac{1}{p}}, \quad \|g\|_q = \left(\int_a^b |g(x)|^p dx\right)^{\frac{1}{q}}, \quad \phi = \frac{f}{\|f\|_p}, \quad \psi = \frac{g}{\|g\|_q}.$$

Then by the Young inequality, $|\phi\psi| \leq \dfrac{1}{p}|\phi|^p + \dfrac{1}{q}|\psi|^q$. By the obvious equality

$$\int_a^b |\phi|^p dx = \frac{\displaystyle\int_a^b |f|^p dx}{\|f\|_p^p} = 1$$

and the similar one for ψ, this implies

$$\begin{aligned}\int_a^b |f(x)g(x)|dx &= \|f\|_p\|g\|_q\int_a^b |\phi(x)\psi(x)|dx \leq \|f\|_p\|g\|_q\left(\frac{1}{p}\int_a^b |\phi|^p dx + \frac{1}{q}\int_a^b |\psi|^q dx\right) \\ &= \|f\|_p\|g\|_q\left(\frac{1}{p}+\frac{1}{q}\right) = \|f\|_p\|g\|_q.\end{aligned}$$

This completes the proof of the Hölder inequality.

Applying Hölder inequality to f and $(|f+g|)^{p-1}$, we get

$$\int_a^b |f(x)||f(x)+g(x)|^{p-1}dx \leq \left(\int_a^b |f(x)|^p dx\right)^{\frac{1}{p}}\left(\int_a^b |f(x)+g(x)|^{(p-1)q}dx\right)^{\frac{1}{q}}$$

BY exchanging f and g, we get another inequality. Addint the two equalities and using $(p-1)q = p$, we get

$$\begin{aligned}\int_a^b |f(x)+g(x)|^p dx &\leq \int_a^b |f(x)||f(x)+g(x)|^{p-1}dx + \int_a^b |g(x)||f(x)+g(x)|^{p-1}dx \\ &\leq \left(\left(\int_a^b |f(x)|^p dx\right)^{\frac{1}{p}} + \left(\int_a^b |g(x)|^p dx\right)^{\frac{1}{p}}\right)\left(\int_a^b |f(x)+g(x)|^p dx\right)^{\frac{1}{q}}.\end{aligned}$$

By $1-\dfrac{1}{q}=\dfrac{1}{p}$, this is the same as the Minkowski inequality.

Exercise 3.1.51

Let $M=\max_{(a,b)} f'$ and $m=\min_{(a,b)} f'$. Then for any $x\in[a,b]$, by the mean value theorem, we have $m(x-a)\le f(x)-f(a)=f'(y)(x-a)\le M(x-a)$. This implies

$$\int_a^b f(x)dx-(b-a)f(a)=\int_a^b (f(x)-f(a))dx\le\int_a^b M(x-a)dx=\frac{(b-a)^2}{2}M.$$

We have similar estimation for the lower bound m.

The inequalities above means

$$m\le I=\frac{2}{(b-a)^2}\left((b-a)f(b)-\int_a^b f(x)dx\right)\le M$$

By Exercise 2.2.34, this implies $I=f'(c)$ for some $c\in[a,b]$. Moreover, by Exercise 3.1.18, $I=m$ or M only if f' is constant, in which case c can be chosen in (a,b). If $I\ne m$ and M, then c can always be chosen in (a,b).

Exercise 3.1.52

Using the set up in Exercise 3.1.51, we have

$$\int_a^b f(x)g(x)dx-f(a)\int_a^b g(x)dx=\int_a^b (f(x)-f(a))g(x)dx\le\int_a^b M(x-a)g(x)dx=M\int_a^b (x-a)g(x)dx.$$

We have similar estimation for the lower bound m. Therefore we have

$$m\int_a^b (x-a)g(x)dx\le\int_a^b f(x)g(x)dx-f(a)\int_a^b g(x)dx\le\int_a^b (x-a)g(x)dx.$$

The rest of the argument is similar.

Exercise 3.1.53

Let $d=\dfrac{a+b}{2}$. Applying Exercise 3.1.51 to $f(x)$ on $[a,d]$ and using $d-a=\dfrac{b-a}{2}$, we get

$$\frac{\inf_{(a,d)} f'}{8}(b-a)^2\le\int_a^d f(x)dx-\frac{f(a)}{2}(b-a)\le\frac{\sup_{(a,d)} f'}{8}(b-a)^2.$$

By considering $L(x)=f(b)+m(x-a)$ in Exercise 3.1.51, we can similarly prove

$$\frac{\inf_{(a,b)} f'}{2}(b-a)^2\le f(b)(b-a)-\int_a^b f(x)dx\le\frac{\sup_{(a,b)} f'}{2}(b-a)^2.$$

By applying the inequality to $f(x)$ on $[d,b]$ and using $b-d=\dfrac{b-a}{2}$, we get

$$\frac{\inf_{(d,b)} f'}{8}(b-a)^2\le\frac{f(b)}{2}(b-a)-\int_d^b f(x)dx\le\frac{\sup_{(d,b)} f'}{8}(b-a)^2.$$

The two estimations on half intervals can be combined to give us

$$\begin{aligned}\int_a^b f(x)dx - \frac{f(a)+f(b)}{2}(b-a) &= \left(\int_a^d f(x)dx - \frac{f(a)}{2}(b-a)\right) - \left(\frac{f(b)}{2}(b-a) - \int_d^b f(x)dx\right) \\ &\le \frac{\sup_{(a,d)} f' - \inf_{(d,b)} f'}{2}(b-a)^2 \le \frac{\sup_{(a,b)} f' - \inf_{(a,b)} f'}{2}(b-a)^2 \\ &= \frac{\omega_{(a,b)}(f')}{8}(b-a)^2,\end{aligned}$$

and

$$\begin{aligned}\int_a^b f(x)dx - \frac{f(a)+f(b)}{2}(b-a) &= \left(\int_a^d f(x)dx - \frac{f(a)}{2}(b-a)\right) - \left(\frac{f(b)}{2}(b-a) - \int_d^b f(x)dx\right) \\ &\ge \frac{\inf_{(a,d)} f' - \sup_{(d,b)} f'}{2}(b-a)^2 \ge \frac{\inf_{(a,b)} f' - \sup_{(a,b)} f'}{2}(b-a)^2 \\ &= -\frac{\omega_{(a,b)}(f')}{8}(b-a)^2.\end{aligned}$$

Exercise 3.1.54

Let $d = \dfrac{a+b}{2}$. Let $M = \sup_{[a,b]} f''$ and $m = \inf_{[a,b]} f''$. Then for any $x \in [a,b]$, by the remainder for the second order Taylor expansion, we have

$$\frac{m}{2}(x-d)^2 \le f(x) - f(d) - f'(d)(x-d) = \frac{f''(y)}{2}(x-d)^2 \le \frac{M}{2}(x-d)^2.$$

Integrating from a to d and from d to b, this implies

$$\begin{aligned}\frac{m}{6}(d-a)^3 &\le \int_a^d f(x)dx - (d-a)f(d) + \frac{f'(d)}{2}(d-a)^2 \le \frac{M}{6}(c-a)^3, \\ \frac{m}{6}(b-d)^3 &\le \int_d^b f(x)dx - (b-d)f(d) - \frac{f'(d)}{2}(b-d)^2 \le \frac{M}{6}(b-d)^3.\end{aligned}$$

Adding the inequalities together and substituting $d = \dfrac{a+b}{2}$, we get

$$\frac{m}{24}(b-a)^3 \le \int_a^b f(x)dx - (b-a)f(d) \le \frac{M}{24}(b-a)^3.$$

Then by Exercise 2.2.34 and an argument similar to Exercise 3.1.51, we get

$$\int_a^b f(x)dx - (b-a)f(d) = \frac{f''(c)}{24}(b-a)^3$$

for some $a < c < b$.

Exercise 3.2.1

(1) $\int_0^b x^n dx = \frac{b^{n+1}}{n+1}$.

(2) $\int_1^b x^\alpha dx = \frac{b^{\alpha+1}-1}{\alpha+1}$.

(3) $\int_0^b \sin x dx = 1 - \cos b$.

(4) $\int_0^b \cos 2x dx = \int_0^b \frac{1+\cos 2x}{2} dx = \frac{2b + \sin 2b}{4}$.

(5) $\int_0^b e^x dx = e^b - 1$.

(6) $\int_0^b 2^x dx = \frac{2^x - 1}{\log 2}$.

Exercise 3.2.2

(1) $\left(\int_0^x \sin t^2 dt\right)' = \sin x^2$.

(2) $\left(\int_0^{x^2} \sin t dt\right)' = 2x \sin x^2$.

(3) $\left(\int_0^{\sin x} t^2 dt\right)' = \sin^2 x \cos x$.

(4) $\left(\int_{\sin x}^x \sin t^2 dt\right)' = \sin x^2 - (\sin \sin^2 x) \cos x$.

(5) $\left(\int_0^{|x|} \sin t^2 dt\right)' = \sin x^2$ for $x \geq 0$, $= -\sin x^2$ for $x < 0$.

(6) $\left(\int_0^{\sin x} |t| dt\right)' = |\sin x| \cos x$.

Exercise 3.2.3

For any integer k, we have

$$\int_0^{2\pi} \cos kx dx = \begin{cases} \frac{\sin 2k\pi - \sin 0}{k} = 0 & \text{if } k \neq 0 \\ 2\pi & \text{if } k = 0 \end{cases},$$

$$\int_0^{2\pi} \sin kx dx = \begin{cases} \frac{-\cos 2k\pi + \cos 0}{k} = 0 & \text{if } k \neq 0 \\ 0 & \text{if } k = 0 \end{cases}.$$

For non-negative integers m and n, we then have

$$\int_0^{2\pi} \cos mx \sin nx dx = \frac{1}{2}\int_0^{2\pi} (\sin(m+n)x - \sin(m-n)x) dx = 0.$$

$$\int_0^{2\pi} \cos mx \cos nx dx = \frac{1}{2}\int_0^{2\pi} (\cos(m+n)x + \cos(m-n)x) dx = \begin{cases} 0 & \text{if } m \neq n \\ \pi & \text{if } m = n \neq 0 \\ 2\pi & \text{if } m = n = 0 \end{cases},$$

$$\int_0^{2\pi} \sin mx \sin nx dx = \frac{1}{2}\int_0^{2\pi} (\cos(m+n)x - \cos(m-n)x) dx = \begin{cases} 0 & \text{if } m \neq n \\ \pi & \text{if } m = n \end{cases}.$$

Exercise 3.2.4

(1) Taking the partition P_n of $[0,1]$ to consists of $x_i = \dfrac{i^2}{n^2}$, $0 \le i \le n$ and choosing $x_i^* = x_i$, by $\Delta x_i = \dfrac{2i-1}{n^2}$ we get

$$\lim_{n\to\infty} \frac{1^2\cdot 1 + 2^2 \cdot 3 + \cdots + n^2\cdot(2n-1)}{n^4} = \lim_{n\to\infty} S(P_n, x^2) = \int_0^1 x^2 dx = \frac{1}{3}.$$

(2) Taking the partition P_n of $[0,1]$ to consists of $x_i = \dfrac{i\pi}{n}$, $0 \le i \le n$ and choosing $x_i^* = x_{i-1}$, we get

$$\lim_{n\to\infty} \frac{1}{n}\left(\cos\frac{\pi}{n} + \cos\frac{2\pi}{n} + \cdots + \cos\frac{(n-1)\pi}{n}\right) = \lim_{n\to\infty} S(P_n, \cos x) = \frac{1}{\pi}\int_0^{\pi} \cos x dx = 0.$$

(3) Taking log, we get

$$\lim_{n\to\infty} \log\frac{\sqrt[n]{n!}}{n} = \lim_{n\to\infty}\frac{1}{n}\left(\log\frac{1}{n} + \log\frac{2}{n} + \cdots + \log\frac{n}{n}\right) = \int_0^1 \log x dx = -1.$$

Then taking the exponential, we get $\lim_{n\to\infty} \dfrac{\sqrt[n]{n!}}{n} = e^{-1}$.

(4, 5) Taking the partition P_n of $[0,1]$ to consists of $x_i = \dfrac{2i}{n}$, $0 \le i \le n$ and choosing $x_i^* = \dfrac{2i-1}{n}$, we get

$$\lim_{n\to\infty} \frac{1^\alpha + 3^\alpha + \cdots + (2n+1)^\alpha}{n^{\alpha+1}} = \lim_{n\to\infty}\left(\frac{1}{2}S(P_n, x^\alpha) + \frac{(2n+1)^\alpha}{n^{\alpha+1}}\right) = \frac{1}{2}\int_0^1 x^\alpha dx = \frac{1}{2(\alpha+1)}.$$

Moreover,

$$\lim_{n\to\infty} \frac{1^\alpha + 3^\alpha + \cdots + (2n-1)^\alpha}{n^{\alpha+1}} = \lim_{n\to\infty} \frac{1^\alpha + 3^\alpha + \cdots + (2n+1)^\alpha}{n^{\alpha+1}} - \lim_{n\to\infty}\frac{(2n+1)^\alpha}{n^{\alpha+1}} = \frac{1}{2(\alpha+1)}.$$

Exercise 3.2.5

Not true. For $f(x) = \begin{cases} 0 & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$, the function $F(x) = \displaystyle\int_0^x f(x)dx = 0$ is differentiable everywhere, although $f(x)$ is not continuous at 0.

EXERCISE 3.2.6

For any $\epsilon > 0$. there is $\delta > 0$, such that $|f(x) - f(x_0^-)| < \epsilon$ for $-\delta < x - x_0 < 0$. Then

$$\left|F(x) - F(x_0) - f(x_0^-)(x - x_0)\right| = \left|\int_{x_0}^{x} (f(t) - f(t_0^-))dt\right| \le \int_{x_0}^{x} |f(t) - f(t_0^-)|dt \le \epsilon|x - x_0|.$$

This shows that $F'_-(x_0) = f(x_0^-)$.

EXERCISE 3.2.7

The function $F(x) = \int_a^x f(t)dt$ is continuous on $[a, b]$. Since the value $\frac{1}{2}\int_a^b f(x)dx = \frac{1}{2}F(b)$ lies between $F(a) = 0$ and $F(b)$, by intermediate value theorem, we get $\frac{1}{2}F(b) = F(c)$ for some $a < c < b$.

EXERCISE 3.2.8

(1) Taking derivative on both sides of $\int_0^x f(t)dt = \int_x^1 f(t)dt$, we get $f(x) = -f(x)$. This implies $f(x) = 0$.

(2) Taking derivative on both sides of $A\int_0^x tf(t)dt = x\int_0^x f(t)dt$, we get $xf(x) = xf(x) + \int_0^x f(t)dt$. Thus $\int_0^x f(t)dt = 0$. Taking derivative again, we get $f(x) = 0$.

(3) Since $f(x)$ is continuous, $f(x)^2$ is differentiable. Thus $f(x)$ is differentiable at places where $f(x) \neq 0$. Taking derivative on both sides of $(f(x))^2 = 2\int_0^x f(t)dt$, we get $2f(x)f'(x) = 2f(x)$ at places where $f(x) \neq 0$. This implies $f'(x) = 1$ at places where $f(x) \neq 0$.

Suppose $f(x)$ is positive at a. Then by the continuity of $f(x)$, we have $f(x) = x + f(a) - a$ for $x \ge a - f(a)$. Since $f(0)^2 = \int_0^0 f(t)dt = 0$, we also have $f(0) = 0$. If $f(a) > a$, then $0 > a - f(a)$ and $0 = f(0) = 0 + f(a) - a$ tells us $a = f(a)$. The contradiction shows $f(a) \le a$, then $f(x) = x - b$ for $x \ge b = a - f(a) \ge 0$. We also note the following.

- We must have $a \ge f(a) > 0$. Therefore positive values can only happen on $(0, +\infty)$.
- If there is another b', such that $f(x) = x - b'$ for $x \ge b' > 0$. Then evaluating $f(x)$ at a big $x > b, b'$ would give us $x - b' = x - b$. This shows that b is unique.

Similar discussion can be made when $f(x)$ is negative somewhere. In particular, we know that $f(x)$ cannot be negative on $(0, +\infty)$. Therefore we conclude that on $[0, +\infty)$, either $f(x) = 0$ or

$$f(x) = \begin{cases} x - b & \text{if } x \ge b \\ 0 & \text{if } 0 \le x < b \end{cases} \quad \text{for some } b \ge 0.$$

Both satisfy the equation $(f(x))^2 = 2\int_0^x f(t)dt$. Combined with the similar conclusion for $x \le 0$, we find all four possibilities.

1. $f(x) = 0$ for all x.
2. $f(x) = \begin{cases} x - b & \text{if } x \ge b \\ 0 & \text{if } x < b \end{cases}$ for some $b \ge 0$.

3. $f(x) = \begin{cases} x-b & \text{if } x \le b \\ 0 & \text{if } x > b \end{cases}$ for some $b \le 0$.

4. $f(x) = \begin{cases} x-b & \text{if } x \ge b \\ 0 & \text{if } b' < x < b \\ x-b' & \text{if } x \le b' \end{cases}$ for some $b \ge 0$ and $b' \le 0$.

EXERCISE 3.2.9

Since $\displaystyle\int_x^{bx} f(t)dt$ is independent of x, taking the derivative in x, we get $bf(bx) - f(x) = 0$. Since this holds for any b, by taking $b = \dfrac{1}{x}$ we get $f(x) = \dfrac{1}{x}f(1) = \dfrac{c}{x}$.

EXERCISE 3.2.10

Consider $F(x) = \left(\displaystyle\int_a^x f(t)dt\right)^2 - \displaystyle\int_a^x f(t)^3dt$. We have $F'(x) = 2f(x)\displaystyle\int_a^x f(t)dt - f(x)^3 = fG$, where $G(x) = 2\displaystyle\int_a^x f(t)dt - f(x)^2$. We further have $G' = 2f - 2ff' = 2f(1-f')$. Note that $f' \ge 0$ implies $f(x) \ge f(a) = 0$ for $x > a$. Therefore combined with $f' \le 1$, we get $G' \ge 0$. This implies $G(x) \ge G(a) = 0$ for $x > a$. Therefore $F' \ge 0$. This further implies $F(x) \ge F(a) = 0$ for $x > a$.

EXERCISE 3.2.11

Suppose $F(x)$ is continuous on $[a,b]$, and is differentiable on (a,c) and (c,b). Then by the fundamental theorem, we have

$$\int_a^c f(x)dx = F(c) - F(a), \quad \int_c^b f(x)dx = F(b) - F(c).$$

Adding up, we get $\displaystyle\int_a^b f(x)dx = F(b) - F(a)$. The more general case of more than one point is similar.

EXERCISE 3.2.12

Suppose $f'(x)$ is integrable. Then by Theorem 3.2.2, we have $f(x) = f(a) + \displaystyle\int_a^x f'(t)dt$. In other words, we can take $g(x) = f'(x)$.

Conversely, suppose f is differentiable and $f(x) = f(a) + \displaystyle\int_a^x g(t)dt$ for some integrable $g(t)$ (Note that this does not imply $f' = g$ because the values of g can be changed in finitely many places). Suppose $m \le g \le M$ on $[c,d]$. Then for $x, y \in [c,d]$, we have

$$m \le \frac{f(x)-f(y)}{x-y} = \frac{1}{x-y}\int_x^y g(t)dt \le M.$$

This implies that $m \le f' \le M$ on $[c,d]$. Therefore we conclude $\omega_{[c,d]}(f') \le \omega_{[c,d]}(g)$. Based on this, it is easy to see that the integrability of g implies the integrability of f'.

EXERCISE 3.2.13

$\displaystyle\int_a^b [x]f'(x)dx = \int_a^{[a]+1} [a]f'(x)dx + \int_{[a]+1}^{[a]+2} ([a]+1)f'(x)dx + \int_{[a]+2}^{[a]+3} ([a]+2)f'(x)dx + \cdots + \int_{[b]-1}^{[b]} ([b]-1)f'(x)dx + \int_{[b]}^{b} [b]f'(x)dx = [a](f([a]+1) - f(a)) + ([a]+1)(f([a]+2) - f([a]+1)) + ([a]+2)(f([a]+3) - f([a]+2)) + \cdots + ([b]-1)(f([b]) - f([b]-1)) + [b](f(b) - f([b])) = [b]f(b) - [a]f(a) - f([a]+1) - f([a]+2) - \cdots - f([b])$.

Exercise 3.2.14

Suppose $f(c) = 0$. Then $\displaystyle |f(x)| = |f(x) - f(c)| = \left|\int_c^x f'(x)dx\right| \le \left|\int_c^x |f'(x)|dx\right| \le \int_a^b |f'(x)|dx$.

The first part implies that if $f(x)$ vanishes somewhere, then $\displaystyle\int_a^b |f(x)|dx \le \int_a^b \left(\int_a^b |f'(x)|dx\right) dx = (b-a)\int_a^b |f'(x)|dx$. On the other hand, if $f(x)$ does not vanish, then by the continuity and the intermediate value theorem, $f(x)$ does not change sign, so that $\displaystyle\int_a^b |f(x)|dx = \left|\int_a^b f(x)dx\right|$. The second part thus follows.

Exercise 3.2.15

By Hölder inequality, we get

$$(b-a)\int_a^b f'(x)^2dx = \int_a^b 1^2dx \int_a^b f'(x)^2dx \ge \left(\int_a^b 1\cdot f'(x)dx\right)^2 = (f(b)-f(a))^2.$$

(1) In case $f(a) = 0$, we have $\displaystyle f(x)^2 = (f(x) - f(a))^2 \le (x-a)\int_a^x f'(t)^2dt \le (x-a)\int_a^b f'(t)^2dt$. Integrating on $[a,b]$, we get

$$\int_a^b f(x)^2dx \le \int_a^b (x-a)dx \int_a^b f'(t)^2dt \le \frac{(b-a)^2}{2}\int_a^b f'(t)^2dt.$$

(2) In case $f(a) = f(b) = 0$, we let $c = \dfrac{a+b}{2}$ and get

$$\int_a^c f(x)^2dx \le \frac{(c-a)^2}{2}\int_a^c f'(t)^2dt = \frac{(b-a)^2}{8}\int_a^c f'(t)^2dt$$

and the similar inequality for the integration on $[c,b]$. Adding the two inequalities together, we get $\displaystyle\int_a^b f(x)^2dx \le \frac{(b-a)^2}{4}\int_a^b f'(x)^2dx$.

(3) In case $f(c) = 0$, $c = \dfrac{a+b}{2}$, we have

$$\int_a^c f(x)^2dx \le \frac{(c-a)^2}{2}\int_a^c f'(t)^2dt = \frac{(b-a)^2}{8}\int_a^c f'(t)^2dt$$

and the similar inequality for the integration on $[c,b]$. Adding the two inequalities together, we get $\displaystyle\int_a^b f(x)^2dx \le \frac{(b-a)^2}{4}\int_a^b f'(x)^2dx$.

Exercise 3.2.16

(1) $\frac{d}{dx}|\log(x+\sqrt{x^2+a})| = \dfrac{1+\dfrac{2x}{2\sqrt{x^2+a}}}{x+\sqrt{x^2+a}} = \dfrac{1}{\sqrt{x^2+a}}.$

(2) $\frac{1}{2}\frac{d}{dx}(\arcsin x + x\sqrt{1-x^2}) = \frac{1}{2}\left(\frac{1}{\sqrt{1-x^2}} + \sqrt{1-x^2} + x\frac{-2x}{2\sqrt{1-x^2}}\right) = \sqrt{1-x^2}.$

(3) $\frac{d}{dx}\left(e^{ax}\frac{a\cos bx + b\sin bx}{a^2+b^2}\right) = ae^{ax}\frac{a\cos bx + b\sin bx}{a^2+b^2} + e^{ax}\frac{-ab\sin bx + b^2\cos bx}{a^2+b^2} = e^{ax}\cos bx.$

(4) $\frac{d}{dx}(x\log|x| - x) = \log|x| + x\frac{1}{x} - 1 = \log|x|.$

Exercise 3.2.17

(1) $\int \frac{dx}{x(1+x)} = \int\left(\frac{1}{x} - \frac{1}{1+x}\right)dx = \log\left|\frac{x}{1+x}\right| + C.$

(2) $\int x(1+x)^9dx = \int(1+x)^{10}dx - \int(1+x)^9dx = \frac{(1+x)^{11}}{11} - \frac{(1+x)^{10}}{10} = \frac{(1+x)^{10}(10x-1)}{110} + C.$

(3) $\int \frac{x^2}{(1+x)^9}dx = \int\frac{(1+x)^2 - 2(1+x) + 1}{(1+x)^9}dx = -\frac{(1+x)^{-6}}{6} + 2\frac{(1+x)^{-7}}{7} - \frac{(1+x)^{-8}}{8} = -\frac{(1+x)^{-8}(28x^2+8x+1)}{168} + C.$

(4) $\int x(1+x)^ndx = \int(1+x)^{n+1}dx - \int(1+x)^ndx = \frac{(1+x)^{n+2}}{n+2} - \frac{(1+x)^{n+1}}{n+1} + C.$

(5) $\int\frac{x^2-x+1}{(1+x)^n}dx = \int\frac{(x+1)^2-3x}{(1+x)^n}dx = \int\frac{(x+1)^2-3(x+1)+3}{(1+x)^n}dx = -\frac{1}{(n-3)(1+x)^{n-3}} + 3\frac{1}{(n-2)(1+x)^{n-2}} - 3\frac{1}{(n-1)(1+x)^{n-1}} + C.$ In case one of $n = 1, 2$ or 3, one of the antiderivatives is log.

(6) $\int(2^x+2^{-x})^2dx = \int(4^x+4^{-x}+2)dx = \frac{4^x-4^{-x}}{\log 4} + 2x + C.$

(7) $\int|x|dx = C + \begin{cases}\frac{x^2}{2} & \text{if } x \ge 0\\ -\frac{x^2}{2} & \text{if } x < 0\end{cases}.$

(8) $\int\sin 2x\cos 3xdx = \frac{1}{2}\int(\sin 5x - \sin x)dx = -\frac{1}{10}\cos 5x + \frac{1}{2}\cos x + C.$

(9) $\int\cos^2 xdx = \frac{1}{2}\int(1+\cos 2x)dx = \frac{2x+\sin 2x}{4} + C.$

(10) $\int\tan^2 xdx = \int(\sec^2 x - 1)dx = \tan x - x + C.$

(11) $\int\sin^3 xdx = \frac{1}{2}\int(1-\cos 2x)\sin xdx = \frac{1}{4}\int(2\sin x - \sin 3x + \sin x)dx = -\frac{3}{4}\cos x + \frac{1}{12}\cos 3x + C.$

(12) $\int\frac{\cos 2xdx}{\sin^2 x\cos^2 x} = \int\frac{(\cos^2 x - \sin^2 x)dx}{\sin^2 x\cos^2 x} = \int(\csc^2 x - \sec^2 x)dx = -\cot x - \tan x + C = -\frac{1}{\sin x\cos x} + C.$

Exercise 3.2.18

(1, 2) We have

$$I = \int e^{ax}\sin bx dx = a^{-1}\int \sin bx de^{ax} = a^{-1}e^{ax}\sin bx - a^{-1}b\int e^{ax}\cos bx dx = a^{-1}e^{ax}\sin bx - a^{-1}bJ.$$

$$J = \int e^{ax}\cos bx dx = a^{-1}\int \cos bx de^{ax} = a^{-1}e^{ax}\cos bx + a^{-1}b\int e^{ax}\sin bx dx = a^{-1}e^{ax}\cos bx + a^{-1}bI.$$

Solving $I + a^{-1}bJ = a^{-1}e^{ax}\sin bx$ and $-a^{-1}bI + J = a^{-1}e^{ax}\cos bx$, we get

$$\begin{aligned}\int e^{ax}\sin bx dx = I = \frac{ae^{ax}\sin bx - be^{ax}\cos bx}{a^2+b^2} + C,\\ \int e^{ax}\cos bx dx = J = \frac{be^{ax}\sin bx + ae^{ax}\cos bx}{a^2+b^2} + C.\end{aligned}$$

Moreover,

$$\begin{aligned}\int a^x\cos bx dx = \int e^{x\log a}\cos bx dx =& \frac{be^{x\log a}\sin bx + (\log a)e^{x\log a}\cos bx}{(\log a)^2+b^2} + C\\ =& \frac{ba^x\sin bx + (\log a)a^x\cos bx}{(\log a)^2+b^2} + C\end{aligned}$$

(3) $\displaystyle\int x^2 2^x dx = \frac{1}{\log 2}\int x^2 d2^x = \frac{1}{\log 2}x^2 2^x - \frac{2}{\log 2}\int x2^x dx = \frac{1}{\log 2}x^2 2^x - \frac{2}{(\log 2)^2}\int xd2^x = \frac{1}{\log 2}x^2 2^x - \frac{2}{(\log 2)^2}x2^x + \frac{2}{(\log 2)^2}\int 2^x dx = \frac{1}{\log 2}x^2 2^x - \frac{2}{(\log 2)^2}x2^x + \frac{2}{(\log 2)^3}2^x.$

(4) Using the result of (1),

$$\begin{aligned}\int xe^x\sin x dx = \int xd\frac{e^x(\sin x - \cos x)}{2} &= \frac{x}{2}e^x(\sin x - \cos x) - \frac{1}{2}\int e^x(\sin x - \cos x)dx\\ &= \frac{x}{2}e^x(\sin x - \cos x) - \frac{1}{2}\left(\frac{e^x(\sin x - \cos x)}{2} - \frac{e^x(\sin x + \cos x)}{2}\right) + C\\ &= \frac{1}{2}(xe^x\sin x - (x-1)e^x\cos x) + C.\end{aligned}$$

(5) Using the result of (1),

$$\begin{aligned}\int x^2e^{-x}\cos 2x dx = \int x^2 d\frac{e^{-x}(2\sin 2x - \cos 2x)}{5}&\\ &= \frac{x^2}{5}e^{-x}(2\sin 2x - \cos 2x) - \frac{2}{5}\int xe^{-x}(2\sin 2x - \cos 2x)dx\\ &= \frac{x^2}{5}e^{-x}(2\sin 2x - \cos 2x) - \frac{2}{5}\int xd\frac{e^{-x}(-4\sin 2x - 3\cos 2x)}{5}\\ &= \frac{x^2}{5}e^{-x}(2\sin 2x - \cos 2x) + \frac{2x}{25}e^{-x}(4\sin 2x + 3\cos 2x)\\ &\quad - \frac{2}{25}\int e^{-x}(4\sin 2x + 3\cos 2x)dx\\ &= \frac{x^2}{5}e^{-x}(2\sin 2x - \cos 2x) + \frac{2x}{25}e^{-x}(4\sin 2x + 3\cos 2x)\\ &\quad - \frac{2}{125}e^{-x}(2\sin 2x - 11\cos 2x)dx + C\\ &= \frac{50x^2 + 40x - 4}{125}e^{-x}\sin 2x + \frac{-25x^2 + 30x + 22}{125}e^{-x}\cos 2x + C.\end{aligned}$$

(6) We have

$$\int x^n \arctan x dx = \frac{1}{n+1}\int \arctan x dx^{n+1} = \frac{x^{n+1}\arctan x}{n+1} - \frac{1}{n+1}\int \frac{x^{n+1}dx}{1+x^2}.$$

By

$$\frac{x^{n+1}}{1+x^2} = \frac{(1+x^2)(x^{n-1}-x^{n-3}+x^{n-5}+\cdots)+y}{1+x^2} = x^{n-1}-x^{n-3}+x^{n-5}+\cdots+\frac{y}{1+x^2},$$

where the sum ends at $(-1)^{k-1}x$ and $y=(-1)^k$ if $n=2k-1$, and ends at $(-1)^{k-1}x$ and $y=(-1)^kx$ if $n=2k$. We have

$$\begin{aligned}\int x^n \arctan x dx = &\frac{x^{n+1}\arctan x}{n+1} - \frac{x^n}{(n+1)n} + \frac{x^{n-2}}{(n+1)(n-2)} - \frac{x^{n-4}}{(n+1)(n-4)} \\ &+\cdots+\begin{cases}(-1)^k\dfrac{x}{n+1}+(-1)^{k+1}\dfrac{\arctan x}{n+1}+C & \text{if } x=2k-1 \\ (-1)^k\dfrac{x^2}{2(n+1)}+(-1)^{k+1}\dfrac{\log(1+x^2)}{2(n+1)}+C & \text{if } x=2k\end{cases}.\end{aligned}$$

EXERCISE 3.2.19

(1) We have

$$\begin{aligned}\int \sin^n xdx &= -\int \sin^{n-1} xd\cos x = -\sin^{n-1}x\cos x + \int \cos xd\sin^{n-1}x \\ &= -\sin^{n-1}x\cos x + (n-1)\int \sin^{n-2}x\cos^2 xdx \\ &= -\sin^{n-1}x\cos x + (n-1)\int \sin^{n-2}x(1-\sin^2 x)dx \\ &= -\sin^{n-1}x\cos x + (n-1)\int \sin^{n-2}xdx - (n-1)\int \sin^n xdx.\end{aligned}$$

Solving for $\int \sin^n xdx$, we get

$$\int \sin^n xdx = -\frac{1}{n}\sin^{n-1}x\cos x + \frac{n-1}{n}\int \sin^{n-2}x.$$

For $n=6$, we have

$$\begin{aligned}\int \sin^6 xdx &= -\frac{1}{6}\sin^5 x\cos x + \frac{5}{6}\int \sin^4 xdx \\ &= -\frac{1}{6}\sin^5 x\cos x - \frac{5}{6\cdot 4}\sin^3 x\cos x + \frac{5\cdot 3}{6\cdot 4}\int \sin^2 xdx \\ &= -\frac{1}{6}\sin^5 x\cos x - \frac{5}{6\cdot 4}\sin^3 x\cos x - \frac{5\cdot 3}{6\cdot 4\cdot 2}\sin x\cos x + \frac{5\cdot 3\cdot 1}{6\cdot 4\cdot 2}\int dx \\ &= -\frac{1}{6}\sin^5 x\cos x - \frac{5}{24}\sin^3 x\cos x - \frac{5}{16}\sin x\cos x + \frac{5}{16}x + C.\end{aligned}$$

Moreover, we have

$$\begin{aligned}\int \sin^5 x \cos^4 x dx &= \int \sin^5 x(1-\sin^2 x)^2 dx = \int (\sin^9 x - 2\sin^7 x + \sin^5 x)dx\\ &= -\frac{1}{9}\sin^8 x\cos x + \int \left(\left(\frac{8}{9}-2\right)\sin^7 x + \sin^5 x\right) dx\\ &= -\frac{1}{9}\sin^8 x\cos x + \frac{10}{9}\frac{1}{7}\sin^6 x\cos x + \int \left(-\frac{10}{9}\frac{6}{7}+1\right)\sin^5 x dx\\ &= -\frac{1}{9}\sin^8 x\cos x + \frac{10}{63}\sin^6 x\cos x - \frac{1}{21}\frac{1}{5}\sin^4 x\cos x + \frac{1}{21}\frac{4}{5}\int \sin^3 x dx\\ &= -\frac{1}{9}\sin^8 x\cos x + \frac{10}{63}\sin^6 x\cos x - \frac{1}{105}\sin^4 x\cos x\\ &\quad - \frac{1}{21}\frac{4}{5}\frac{1}{3}\sin^2 x\cos x + \frac{1}{21}\frac{4}{5}\frac{2}{3}\int \sin x dx\\ &= -\frac{1}{9}\sin^8 x\cos x + \frac{10}{63}\sin^6 x\cos x - \frac{1}{105}\sin^4 x\cos x\\ &\quad - \frac{4}{315}\sin^2 x\cos x - \frac{8}{315}\cos x + C.\end{aligned}$$

(2) We have

$$\begin{aligned}\int \cos^n x dx &= \int \cos^{n-1} x d\sin x = \cos^{n-1} x\sin x - \int \sin x d\cos^{n-1} x\\ &= \cos^{n-1} x\sin x - (n-1)\int \cos^{n-2} x\sin^2 x dx\\ &= \cos^{n-1} x\sin x - (n-1)\int \cos^{n-2} x(1-\cos^2 x)dx\\ &= \cos^{n-1} x\sin x - (n-1)\int \cos^{n-2} x dx + (n-1)\int \cos^n x dx.\end{aligned}$$

Solving for $\int \cos^n x dx$, we get

$$\int \cos^n x dx = \frac{1}{n}\cos^{n-1} x\sin x + \frac{n-1}{n}\int \cos^{n-2} x dx.$$

(3) We have

$$\begin{aligned}\int \tan^n x dx &= \int \tan^{n-2} x(\sec^2 x - 1)dx = \int \tan^{n-2} x\sec^2 x dx - \int \tan^{n-2} x dx\\ &= \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x dx.\end{aligned}$$

Then

$$\begin{aligned}\int \tan^6 x dx &= \frac{\tan^5 x}{5} - \int \tan^4 x dx = \frac{\tan^5 x}{5} - \frac{\tan^3 x}{3} + \int \tan^2 x dx\\ &= \frac{\tan^5 x}{5} - \frac{\tan^3 x}{3} + \frac{\tan x}{1} - \int dx = \frac{\tan^5 x}{5} - \frac{\tan^3 x}{3} + \tan x - x + C.\end{aligned}$$

and

$$\int \tan^{-6} x dx = \frac{\tan^{-5} x}{-5} - \int \tan^{-4} x dx = \frac{\tan^{-5} x}{-5} - \frac{\tan^{-3} x}{-3} + \int \tan^{-2} x dx$$

$$= \frac{\tan^{-5} x}{-5} - \frac{\tan^{-3} x}{-3} + \frac{\tan^{-1} x}{-1} - \int dx = -\frac{1}{5\tan^5 x} + \frac{1}{3\tan^3 x} - \frac{1}{\tan x} - x + C.$$

(4) We have

$$\begin{aligned}\int \sec^n x dx = \int \sec^{n-2} x d(\tan x) &= \sec^{n-2} x \tan x - (n-2)\int \tan x \sec^{n-2} x \tan x dx\\ &= \sec^{n-2} x \tan x - (n-2)\int \sec^{n-2} x(\sec^2 x - 1)dx\\ &= \sec^{n-2} x \tan x - (n-2)\int \sec^n x dx + (n-2)\int \sec^{n-2} x.\end{aligned}$$

Therefore

$$\int \sec^n x dx = \frac{\sec^{n-2} x \tan x}{n-1} - \frac{n-2}{n-1}\int \sec^{n-2} x dx.$$

(5) We have

$$\int x^n e^x dx = \int x^n de^x = x^n e^x - n\int x^{n-1} e^x dx.$$

Then

$$\begin{aligned}\int x^3(x+1)e^x dx = \int x^4 e^x dx + \int x^3 e^x dx &= x^4 e^x - 4\int x^3 e^x dx + \int x^3 e^x dx\\ &= x^4 e^x - 3x^3 e^x + 3\cdot 3\int x^2 e^x dx\\ &= x^4 e^x - 3x^3 e^x + 3\cdot 3x^2 e^x - 3\cdot 3\cdot 2\int x e^x dx\\ &= x^4 e^x - 3x^3 e^x + 3\cdot 3x^2 e^x - 3\cdot 3\cdot 2x e^x + 3\cdot 3\cdot 2\cdot 1\int e^x dx\\ &= (x^4 - 3x^3 + 9x^2 - 18x + 18)e^x + C.\end{aligned}$$

(6) We have

$$\int x^\alpha(\log|x|)^n dx = \frac{1}{\alpha+1}\int(\log|x|)^n dx^{\alpha+1} = \frac{1}{\alpha+1}x^{\alpha+1}(\log|x|)^n - \frac{1}{\alpha+1}\int x^{\alpha+1} d(\log|x|)^n$$

$$= \frac{1}{\alpha+1}x^{\alpha+1}(\log|x|)^n - \frac{n}{\alpha+1}\int x^{\alpha+1}(\log|x|)^{n-1}\frac{dx}{x}$$

$$= \frac{1}{\alpha+1}x^{\alpha+1}(\log|x|)^n - \frac{n}{\alpha+1}\int x^\alpha(\log|x|)^{n-1} dx.$$

Then

$$\begin{aligned}\int \sqrt{x}(\log|x|)^2 dx &= \frac{2}{3}x^{\frac{3}{2}}(\log|x|)^2 - \frac{4}{3}\int \sqrt{x}\log|x| dx\\ &= \frac{2}{3}x^{\frac{3}{2}}(\log|x|)^2 - \frac{4}{3}\left(\frac{2}{3}x^{\frac{3}{2}}\log|x| - \frac{4}{3}\int \sqrt{x} dx\right)\\ &= x^{\frac{3}{2}}\left(\frac{2}{3}(\log|x|)^2 - \frac{4}{3}\frac{2}{3}\log|x| + \frac{4}{3}\frac{4}{3}\frac{2}{3}\right) + C.\end{aligned}$$

(7) We have

$$\int e^x \sin^n x dx = \int \sin^n x de^x = e^x \sin^n x - n \int e^x \sin^{n-1} \cos x dx = e^x \sin^n x - n \int \sin^{n-1} \cos x de^x$$
$$= e^x \sin^n x - ne^x \sin^{n-1} x \cos x - n \int e^x ((n-1) \sin^{n-2} x \cos^2 x - \sin^n x) dx$$
$$= e^x \sin^{n-1} x (\sin x - n \cos x) - n \int e^x ((n-1) \sin^{n-2} x - n \sin^n x) dx.$$

Therefore

$$\int e^x \sin^n x dx = \frac{1}{n^2-1} e^x \sin^{n-1} x (n \cos x - \sin x) + \frac{n}{n+1} \int e^x \sin^{n-2} x dx.$$

Then

$$\int e^x \sin^4 x dx = \frac{1}{15} e^x \sin^3 x (4 \cos x - \sin x) + \frac{4}{5} \int e^x \sin^2 x dx$$
$$= \frac{1}{15} e^x \sin^3 x (4 \cos x - \sin x) + \frac{4}{5} \left(\frac{1}{3} e^x \sin x (2 \cos x - \sin x) + \int e^x dx \right)$$
$$= \frac{e^x}{15} (4 \sin^3 x \cos x - \sin^4 x + 8 \sin x \cos x - 4 \sin^2 x + 12) + C.$$

(8, 9) We have

$$\int x^n \sin x dx = - \int x^n d \cos x = -x^n \cos x + n \int x^{n-1} \cos x dx,$$
$$\int x^n \cos x dx = \int x^n d \sin x = x^n \sin x - n \int x^{n-1} \sin x dx.$$

Combining the two, we get

$$\int x^n \sin x dx = -x^n \cos x + n \int x^{n-1} \cos x dx = -x^n \cos x + nx^{n-1} \sin x - n(n-1) \int x^{n-2} \sin x dx,$$
$$\int x^n \cos x dx = x^n \sin x - n \int x^{n-1} \sin x dx = x^n \sin x + nx^{n-1} \cos x - n(n-1) \int x^{n-2} \cos x dx.$$

Then

$$\int x^4 \sin x dx = -x^4 \cos x + 4x^3 \sin x + 4 \cdot 3x^2 e^x \cos x - 4 \cdot 3 \cdot 2xe^x \sin x - 4 \cdot 3 \cdot 2 \cdot 1 e^x \cos x + C.$$

(10) We have

$$\int (1+ax^2)^n dx = x(1+ax^2)^n - \int x d(1+ax^2)^n$$
$$= x(1+ax^2)^n - 2na \int x^2 (1+ax^2)^{n-1} dx$$
$$= x(1+ax^2)^n - 2n \int [(1+ax^2)^n - (1+ax^2)^{n-1}] dx.$$

Therefore

$$\int (1+ax^2)^n dx = \frac{x(1+ax^2)^n}{2n+1} + \frac{2n}{2n+1}\int (1+ax^2)^{n-1}dx.$$

Replacing n by $-n$, we get

$$\int \frac{dx}{(x^2+1)^n} = \frac{x}{2(n-1)(x^2+1)^{n-1}} + \frac{2n-3}{2(n-1)}\int \frac{dx}{(x^2+1)^{n-1}}.$$

Then

$$\int \frac{dx}{(1+x^2)^2} = \frac{x}{2(x^2+1)} + \frac{1}{2}\int \frac{dx}{1+x^2} = \frac{x}{2(x^2+1)} + \frac{1}{2}\arctan x + C.$$

Exercise 3.2.20

If $m \neq 1$, then

$$\begin{aligned}
I(m,n) &= \int \cos^m x \sin^n x dx = -\int \cos^m x \sin^{n-1} x d\cos x \\
&= -\frac{1}{m+1}\int \sin^{n-1} x d\cos^{m+1} x = -\frac{\cos^{m+1} x \sin^{n-1} x}{m+1} + \frac{1}{m+1}\int \cos^{m+1} x d\sin^{n-1} x \\
&= -\frac{\cos^{m+1} x \sin^{n-1} x}{m+1} + \frac{n-1}{m+1}\int \cos^{m+1} x \sin^{n-2} x \cos x dx \\
&= -\frac{\cos^{m+1} x \sin^{n-1} x}{m+1} + \frac{n-1}{m+1} I(m+1, n-2).
\end{aligned}$$

If $m+n \neq 0$, the same computation still tells us

$$\begin{aligned}
(m+1)I(m,n) &= -\cos^{m+1} x \sin^{n-1} x + (n-1)\int \cos^{m+2} x \sin^{n-2} x dx \\
&= -\cos^{m+1} x \sin^{n-1} x + (n-1)\int \cos^m x (1-\sin^2 x)\sin^{n-2} x \cos x dx \\
&= -\cos^{m+1} x \sin^{n-1} x + (n-1)I(m,n-2) - (n-1)I(m,n).
\end{aligned}$$

Solving for $I(m,n)$, we get

$$\int \cos^m x \sin^n x dx = -\frac{\cos^{m+1} x \sin^{n-1} x}{m+n} + \frac{n-1}{m+n}\int \cos^m x \sin^{n-2} x dx.$$

In particular,

$$\begin{aligned}
\int \cos^4 x \sin^5 x dx &= -\frac{1}{9}\cos^5 x \sin^4 x + \frac{4}{9}\int \cos^4 x \sin^3 x dx \\
&= -\frac{1}{9}\cos^5 x \sin^4 x - \frac{4}{9}\frac{1}{7}\cos^5 x \sin^2 x + \frac{4}{9}\frac{2}{7}\int \cos^4 x \sin x dx \\
&= -\frac{1}{9}\cos^5 x \sin^4 x - \frac{4}{9}\frac{1}{7}\cos^5 x \sin^2 x - \frac{4}{9}\frac{2}{7}\frac{1}{5}\cos^5 x + C.
\end{aligned}$$

Similarly,

$$\int \cos^m x \sin^n x dx = \frac{\cos^{m-1} x \sin^{n+1} x}{m+n} + \frac{m-1}{m+n}\int \cos^{m-2} x \sin^n x dx,$$

or

$$\int \cos^m x \sin^n x dx = -\frac{\cos^{m+1} x \sin^{n+1} x}{m+1} + \frac{m+n+2}{m+1}\int \cos^{m+2} x \sin^n x dx.$$

In particular,

$$\begin{aligned}\int \frac{\sin^4 x}{\cos^3 x}dx = \int \cos^{-3} x \sin^4 x dx &= -\frac{1}{-2}\cos^{-2} x \sin^5 x + \frac{3}{-2}\int \cos^{-1} x \sin^4 x dx \\ &= \frac{1}{2}\cos^{-2} x \sin^5 x + \frac{3}{2}\frac{1}{3}\sin^3 x - \frac{3}{2}\frac{3}{3}\int \cos^{-1} x \sin^2 x dx \\ &= \frac{\sin^5 x}{2\cos^2 x} + \frac{\sin^3 x}{2} - \frac{3}{2}\int \left(\frac{1}{2(1+\sin x)} + \frac{1}{2(1-\sin x)} - 1\right) d\sin x. \\ &= \frac{\sin^5 x}{2\cos^2 x} + \frac{\sin^3 x}{2} - \frac{3}{4}\log\left|\frac{1+\sin x}{1-\sin x}\right| + \frac{3}{2}\sin x + C.\end{aligned}$$

Exercise 3.2.21

We have

$$\begin{aligned}I(m,n) = \int (x-a)^m (x-b)^n dx &= \frac{1}{n+1}\int (x-a)^m d(x-b)^{n+1} \\ &= \frac{1}{n+1}(x-a)^m(x-b)^{n+1} - \frac{1}{n+1}\int (x-b)^{n+1} d(x-a)^m \\ &= \frac{1}{n+1}(x-a)^m(x-b)^{n+1} - \frac{m}{n+1}I(m-1,n+1).\end{aligned}$$

Then

$$\begin{aligned}\int (x-1)^3(x+1)^{10}dx = I(3,10) &= \frac{1}{11}(x-1)^3(x+1)^{11} - \frac{3}{11}I(2,11) \\ &= \frac{1}{11}(x-1)^3(x+1)^{11} - \frac{3}{11}\frac{1}{12}(x-1)^2(x+1)^{12} + \frac{3}{11}\frac{2}{12}I(1,12) \\ &= \frac{1}{11}(x-1)^3(x+1)^{11} - \frac{3}{11}\frac{1}{12}(x-1)^2(x+1)^{12} \\ &\quad + \frac{3}{11}\frac{2}{12}\frac{1}{13}(x-1)(x+1)^{13} - \frac{3}{11}\frac{2}{12}\frac{1}{13}I(0,13) \\ &= \frac{1}{11}(x-1)^3(x+1)^{11} - \frac{3}{11}\frac{1}{12}(x-1)^2(x+1)^{12} \\ &\quad + \frac{3}{11}\frac{2}{12}\frac{1}{13}(x-1)(x+1)^{13} - \frac{3}{11}\frac{2}{12}\frac{1}{13}\frac{1}{14}(x+1)^{14} + C.\end{aligned}$$

and

$$\int_{-1}^{1}(x-1)^3(x+1)^{10}dx = -\frac{3}{11}\frac{2}{12}\frac{1}{13}\frac{1}{14}2^{14} = -\frac{2^{12}}{7\cdot 11\cdot 13}.$$

Exercise 3.2.22

(1) $\displaystyle\int \frac{f'(x)}{f(x)^\alpha}dx = \int \frac{df(x)}{f(x)^\alpha} = \frac{1}{(1-\alpha)f(x)^{\alpha-1}} + C.$

(2) $\displaystyle\int \frac{f'(x)}{1+f(x)^2}dx = \int \frac{df(x)}{1+f(x)^2} = \arctan f(x) + C.$

(3) $\displaystyle\int 2^{f(x)}f'(x)dx = \int 2^{f(x)}df(x) = 2^{f(x)}\log 2 + C.$

Exercise 3.2.23

(1) By $x = y^6$, we have

$$\begin{aligned}\int \frac{dx}{\sqrt{x}+\sqrt[3]{x}} &= \int \frac{6y^5dy}{y^3+y^2} = 6\int\left(y^2-y+1-\frac{1}{y+1}\right)dy = 2y^3-3y^2+6y-6\log|y+1|+C\\ &= 2\sqrt{x}-3\sqrt[3]{x}+6\sqrt[6]{x}-6\log(\sqrt[6]{x}+1)+C.\end{aligned}$$

(2) By $x = y^6$, we have

$$\begin{aligned}\int \frac{\sqrt{x}dx}{1-\sqrt[3]{x}} &= \int \frac{y^3 6y^5dy}{1-y^2} = 6\int\left(-1-y^2-y^4-y^6+\frac{1}{2(1-y)}+\frac{1}{2(1+y)}\right)dy\\ &= -6y-2y^3-\frac{6}{5}y^5-\frac{6}{7}y^7-6\log|1-y|+6\log|1+y|+C\\ &= -6\sqrt[6]{x}-2\sqrt{x}-\frac{6}{5}\sqrt[6]{x^5}-\frac{6}{7}\sqrt[6]{x^7}+6\log\left|\frac{1-\sqrt[6]{x}}{1+\sqrt[6]{x}}\right|+C.\end{aligned}$$

(3) By $x = (y-1)^3$, we have

$$\begin{aligned}\int (1+\sqrt[3]{x})^{10}dx &= 3\int y^{10}(y-1)^2dy = 3\left(\frac{y^{13}}{13}-2\frac{y^{12}}{12}+\frac{y^{11}}{11}\right)+C\\ &= 3(1+\sqrt[3]{x})^{11}\left(\frac{(1+\sqrt[3]{x})^2}{13}-2\frac{1+\sqrt[3]{x}}{12}+\frac{1}{11}\right)+C\\ &= \frac{1}{286}(1+\sqrt[3]{x})^{11}(66\sqrt[3]{x^2}-143\sqrt[3]{x}+1)+C.\end{aligned}$$

(4-8) By $x+1=2t$, we have

$$\begin{aligned}
\int \frac{dx}{x^2+2x+5} &= \int \frac{dx}{(x+1)^2+4} = \int \frac{2dt}{4(t^2+1)} = \frac{1}{2}\arctan\frac{x+1}{2}+C,\\
\int \frac{xdx}{x^2+2x+5} &= \int \frac{2(t-1)2dt}{4(t^2+1)} = \frac{1}{2}\log(t^2+1)-\arctan t+C\\
&= \frac{1}{2}\log(x^2+2x+5)-\arctan\frac{x+1}{2}+C,\\
\int \frac{x^3dx}{x^2+2x+5} &= x-2+\int\frac{-x+10}{x^2+2x+5}dx = \frac{1}{2}x^2-2x+\int\frac{(-2t+11)2dt}{4(t^2+1)}\\
&= \frac{1}{2}x^2-2x-\frac{1}{2}\log(t^2+1)+\frac{11}{2}\arctan t+C\\
&= \frac{1}{2}x^2-2x-\frac{1}{2}\log(x^2+2x+5)+\frac{11}{2}\arctan\frac{x+1}{2}+C,
\end{aligned}$$

$$\begin{aligned}
\int \frac{dx}{(x^2+2x+5)^2} &= \int \frac{2dt}{16(t^2+1)^2} = \frac{1}{8}\left(\frac{t}{2(t^2+1)}+\frac{1}{2}\arctan t\right)+C \ \ \text{(Exercise 3.2.20(10))}\\
&= \frac{x+1}{8(x^2+2x+5)}+\frac{1}{16}\arctan\frac{x+1}{2}+C,\\
\int \frac{x^3dx}{(x^2+2x+5)^2} &= \int\frac{x-2}{x^2+2x+5}dx+\int\frac{-x+10}{(x^2+2x+5)^2}dx = \int\frac{(2t-3)2dt}{4(t^2+1)}+\int\frac{(-2t+11)2dt}{16(t^2+1)^2}\\
&= \frac{1}{2}\log(t^2+1)-\frac{3}{2}\arctan t+\frac{1}{8(t^2+1)}+\frac{11}{8}\left(\frac{t}{2(t^2+1)}+\frac{1}{2}\arctan t\right)+C\\
&= \frac{2+11t}{16(t^2+1)}+\frac{1}{2}\log(t^2+1)-\frac{13}{16}\arctan t+C\\
&= \frac{1}{2}\log(x^2+2x+5)+\frac{1}{2(x^2+2x+5)}-3\int\frac{dx}{x^2+2x+5}+11\int\frac{dx}{(x^2+2x+5)^2}\\
&= \frac{11x+12}{2(x^2+2x+5)}+\frac{1}{2}\log(x^2+2x+5)-\frac{13}{16}\arctan\frac{x+1}{2}+C.
\end{aligned}$$

(9) By $x+1=2t$, we have

$$\begin{aligned}
\int\sqrt{x^2+2x+5}dx &= \int\sqrt{4(t^2+1)}2dt = 4\left(\frac{t\sqrt{t^2+1}}{2}+\frac{1}{2}\int\frac{dt}{\sqrt{t^2+1}}\right) \ \ \text{(Exercise 3.2.19(10))}\\
&= 2t\sqrt{t^2+1}+2\log(t+\sqrt{t^2+1})+C \ \ \text{(Exercise 3.2.16(1))}\\
&= \frac{1}{2}(x+1)\sqrt{x^2+2x+5}+2\log(x+1+\sqrt{x^2+2x+5})+C.
\end{aligned}$$

(10) $\displaystyle\int\frac{dx}{\sqrt{2x-x^2}} = \int\frac{d(x-1)}{\sqrt{1-(x-1)^2}} = \arcsin(x-1)+C.$

(11) We have

$$\begin{aligned}
\int x\sqrt{5+4x-x^2}dx &= -\frac{1}{2}\int[(5+4x-x^2)'-4]\sqrt{5+4x-x^2}dx\\
&= -\frac{1}{3}(5+4x-x^2)^{\frac{3}{2}}+2\int\sqrt{9-(x-2)^2}dx.
\end{aligned}$$

By $x-2=3\sin t$, we have

$$\begin{aligned}\int\sqrt{9-(x-2)^2}dx &= \int 3\cos t 3\cos t dt = \frac{9}{4}(2t+\sin 2t)+C = \frac{9}{2}(t+\sin t\cos t)+C\\ &= \frac{9}{2}\left(\arcsin\frac{x-2}{3}+\frac{x-2}{3}\sqrt{1-\left(\frac{x-2}{3}\right)^2}\right)+C\\ &= \frac{9}{2}\arcsin\frac{x-2}{3}+(x-2)\sqrt{5+4x-x^2}+C.\end{aligned}$$

Thus

$$\begin{aligned}\int x\sqrt{5+4x-x^2}dx &= -\frac{1}{3}(5+4x-x^2)^{\frac{3}{2}}+9\arcsin\frac{x-2}{3}+2(x-2)\sqrt{5+4x-x^2}+C\\ &= \frac{1}{3}(x^2-10x+7)\sqrt{5+4x-x^2}+9\arcsin\frac{x-2}{3}+C.\end{aligned}$$

(12) We have

$$\int\frac{xdx}{(x^2+2x+2)^{\frac{3}{2}}} = \int\frac{\frac{1}{2}(x^2+2x+2)'-1}{(x^2+2x+2)^{\frac{3}{2}}}dx = -\frac{1}{\sqrt{x^2+2x+2}}-\int\frac{dx}{((x+1)^2+1)^{\frac{3}{2}}}.$$

By Exercise 3.2.19(10), we have

$$\int\frac{dt}{(t^2+1)^{\frac{3}{2}}} = \frac{t}{\sqrt{t^2+1}}+C.$$

Therefore

$$\int\frac{xdx}{(x^2+2x+2)^{\frac{3}{2}}} = -\frac{1}{\sqrt{x^2+2x+2}}-\frac{x+1}{\sqrt{x^2+2x+2}}+C = -\frac{x+2}{\sqrt{x^2+2x+2}}+C.$$

(13) $\displaystyle\int\frac{(2x+1)dx}{\sqrt{x(x+1)}} = \int\frac{(x(x+1))'dx}{\sqrt{x(x+1)}} = 2\sqrt{x(x+1)}+C.$

(14) $\displaystyle\int\frac{dx}{x\log x} = \int\frac{d\log x}{\log x} = \log|\log x|+C.$

(15) $\displaystyle\int\frac{\log x}{x}dx = \int\log x d\log x = (\log x)^2+C.$

(16) $\displaystyle\int\frac{dx}{e^x+e^{-x}} = \int\frac{e^x dx}{e^{2x}+1} = \int\frac{de^x}{(e^x)^2+1} = \arctan e^x+C.$

(17) $\displaystyle\int\cot x dx = \int\frac{\cos x}{\sin x}dx = \log|\sin x|+C.$

(18) By $y=\dfrac{\pi}{2}-x$, we have $\displaystyle\int\csc x dx = -\int\sec y dy = -\log|\sec y+\tan y|+C = -\log|\csc x+\cot x|+C.$

(19) $\displaystyle\int\tan^3 x dx = \int\tan x\sec^2 x dx-\int\tan x dx = \int\tan x d\tan x-\int\frac{\sin x}{\cos x}dx = \frac{1}{2}\tan^2 x+\log|\cos x|+C.$

(20) We have $\int \sec^3 x dx = \int \sec x d\tan x = \sec x\tan x - \int \tan^2 x\sec x dx = \sec x\tan x - \int \sec^3 x dx + \int \sec x dx$. Therefore by Example 3.2.14, $\int \sec^3 x dx = \frac{1}{2}\sec x\tan x + \frac{1}{2}\log|\sec x + \tan x| + C$.

(21) By $t = \sin x$, we have

$$\begin{aligned}
\int \frac{\sin^4 x}{\cos^3 x}dx &= \int \frac{\sin^4 x}{\cos^4 x}d(\sin x) = \int \frac{t^4 dt}{(1-t^2)^2} \\
&= \int \left(1 + \frac{3}{4(t-1)} - \frac{3}{4(t+1)} + \frac{1}{4(t-1)^2} + \frac{1}{4(t+1)^2}\right) dt \\
&= t + \frac{3}{4}\log\left|\frac{t-1}{t+1}\right| - \frac{1}{4(t-1)} - \frac{1}{4(t+1)} + C \\
&= \sin x + \frac{3}{4}\log\left|\frac{\sin x - 1}{\sin x + 1}\right| - \frac{\sin x}{2(\sin^2 x - 1)} + C \\
&= \frac{3}{2}\log\left|\frac{\sin x - 1}{\cos x}\right| + \sin x + \frac{\sin x}{2\cos^2 x} + C.
\end{aligned}$$

(22) By $t = \cos x$, we have

$$\begin{aligned}
\int \frac{\sin^5 x}{\cos^3 x}dx &= -\int \frac{\sin^4 x}{\cos^4 x}d(\cos x) = \int \frac{(1-t^2)^2 dt}{t^4} \\
&= -\frac{1}{3t^3} + \frac{2}{t} + t + C = -\frac{1}{3\cos^3 x} + \frac{2}{\cos x} + \cos x + C.
\end{aligned}$$

(23) By $x = \sin t$, we have

$$\begin{aligned}
\int (\arcsin x)^2 dx &= \int t^2 \cos t dt = \int t^2 d\sin t = t^2\sin t - 2\int t\sin t dt \\
&= t^2 \sin t + 2\int t d\cos t = t^2\sin t + 2t\cos t - 2\int \cos t dt \\
&= t^2\sin t + 2t\cos t - 2\sin t + C = x(\arcsin x)^2 - 2x + 2\sqrt{1-x^2}\arcsin x + C.
\end{aligned}$$

We note that we take $|t| \le \frac{\pi}{2}$ in $x = \sin t$, so that $\cos t \ge 0$.

(24) By $x = \sin t$, we have

$$\begin{aligned}
\int x(\arcsin x)^2 dx &= \int t^2 \sin t\cos t dt = -\frac{1}{2}\int t^2 d\cos 2t = -\frac{1}{2}t^2\cos 2t + \int t\cos 2t dt \\
&= -\frac{1}{2}t^2\cos 2t + \frac{1}{2}\int t d\sin 2t = -\frac{1}{2}t^2\cos 2t + \frac{1}{2}t\sin 2t - \frac{1}{2}\int \sin 2t dt \\
&= -\frac{1}{2}t^2\cos 2t + \frac{1}{2}t\sin 2t + \frac{1}{4}\cos 2t + C = \frac{1}{4}(1-2t^2)(1-2\sin^2 t) + t\sin t\cos t \\
&= \frac{1}{4}(1-2x^2)(1-2\arcsin^2 x) + x\sqrt{1-x^2}\arcsin x + C.
\end{aligned}$$

EXERCISE 3.2.24

(1) $\int (ax+b)^\alpha dx = \frac{1}{a}\int (ax+b)^\alpha d(ax+b) = \frac{1}{a(\alpha+1)}(ax+b)^{\alpha+1} + C.$

(2) By $x = at$, $\displaystyle\int \frac{dx}{a^2+x^2} = \int \frac{tdt}{a^2+a^2t^2} = \frac{1}{a}\arctan t + C = \frac{1}{a}\arctan\frac{x}{a} + C.$

(3) By $x = at$ and using Exercise 3.2.19(10), we have

$$\begin{aligned}
\int \frac{dx}{(a^2+x^2)^{\frac{3}{2}}} &= \int \frac{ady}{(a^2+a^2t^2)^{\frac{3}{2}}} = \frac{1}{a^2}\int \frac{dt}{(1+t^2)^{\frac{3}{2}}} \\
&= \frac{1}{a^2}\left(\frac{t}{2\left(\dfrac{3}{2}-1\right)(1+t^2)^{\frac{1}{2}}} + \frac{2\dfrac{3}{2}-3}{2\left(\dfrac{3}{2}-1\right)} \int \frac{dt}{(1+t^2)^{\frac{1}{2}}} \right) \\
&= \frac{t}{a^2\sqrt{1+t^2}} + C = \frac{x}{a^2\sqrt{a^2+x^2}} + C.
\end{aligned}$$

(4) By $x = at$, $\displaystyle\int \frac{dx}{\sqrt{a^2-x^2}} = \int \frac{adt}{\sqrt{a^2-a^2t^2}} = \frac{1}{a}\arcsin t + C = \frac{1}{a}\arcsin\frac{x}{a} + C.$

(5) By $x = a\tan t$ and Example 3.2.14, $\displaystyle\int \frac{dx}{\sqrt{a^2+x^2}} = \frac{1}{a}\int \frac{\sec^2 tdt}{\sec t} = \int \sec tdt = \log|\sec t + \tan t| + C = \log\left|\frac{x}{a} + \sqrt{\frac{x^2}{a^2}+1}\right| + C = \log(x+\sqrt{a^2+x^2}) + C.$

(6) By $x = a\sec t$ and Example 3.2.14, $\displaystyle\int \frac{dx}{\sqrt{x^2-a^2}} = \int \frac{a\sec t\tan tdt}{\sqrt{a^2\sec^2 t - a^2}} = \int \sec tdt = \log|\sec t + \tan t| + C = \log\left|\frac{x}{a} + \sqrt{\frac{x^2}{a^2}-1}\right| + C = \log|x+\sqrt{x^2-a^2}| + C.$

(7) By $x = a\sin t$, $\displaystyle\int \sqrt{a^2-x^2}dx = a^2\int \cos^2 tdt = \frac{a^2}{2}\int(1+\cos 2t)dt = \frac{a^2}{4}(2t+\sin 2t) + C = \frac{a^2}{2}(t+\sin t\cos t) + C = \frac{a^2}{2}\arcsin\frac{x}{a} + \frac{1}{2}x\sqrt{a^2-x^2} + C.$

(8) By $x = a\tan t$ and Exercise 3.2.24(20), we have $\displaystyle\int \sqrt{a^2+x^2}dx = \int a^2\sec^3 tdt = \frac{a^2}{2}\left(\sec t\tan t + \log|\sec t + \tan t|\right) + C = \frac{1}{2}\left(x\sqrt{a^2+x^2} + a^2\log(x+\sqrt{a^2+x^2})\right) + C.$

(9) By $x = a\sec t$ and Example 3.2.14, Exercise 3.2.24(20), we have

$$\begin{aligned}
\int \sqrt{x^2-a^2}dx &= \int a^2\tan t\sec t\tan tdt = a^2\int(\sec^3 t - \sec t)dt \\
&= \frac{a^2}{2}(\sec t\tan t - \log|\sec t + \tan t|) + C \\
&= \frac{1}{2}x\sqrt{x^2-a^2} - \frac{a^2}{2}\log(x+\sqrt{x^2-a^2}) + C.
\end{aligned}$$

(10) $\displaystyle\int \frac{xdx}{\sqrt{x^2-a^2}} = \int \frac{(x^2-a^2)'dx}{2\sqrt{x^2-a^2}} = \sqrt{x^2-a^2} + C.$

(11) By $x = a\sec t$, $\displaystyle\int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a}\int \frac{\sec t\tan tdt}{\sec t\tan t} = \frac{1}{a}t + C = \frac{1}{a}\operatorname{arcsec}\frac{x}{a} + C.$

(12) By $x = a\sec t$, we have

$$\begin{aligned}\int \frac{\sqrt{x^2-a^2}}{x}dx &= \int \frac{a\tan t}{a\sec t} a\sec t\tan t dt = a\int(\sec^2 t - 1)dt \\ &= a\tan t - at + C = \sqrt{x^2-a^2} - a\text{arcsec}\frac{t}{a} + C.\end{aligned}$$

(13) $\displaystyle\int x\sqrt{a^2-x^2}dx = -\frac{1}{2}\int(a^2-x^2)'\sqrt{a^2-x^2}dx = -\frac{1}{3}(a^2-x^2)^{\frac{3}{2}} + C.$

(14) $\displaystyle\int x\sqrt{a^2+x^2}dx = \frac{1}{2}\int(a^2+x^2)'\sqrt{a^2+x^2}dx = \frac{1}{3}(a^2+x^2)^{\frac{3}{2}} + C.$

(15) $\displaystyle\int x\sqrt{x^2-a^2}dx = \frac{1}{2}\int(x^2-a^2)'\sqrt{x^2-a^2}dx = \frac{1}{3}(x^2-a^2)^{\frac{3}{2}} + C$

Exercise 3.2.25

$$\begin{aligned}\int_0^x f(t)dt &= xf(x) - \int_0^x tf'(t)dt = xf(x) - \frac{1}{2}\int_0^x f'(t)dt^2\\ &= xf(x) - \frac{1}{2}x^2f'(x) + \frac{1}{2}\int_0^x t^2f'(t)dt = xf(x) - \frac{1}{2!}x^2f'(x) + \frac{1}{3!}\int_0^x f'(t)dt^3\\ &= xf(x) - \frac{1}{2!}x^2f'(x) + \frac{1}{3!}x^3f''(x) - \frac{1}{3!}\int_0^x t^3f''(t)dt\\ &= \cdots\\ &= xf(x) - \frac{x^2}{2!}f'(x) + \cdots + (-1)^{n-1}\frac{x^n}{n!}f^{(n-1)}(x) + (-1)^n\frac{1}{n!}\int_0^x t^nf^{(n)}(t)dt.\end{aligned}$$

Exercise 3.2.26

We have

$$\begin{aligned}&\left[uv^{(n-1)} - u'v^{(n-2)} + \cdots + (-1)^{n-1}u^{(n-1)}v\right]'\\ =&uv^{(n)} + u'v^{(n-1)} - u'v^{(n-1)} - u''v^{(n-2)} + \cdots + (-1)^{n-1}u^{(n-1)}v' + (-1)^{n-1}u^{(n)}v\\ =&uv^{(n)} + (-1)^{n-1}u^{(n)}v.\end{aligned}$$

Therefore $uv^{(n-1)} - u'v^{(n-2)} + \cdots + (-1)^{n-1}u^{(n-1)}v$ is the antiderivative of $uv^{(n)} + (-1)^{n-1}u^{(n)}v$. By the fundamental theorem of calculus, we get

$$\int_a^b uv^{(n)}dx + (-1)^{n-1}\int_a^b u^{(n)}vdx = \left[uv^{(n-1)} - u'v^{(n-2)} + \cdots + (-1)^{n-1}u^{(n-1)}v\right]_{x=a}^{x=b}.$$

By taking x, u, v, n to be t, $(x-t)^n$, $f(t)$, $n+1$, we get

$$\begin{aligned}&\int_{x_0}^x (x-t)^nf^{(n+1)}(t)dt\\ =&\left[(x-t)^nf^{(n)}(t) + n(x-t)^{n-1}f^{(n-1)}(t) + \cdots + n(n-1)\cdots 1f(t)\right]_{t=x_0}^{t=x}\\ =&n(n-1)\cdots 1f(x) - (x-x_0)^nf^{(n)}(x_0) - n(x-x_0)^{n-1}f^{(n-1)}(x_0) - \cdots - n(n-1)\cdots 1f(x_0)\\ =&n!R_n(x).\end{aligned}$$

For continuous $f^{(n+1)}(x)$,

$$R_n(x) = \frac{1}{n!}\int_{x_0}^x (x-t)^nf^{(n+1)}(t)dt \le \frac{\max_{[x_0,x]}f^{(n+1)}}{n!}\int_{x_0}^x (x-t)^ndt = \frac{\max_{[x_0,x]}f^{(n+1)}}{(n+1)!}(x-x_0)^{n+1}.$$

Similarly, we have $R_n(x) \ge \dfrac{\min_{[x_0,x]}f^{(n+1)}}{(n+1)!}(x-x_0)^{n+1}$. The by the intermediate value theorem for the continuous function $f^{(n+1)}(t)$ on $[x_0,x]$, we get $R_n(x) = \dfrac{f^{(n+1)}(c)}{(n+1)!}(x-x_0)^{n+1}$.

On the other hand, for continuous $f^{(n+1)}(x)$, we apply Exercise 3.1.15 to the continuous function $(x-t)^nf^{(n+1)}(t)$ for $t \in [x_0,x]$ and get $\displaystyle\int_{x_0}^x (x-t)^nf^{(n+1)}(t)dt = (x-x_0)(x-c)^nf^{(n+1)}(c)$ for some $c \in (x_0,x)$. Then $R_n(x) = \dfrac{(x-x_0)(x-c)^nf^{(n+1)}(c)}{n!}$.

EXERCISE 3.2.27

For any $\epsilon > 0$, there is $c > \max 0, a$, such that $|f'(x)| < \epsilon$ for $x \geq c$. Then

$$\begin{aligned}\frac{1}{b}\int_a^b f(x)\sin x dx &= \frac{1}{b}\int_a^c f(x)\sin x dx - \frac{1}{b}\int_c^b f(x) d\cos x \\ &= \frac{1}{b}\int_a^c f(x)\sin x dx - \frac{1}{b}(f(b)\cos b - f(c)\cos c) + \frac{1}{b}\int_c^b f'(x)\cos x dx.\end{aligned}$$

Since a and c are fixed, there is N, such that

$$b > N \implies \left|\frac{1}{b}\int_a^c f(x)\sin x dx\right| < \epsilon, \quad \left|\frac{1}{b}f(c)(\cos b - \cos c)\right| < \epsilon.$$

For $b > c$, we also have

$$\left|\frac{1}{b}\int_c^b f'(x)\cos x dx\right| \leq \frac{b-c}{b}\epsilon \leq \epsilon, \quad \left|\frac{1}{b}(f(b) - f(c))\cos b\right| \leq \frac{|f'(d)|(b-c)}{b} \leq \epsilon.$$

Thus for $b > \max\{N, c\}$, we have

$$\left|\frac{1}{b}\int_a^b f(x)\sin x dx\right| < 4\epsilon.$$

EXERCISE 3.2.28

(1) $\int_0^1 xe^{x^2}dx = \frac{1}{2}\int_0^1 e^{x^2}dx^2 = \frac{1}{2}\int_0^1 e^y dy = \frac{1}{2}(e^1 - e^0) = \frac{e-1}{2}$.

(2) $\int_0^2 \frac{xdx}{1+x^2} = \frac{1}{2}\int_0^2 \frac{dx^2}{1+x^2} = \frac{1}{2}\int_0^4 \frac{dy}{1+y} = \frac{1}{2}(\log(1+4) - \log(1+0)) = \frac{\log 5}{2}$.

(3) $\int_1^3 \frac{dx}{x\sqrt{x+1}} = \int_{\sqrt{2}}^2 \frac{2ydy}{(y^2-1)y} = \int_{\sqrt{2}}^2 \left(\frac{1}{y-1} - \frac{1}{y+1}\right) dy = \log\left|\frac{2-1}{2+1}\right| - \log\left|\frac{\sqrt{2}-1}{\sqrt{2}+1}\right| =$ $\log\frac{(\sqrt{2}+1)^2}{3}$.

EXERCISE 3.2.29

If $f(x)$ is an even function, then

$$\begin{aligned}\int_{-a}^a f(x)dx &= \int_0^a f(x)dx + \int_{-a}^0 f(x)dx = \int_0^a f(x)dx + \int_a^0 f(-x)d(-x)\\ &= \int_0^a f(x)dx - \int_a^0 f(x)dx = \int_0^a f(x)dx + \int_0^a f(x)dx = 2\int_0^a f(x)dx.\end{aligned}$$

If $f(x)$ is an odd function, then

$$\begin{aligned}\int_{-a}^a f(x)dx &= \int_0^a f(x)dx + \int_{-a}^0 f(x)dx = \int_0^a f(x)dx + \int_a^0 f(-x)d(-x)\\ &= \int_0^a f(x)dx + \int_a^0 f(x)dx = 0.\end{aligned}$$

EXERCISE 3.2.30

The only non-trivial proofs are the second or the third items implies the odd function.

Suppose $\int_{-b}^b f(x)dx = 0$ for any b. Then by taking the derivative in b, we get $f(b)+f(-b) = 0$, which means f is odd.

Suppose $\int_{-a}^a f(x)g(x)dx = 0$ for any even continuous function $g(x)$. By

$$\begin{aligned}\int_{-a}^a f(x)g(x)dx &= \int_{-a}^0 f(x)g(x)dx + \int_0^a f(x)g(x)dx = \int_0^a f(-x)g(x)dx + \int_0^a f(x)g(x)dx\\ &= \int_0^a (f(-x)+f(x))g(x)dx,\end{aligned}$$

and the fact that any continuous function on $[0,a]$ can be extended to an even function on $[-a,a]$, we get $\int_0^a (f(-x)+f(x))g(x)dx = 0$ for any continuous function g on $[0,a]$. By Exercise 3.1.18, this implies $f(x)+f(-x) = 0$.

EXERCISE 3.2.31

$\log x + \log y = \int_1^x \frac{dt}{t} + \int_1^x \frac{dt}{t} = \int_1^x \frac{dt}{t} + \int_{1y}^{ty} \frac{d(ty)}{ty} = \int_1^x \frac{dt}{t} + \int_y^{xy} \frac{ydt}{ty} = \int_1^x \frac{dt}{t} + \int_y^{xy} \frac{dt}{t} =$ $\int_1^{xy} \frac{dt}{t} = \log(xy)$.

Exercise 3.2.32

We have

$$\begin{aligned}\int_a^b \frac{f(x+h)-f(x)}{h}dx &= \frac{1}{h}\left(\int_a^b f(x+h)dx - \int_a^b f(x+h)dx\right)\\ &= \frac{1}{h}\left(\int_{a+h}^{b+h} f(x)dx - \int_a^b f(x)dx\right)\\ &= \frac{1}{h}\left(\int_b^{b+h} f(x)dx - \int_a^{a+h} f(x)dx\right).\end{aligned}$$

By Theorem 3.2.1 and the continuity of f at a and b, the limit of the right side as $h \to 0$ is $f(b) - f(a)$.

EXERCISE 3.2.33

By the fundamental theorem, we have $F' = f$. Therefore F has $(n+1)$-st order derivative and has Taylor expansion (and Lagrangian remainder)

$$F(x) = F(a) + F'(a)(x-a) + \frac{F''(a)}{2!}(x-a)^2 + \cdots + \frac{F^{(n)}(a)}{n!}(x-a)^n + \frac{F^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}.$$

We get the expansion of $F(b) = \int_a^b f(x)dx$ by using $F(a) = 0$ and $F^{(k)} = f^{(k-1)}$.

Alternatively, we have the Taylor expansion of f

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n-1)}(a)}{(n-1)!}(x-a)^n + R_n(x).$$

By Lagrange remainder formula, we have $\frac{\inf_{(a,b)} f^{(n)}}{n!}(x-a)^n \le R_n(x) \le \frac{\sup_{(a,b)} f^{(n)}}{n!}(x-a)^n$. Therefore

$$\begin{aligned} F(b) = &\int_a^b \left(f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n-1)}(a)}{(n-1)!}(x-a)^n \right) dx + \int_a^b R_n(x)dx \\ =&f(a)(b-a) + \frac{f'(a)}{2!}(b-a)^2 + \cdots + \frac{f^{(n-1)}(a)}{n!}(b-a)^n + \int_a^b R_n(x)dx. \end{aligned}$$

Moreover,

$$\begin{aligned} \frac{\inf_{(a,b)} f^{(n)}}{(n+1)!}(b-a)^{n+1} =&\frac{\inf_{(a,b)} f^{(n)}}{n!} \int_a^b (x-a)^n dx \le \int_a^b R_n(x)dx \\ \le&\frac{\sup_{(a,b)} f^{(n)}}{n!} \int_a^b (x-a)^n dx = \frac{\sup_{(a,b)} f^{(n)}}{(n+1)!}(b-a)^{n+1}. \end{aligned}$$

By Exercise 2.2.34, we have $\int_a^b R_n(x)dx = \frac{f^{(n)}(c)}{(n+1)!}(b-a)^{n+1}$ for some $c \in (a,b)$.

EXERCISE 3.2.34

Let $d = \frac{a+b}{2}$. Applying Exercise 3.2.33 to (a,d) (expanding at a) and (d,b) (expanding at b) respectively, we get

$$\begin{aligned} \int_a^d f(x)dx =&f(a)(d-a) + \frac{f'(a)}{2!}(d-a)^2 + \cdots + \frac{f^{(n-1)}(a)}{n!}(d-a)^n + \frac{f^{(n)}(c_1)}{(n+1)!}(d-a)^{n+1}, \\ \int_b^d f(x)dx =&f(b)(d-b) + \frac{f'(b)}{2!}(d-b)^2 + \cdots + \frac{f^{(n-1)}(b)}{n!}(d-b)^n + \frac{f^{(n)}(c_2)}{(n+1)!}(d-b)^{n+1}. \end{aligned}$$

By $d - a = \frac{b-a}{2} = -(d-b)$, we get

$$\begin{aligned} \int_a^b f(x)dx =& \int_a^d f(x)dx - \int_b^d f(x)dx \\ =& \sum_{k=0}^{n-1} \frac{f^{(k)}(a) - (-1)^{k+1} f^{(k)}(b)}{(k+1)!2^k}(b-a)^{k+1} + \frac{f^{(n)}(c_1) - (-1)^{n+1} f^{(n)}(c_2)}{(n+1)!2^{n+1}}(b-a)^{n+1}. \end{aligned}$$

If n is odd, then

$$|f^{(n)}(c_1)-(-1)^{n+1}f^{(n)}(c_2)|=|f^{(n)}(c_1)-f^{(n)}(c_2)|\le \omega_{(a,b)}(f^{(n)}).$$

If n is even, then

$$2\inf_{(a,b)} f^{(n)}\le f^{(n)}(c_1)-(-1)^{n+1}f^{(n)}(c_2)=f^{(n)}(c_1)+f^{(n)}(c_2)\le 2\sup_{(a,b)} f^{(n)}.$$

By Exercise 2.2.34, we have $f^{(n)}(c_1)-(-1)^{n+1}f^{(n)}(c_2)=2f^{(n)}(c)$ for some $c\in(a,b)$.

EXERCISE 3.2.35

Let $d=\dfrac{a+b}{2}$. Applying Exercise 3.2.33 to (a,d) (expanding at d) and (d,b) (expanding at d) respectively, we get

$$\begin{aligned}\int_d^a f(x)dx =&f(d)(a-d)+\frac{f'(d)}{2!}(a-d)^2+\cdots+\frac{f^{(2n-1)}(d)}{(2n)!}(a-d)^n+\frac{f^{(2n)}(c_1)}{(2n+1)!}(a-d)^{2n+1},\\ \int_d^b f(x)dx =&f(d)(b-d)+\frac{f'(d)}{2!}(b-d)^2+\cdots+\frac{f^{(2n-1)}(d)}{(2n)!}(b-d)^n+\frac{f^{(2n)}(c_2)}{(2n+1)!}(b-d)^{2n+1}.\end{aligned}$$

By $-(a-d)=\dfrac{b-a}{2}=b-d$, we get

$$\begin{aligned}\int_a^b f(x)dx &=-\int_d^a f(x)dx+\int_d^b f(x)dx\\ &=\sum_{0\le k<2n,k\text{ even}}\frac{f^{(k)}(d)}{(k+1)!2^k}(b-a)^{k+1}+\frac{f^{(2n)}(c_1)+f^{(2n)}(c_2)}{(2n+1)!2^{2n+1}}(b-a)^{2n+1}.\end{aligned}$$

By Exercise 2.2.34, we have $f^{(2n)}(c_1)+f^{(2n)}(c_2)=2f^{(2n)}(c)$ for some $c\in(a,b)$.

EXERCISE 3.2.36

By Exercise 3.1.51, we have $x_{i-1}<x_i^*<x_i$, such that

$$S_{\text{left},n}(f)-\int_a^b f(x)dx=\sum\left(f(x_{i-1})\Delta x_i-\int_{x_{i-1}}^{x_i} f(x)dx\right)=\sum\frac{f'(x_i^*)}{2}\Delta x_i^2=\frac{b-a}{2n}\sum f'(x_i^*)\Delta x_i.$$

Since f' is integrable, by Theorem 3.2.1, this implies

$$\lim_{n\to\infty} n\left(S_{\text{left},n}(f)-\int_a^b f(x)dx\right)=\frac{b-a}{2}\int_a^b f'(x)dx=\frac{b-a}{2}(f(b)-f(a)).$$

EXERCISE 3.2.37

By Exercise 3.1.54, we have $x_{i-1}<x_i^*<x_i$, such that

$$\begin{aligned}\int_a^b f(x)dx-S_{\text{middle},n}(f)&=\sum\left(\int_{x_{i-1}}^{x_i} f(x)dx-f\left(\frac{x_{i-1}+x_i}{2}\right)\Delta x_i\right)\\ &=\sum\frac{f''(x_i^*)}{24}\Delta x_i^3=\frac{(b-a)^2}{24n^2}\sum f''(x_i^*)\Delta x_i.\end{aligned}$$

Since f'' is integrable, by Theorem 3.2.1, this implies

$$\lim_{n\to\infty} n\left(\int_a^b f(x)dx - S_{\text{middle},n}(f)\right) = \frac{(b-a)^2}{24n^2}\int_a^b f''(x)dx = \frac{(b-a)^2}{24n^2}(f'(b)-f'(a)).$$

Exercise 3.2.39

$$\begin{aligned}
\text{Av}_{[a+c,b+c]}(f(x+c)) =& \frac{1}{(b+c)-(a+c)}\int_{a+c}^{b+c} f(x+c)dx = \frac{1}{b-a}\int_a^b f(y)d(y-c)\\
=& \text{Av}_{[a,b]}(f(x)),\\
\text{Av}_{[\lambda a,\lambda b]}(f(\lambda x)) =& \frac{1}{(\lambda b)-(\lambda a)}\int_{\lambda a}^{\lambda b} f(\lambda x)dx\\
=& \frac{1}{\lambda(b-a)}\int_a^b f(y)d(\lambda y) = \text{Av}_{[a,b]}(f(x)),\\
\lambda\text{Av}_{[a,c]}(f) + (1-\lambda)\text{Av}_{[c,b]}(f) =& \lambda\frac{1}{c-a}\int_a^c f(x)dx + (1-\lambda)\frac{1}{b-c}\int_c^b f(x)dx\\
=& \frac{1}{b-a}\int_a^c f(x)dx + \frac{1}{b-a}\int_c^b f(x)dx = \text{Av}_{[a,b]}(f).
\end{aligned}$$

$$f \ge g \implies \int_a^b f(x)dx \ge \int_a^b g(x)dx \implies \text{Av}_{[a,b]}(f) \ge \text{Av}_{[a,b]}(g).$$

If $f(x)$ is continuous, then $\text{Av}_{[a,b]}(f) = f(c)$ by Exercise 3.1.15.

Exercise 3.2.40

By change of variable, we have $g(x) = \int_0^1 f(xt)dt$.

(1) By $\lim_{x\to+\infty} f(x) = l$, $f(x)$ is bounded. Assume $|f(x)| < M$ for all $x \ge a$. For any $\epsilon > 0$, there is $N > 0$, such that $x \ge N$ implies $|f(x) - l| < \epsilon$. Then for $x > \epsilon^{-1}N$, we have $xt > N$ for $t \in [\epsilon, 1]$. Therefore

$$|g(x) - l| \le \left(\int_0^\epsilon + \int_\epsilon^1\right)|f(xt) - l|dx \le \epsilon(M + |l|) + (1-\epsilon)\epsilon.$$

(2) For $0 < x < y$ and $t \ge 0$, we have $xt \le yt$ and $f(xt) \le f(yt)$. Therefore

$$g(x) = \int_0^1 f(xt)dt \le \int_0^1 f(yt)dt = g(y).$$

(3) For $0 \le \lambda \le 1$, $x, y > 0$, we have

$$\lambda g(x) + (1-\lambda)g(y) = \int_0^1 [\lambda f(xt) + (1-\lambda)f(yt)]dt \ge \int_0^1 f(\lambda xt + (1-\lambda)yt)dt = g(\lambda x + (1-\lambda)y).$$

Exercise 3.2.41

$\text{Av}^{\lambda(x+c)}_{[a+c,b+c]}(f(x+c)) = \text{Av}^{\lambda}_{[a,b]}(f(x))$.

$\text{Av}^{\lambda(\mu x)}_{[\mu a,\mu b]}(f(\mu x)) = \text{Av}^{\lambda}_{[a,b]}(f(x))$.

If $c = \lambda a + (1-\lambda)b$, then $\mathrm{Av}^{\lambda}_{[a,b]}(f) = \lambda\mathrm{Av}^{\lambda}_{[a,c]}(f) + (1-\lambda)\mathrm{Av}^{\lambda}_{[c,b]}(f)$.

If $f \ge g$, then $\mathrm{Av}^{\lambda}_{[a,b]}(f) \ge \mathrm{Av}^{\lambda}_{[a,b]}(g)$.

If $f(x)$ is continuous, then $\mathrm{Av}^{\lambda}_{[a,b]}(f) = f(c)$ for some $a < c < b$.

EXERCISE 3.2.42

$\displaystyle\int_a^{a+T} f(x)dx = \int_a^T f(x)dx + \int_T^{a+T} f(x)dx = \int_a^T f(x)dx + \int_T^{a+T} f(x-T)d(x-T) = \int_a^T f(x)dx + \int_0^T f(x-T)d(x-T) = \int_0^T f(x)dx$. The equality $f(x-T) = f(x)$ is used in the second equality.

EXERCISE 3.2.43

Suppose $|f(x)| < M$ on $[0,T]$. By periodic property, we have $|f(x)| < M$ for all x. For any $b > x$, let n be the natural number satisfying $a + nT \le b < a + (n+1)T$. By Exercise 3.2.42,

$$\begin{aligned}\int_a^b f(x)dx &= \int_a^{a+T} f(x)dx + \int_{a+T}^{a+2T} f(x)dx + \cdots + \int_{a+(n-1)T}^{a+nT} f(x)dx + \int_{a+nT}^{b} f(x)dx\\ &= n\int_0^T f(x)dx + \int_0^{b-a-nT} f(x)dx.\end{aligned}$$

Then

$$\left|\frac{1}{b}\int_a^b f(x)dx - \frac{1}{T}\int_0^T f(x)dx\right| \le \left|\frac{n}{b} - \frac{1}{T}\right|\left|\int_0^T f(x)dx\right| + \frac{1}{b}\left|\int_0^{b-a-nT} f(x)dx\right|.$$

The limit follows from $\dfrac{a}{n} \le \dfrac{b}{n} - T < \dfrac{a+T}{n}$ and $\displaystyle\left|\int_0^{b-a-nT} f(x)dx\right| \le M(b-a-nT) \le MT$.

EXERCISE 3.2.44

The periodic integrable function f is bounded. Assume $|f| < M$, $|g| < M$. For any $t > 0$, consider the partition of $[a,b]$ by intervals of length $\delta = \dfrac{T}{t}$

$$P_t\colon a = x_0 \le x_1 = a + \delta < x_2 = a + 2\delta < \cdots < x_n = a + n\delta \le b.$$

The right end of P_t is b. Let P_t' be the partition of $[a, a+n\delta]$ with $a + n\delta$ in place of b as the right end. .

Choose any x_i^* for the partition. Then

$$\begin{aligned}\left|\sum_{i=1}^n g(x_i^*)\int_{x_{i-1}}^{x_i} f(tx)dx - \int_a^{a+n\delta} f(tx)g(x)dx\right| \le&\sum_{i=1}^n \int_{x_{i-1}}^{x_i} |f(tx)||g(x_i^*) - g(x)|dx\\ \le& M\sum_{i=1}^n \omega_{[x_{i-1},x_i]}(g)\Delta x_i.\end{aligned}$$

By Exercise 3.2.42, we have

$$\int_{x_{i-1}}^{x_i} f(tx)dx = \int_{x_{i-1}}^{x_{i-1}+\delta} f(tx)dx = \frac{1}{t}\int_{x_{i-1}}^{x_{i-1}+T} f(x)dx = \frac{\delta}{T}\int_0^T f(x)dx.$$

By $\Delta x_i = \delta$, we get

$$\sum_{i=1}^{n} g(x_i^*) \int_{x_{i-1}}^{x_i} f(tx)dx = \sum_{i=1}^{n} g(x_i^*) \frac{\delta}{T} \int_0^T f(x)dx = \left(\frac{1}{T} \int_0^T f(x)dx \right) S(P'_t, g).$$

Moreover,

$$\begin{aligned} |S(P'_t, g) - S(P_t, g)| &= |g(x_n^*)|(b - x_n) \le M\delta, \\ \left| \frac{\delta}{T} \int_0^T f(x)dx \right| &\le M, \\ \left| \int_a^{a+n\delta} f(tx)g(x)dx - \int_a^b f(tx)g(x)dx \right| &\le M^2(b - x_n) \le M^2\delta. \end{aligned}$$

Combining all the estimations together, we get

$$\left| \left(\frac{1}{T} \int_0^T f(x)dx \right) S(P_t, g) - \int_a^b f(tx)g(x)dx \right| \le M \sum_{i=1}^{n} \omega_{[x_{i-1}, x_i]}(g)\Delta x_i + 2M^2\delta.$$

By the integrability of g, the estimation implies that

$$\lim_{t\to\infty} \int_a^b f(tx)g(x)dx = \left(\frac{1}{T} \int_0^T f(x)dx \right) \lim_{\delta\to 0} S(P_t, g) = \frac{1}{T} \int_0^T f(x)dx \int_a^b g(x)dx.$$

Exercise 3.2.45

(1) Taking derivative of both sides, we get

$$\begin{aligned} \frac{1}{(a\sin x + b\cos x)^n} = \frac{A\cos x - B\sin x}{(a\sin x + b\cos x)^{n-1}} &- \frac{(n-1)(A\sin x + B\cos x)(a\cos x - b\sin x)}{(a\sin x + b\cos x)^n} \\ + C\frac{1}{(a\sin x + b\cos x)^{n-2}}&. \end{aligned}$$

Comparing the two sides, we get

$$1 = (Av - Bu)(au + bv) - (n-1)(Au + Bv)(av - bu) + C(au + bv)^2$$

for all u, v satisfying $u^2 + v^2 = 1$. Comparing the coefficients, we get

$$\begin{array}{cccc} -(n-2)aA & +(n-2)bB & +2abC & = 0 \\ (n-1)bA & -aB & +a^2C & = 1 \\ bA & -(n-1)aB & +b^2C & = 1 \end{array}$$

The solution is

$$A = \frac{b}{(n-1)(a^2+b^2)}, \quad B = \frac{-a}{(n-1)(a^2+b^2)}, \quad C = \frac{n-2}{(n-1)(a^2+b^2)}.$$

(2) Taking derivative of both sides and multiplying the common denominator, we get

$$1 = A\cos x(a + b\cos x) + (n-1)bA\sin^2 x + B(a + b\cos x) + C(a + b\cos x)^2.$$

Comparing the two sides, we get the equations

$$aA+bB=2abC=0,\ \ bA+b^2C=(n-1)bA,\ \ (n-1)bA+aB+a^2C=1.$$

The solution is

$$A=\frac{b}{(n-1)(b^2-a^2)},\ \ B=\frac{-(2n-3)a}{(n-1)(b^2-a^2)},\ \ C=\frac{n-2}{(n-1)(b^2-a^2)}.$$

Exercise 3.2.46

If $b-a\neq n\pi$, then by

$$\tan(x+a)-\tan(x+b)=\frac{\sin(a-b)}{\cos(x+a)\cos(x+b)},$$

we have

$$\int\frac{dx}{\cos(x+a)\cos(x+b)}=\frac{1}{\sin(a-b)}\int(\tan(x+a)-\tan(x+b))dx=\frac{1}{\sin(a-b)}\log\left|\frac{\cos(x+b)}{\cos(x+a)}\right|+C.$$

If $b-a=n\pi$, then

$$\int\frac{dx}{\cos(x+a)\cos(x+b)}=(-1)^n\int\frac{dx}{\cos^2(x+a)}=(-1)^n\int\sec^2(x+a)dx=(-1)^n\tan(x+a)+C.$$

Similarly, by

$$\cot(x+a)+\tan(x+b)=\frac{\cos(a-b)}{\sin(x+a)\cos(x+b)},$$

we have

$$\int\frac{dx}{\sin(x+a)\cos(x+b)}=\begin{cases}\dfrac{1}{\cos(a-b)}\log\left|\dfrac{\sin(x+b)}{\cos(x+a)}\right|+C & \text{if } b-a\neq\left(n+\frac{1}{2}\right)\pi\\ (-1)^n\cot(x+a)+C & \text{if } b-a=\left(n+\frac{1}{2}\right)\pi\end{cases}.$$

By

$$\tan(x+a)\tan(x+b)+1=\frac{\cos(a-b)}{\cos(x+a)\cos(x+b)}=\frac{\cos(a-b)}{\sin(a-b)}(\tan(x+a)-\tan(x+b)),$$

we get

$$\int\tan(x+a)\tan(x+b)dx=\begin{cases}\cot(a-b)\log\left|\dfrac{\sin(x+b)}{\cos(x+a)}\right|-x+C & \text{if } b-a\neq\left(n+\frac{1}{2}\right)\pi\\ -x+C & \text{if } b-a=\left(n+\frac{1}{2}\right)\pi\end{cases}.$$

Then

$$\int\frac{dx}{\sin x-\sin a}=4\int\frac{d\dfrac{x}{2}}{\sin\dfrac{x+a}{2}\cos\dfrac{x-a}{2}}=\begin{cases}\dfrac{4}{\cos a}\log\left|\dfrac{\sin\dfrac{x-a}{2}}{\cos\dfrac{x+a}{2}}\right|+C & \text{if } a\neq\left(n+\frac{1}{2}\right)\pi\\ (-1)^n\cot\dfrac{x+a}{2}+C & \text{if } a=\left(n+\frac{1}{2}\right)\pi\end{cases},$$

and

$$\int \frac{dx}{\cos x + \cos a} = 4\int \frac{d\frac{x}{2}}{\cos\frac{x+a}{2}\cos\frac{x-a}{2}} = \begin{cases} \dfrac{4}{\sin a}\log\left|\dfrac{\cos\frac{x-a}{2}}{\cos\frac{x+a}{2}}\right| + C & \text{if } a \neq \left(n+\frac{1}{2}\right)\pi \\ (-1)^n\tan\dfrac{x+a}{2} + C & \text{if } a = \left(n+\frac{1}{2}\right)\pi \end{cases}.$$

Exercise 3.2.47

$w(p,q) = \int_0^1 (1-x^{\frac{1}{p}})^{q-1}dx - \int_0^1 (1-x^{\frac{1}{p}})^{q-1}x^{\frac{1}{p}}dx = w(p,q-1) + \int_0^1 \frac{p}{q}xd(1-x^{\frac{1}{p}})^q = w(p,q-1) + \frac{p}{q}\,x(1-x^{\frac{1}{p}})^q\Big|_{x=0}^{x=1} - \frac{p}{q}\int_0^1 (1-x^{\frac{1}{p}})^q dx = w(p,q-1) - \frac{p}{q}w(p,q).$

Exercise 3.2.48

For $y=(1-x^{\frac{1}{p}})^q$, we have $x=(1-y^{\frac{1}{q}})^p$ and $w(p,q) = \int_0^1 (1-x^{\frac{1}{p}})^q dx = \int_1^0 yd(1-y^{\frac{1}{q}})^p = -\int_1^0 yp(1-y^{\frac{1}{q}})^{p-1}\frac{1}{q}y^{\frac{1}{q}-1}dy = \frac{p}{q}\int_0^1 (1-y^{\frac{1}{q}})^{p-1}y^{\frac{1}{q}}dy = \frac{p}{q}\int_0^1 [(1-y^{\frac{1}{q}})^{p-1} - (1-y^{\frac{1}{q}})^p]dy = \frac{p}{q}(w(q,p-1)-w(q,p)) = \frac{p}{q}\left(\frac{p+q}{p}w(q,p) - w(q,p)\right) = w(q,p).$

Exercise 3.2.49 By Exercise 3.2.47,

$$\begin{aligned} w(m,n) =& \frac{n}{m+n}w(m,n-1) = \frac{n(n-1)}{(m+n)(m+n-1)}w(m,n-2) \\ =& \cdots = \frac{n(n-1)\cdots 1}{(m+n)(m+n-1)\cdots(m+1)}w(m,0) = \frac{m!n!}{(m+n)!}1 = \frac{m!n!}{(m+n)!}. \end{aligned}$$

Exercise 3.2.50

w is strictly decreasing in q because $(1-x^{\frac{1}{p}})^q$ is strictly decreasing in q (since $1-x^{\frac{1}{p}} < 1$). By symmetry, w is also strictly decreasing in p.

Exercise 3.2.51

$w\left(\frac{1}{2},\frac{1}{2}\right) = \int_0^1 \sqrt{1-x^2}dx = \int_0^{\frac{\pi}{2}} \cos t d(\sin t) = \int_0^{\frac{\pi}{2}} \cos^2 t dt = \frac{1}{2}\int_0^{\frac{\pi}{2}} (1+\cos 2t)dt = \frac{\pi}{4}.$

Exercise 3.2.52

By Exercises 3.2.47 and 3.2.48,

$$\begin{aligned} w\left(n+\frac{1}{2}, n+\frac{1}{2}\right) =& \frac{n+\frac{1}{2}}{n+\frac{1}{2}+n+\frac{1}{2}}w\left(n+\frac{1}{2},n-\frac{1}{2}\right) = \frac{1}{2}\frac{n+\frac{1}{2}}{n+\frac{1}{2}+n-\frac{1}{2}}w\left(n-\frac{1}{2},n-\frac{1}{2}\right) \\ =& \frac{2n+1}{8n}w\left(n-\frac{1}{2},n-\frac{1}{2}\right) = \frac{(2n+1)(2n-1)}{8^2n(n-1)}w\left(n-\frac{3}{2},n-\frac{3}{2}\right) \\ =& \cdots = \frac{(2n+1)(2n-1)\cdots 3}{8^n n(n-1)\cdots 1}w\left(\frac{1}{2},\frac{1}{2}\right). \end{aligned}$$

Then by Exercises 3.2.49, 3.2.50, 3.2.51,

$$w(n,n) = \frac{(n!)^2}{(2n)!} > w\left(n+\frac{1}{2},n+\frac{1}{2}\right) = \frac{(2n+1)(2n-1)\cdots 3}{8^n n(n-1)\cdots 1}\frac{\pi}{4}.$$

By $2n(2n-2)\cdots 2 = 2^n n!$, this is the same as

$$\frac{\pi}{2} < \frac{(2n(2n-2)\cdots 2)^3}{(2n)!(2n+1)(2n-1)\cdots 3} = \frac{(2n(2n-2)\cdots 2)^2}{(2n+1)((2n-1)\cdots 3)^2}.$$

The other inequality is similar.

EXERCISE 3.2.54

$$\begin{aligned}
&\left|S(P,Fg) - \sum_{i=1}^n S_i(P,f)g(x_i)\Delta x_i\right| = \left|\sum_{i=1}^n F(x_i)g(x_i)\Delta x_i - \sum_{i=1}^n S_i(P,f)g(x_i)\Delta x_i\right| \\
\le &\sum_{i=1}^n |F(x_i) - S_i(P,f)||g(x_i)|\Delta x_i \le \sum_{i=1}^n \left(\sum_{j=1}^i \omega_{[x_{j-1},x_j]}(f)\Delta x_j\right)|g(x_i)|\Delta x_i \\
\le &\left(\sum_{i=1}^n \omega_{[x_{i-1},x_i]}(f)\Delta x_i\right)\sum_{i=1}^n |g(x_i)|\Delta x_i = \left(\sum_{i=1}^n \omega_{[x_{i-1},x_i]}(f)\Delta x_i\right)S(P,|g|),
\end{aligned}$$

where the second inequality follows from the second inequality in Exercise 3.1.16.

EXERCISE 3.2.55

$$\begin{aligned}
&\sum_{i=1}^n S_i(P,f)g(x_i)\Delta x_i + \sum_{i=1}^n f(x_i)\Delta x_i S_i(g,P) \\
= &\sum_{1\le j\le i\le n} f(x_j)\Delta x_j g(x_i)\Delta x_i + \sum_{1\le j\le i\le n}^n f(x_i)\Delta x_i g(x_j)\Delta x_j \\
= &\sum_{1\le i\le j\le n} f(x_i)\Delta x_i g(x_j)\Delta x_j + \sum_{1\le j<i\le n}^n f(x_i)\Delta x_i g(x_j)\Delta x_j + \sum_{1\le i\le n}^n f(x_i)\Delta x_i g(x_i)\Delta x_i \\
= &\left(\sum_{1\le i\le n} f(x_i)\Delta x_i\right)\left(\sum_{1\le j\le n} g(x_j)\Delta x_j\right) + \sum_{1\le i\le n}^n f(x_i)g(x_i)\Delta x_i^2 \\
= &S(P,f)S(P,g) + \sum_{i=1}^n f(x_i)g(x_i)\Delta x_i^2.
\end{aligned}$$

[In fact, the computation holds for general choices of x_i^*].

EXERCISE 3.2.56

By Exercises 3.2.54 and 3.2.55, the difference between $S(P,Fg)+S(P,fG)$ and $S(P,f)S(P,g)$ is bounded by the three quantities: $\left(\sum_{i=1}^n \omega_{[x_{i-1},x_i]}(f)\Delta x_i\right)S(P,|g|)$, $\left(\sum_{i=1}^n \omega_{[x_{i-1},x_i]}(g)\Delta x_i\right)S(P,|f|)$, and $\sum_{i=1}^n f(x_i)g(x_i)\Delta x_i^2$. If we can shows that the three quantities approach 0 as $\|P\| \to 0$, then we get

$$\begin{aligned}
\int_a^b (F(x)g(x) + f(x)G(x))dx &= \lim_{\|P\|\to 0}(S(P,Fg) + S(P,fG)) = \lim_{\|P\|\to 0} S(P,f)S(P,g) \\
&= \lim_{\|P\|\to 0} S(P,f) \lim_{\|P\|\to 0} S(P,g) = F(b)G(b).
\end{aligned}$$

Since f and g are integrable, their absolute values are also integrable. Therefore $S(P,|f|)$ and $S(P,|g|)$ are bounded, and $\sum_{i=1}^n \omega_{[x_{i-1},x_i]}(f)$ and $\sum_{i=1}^n \omega_{[x_{i-1},x_i]}(g)$ approach 0 as $\|P\| \to 0$. Thus the first two quantities approach 0.

Suppose $|g| < M$ (this happens becuase g is integrable). Then $|\sum_{i=1}^{n} f(x_i)g(x_i)\Delta x_i^2| \leq M\|P\| \sum |f(x_i)|\Delta x_i = M\|P\|S(P, |f|)$. Then by the boundedness of $S(P, |f|)$, we see $\sum_{i=1}^{n} f(x_i)g(x_i)\Delta x_i^2$ approaches 0 as $\|P\| \to 0$.

Exercise 3.2.57

We have $c = F(a)$, $d = G(a)$. Exercise 3.2.56 tells us

$$\int_a^b ((F(x) - c)g(x) + f(x)(G(x) - d))dx = (F(b) - c)(G(b) - d),$$

Therefore

$$\begin{aligned}\int_a^b (F(x)g(x) + f(x)G(x))dx = &\int_a^b (cg(x) + df(x))dx + (F(b) - c)(G(b) - d)\\ =&cG(b) + dF(b) + (F(b) - c)(G(b) - d)\\ =&F(b)G(b) - cd = F(b)G(b) - F(a)G(a).\end{aligned}$$

Exercise 3.2.58

By integration by parts, we have

$$\begin{aligned}\int_a^b f(x)g(x)dx = &\int_a^b f(x)dG(x) = f(b)G(b) - \int_a^b f'(x)G(x)dx \leq f(b)G(b) - \int_a^b f'(x)Mdx\\ =&f(b)G(b) - M(f(b) - f(a)) = f(b)(G(b) - M) + Mf(a) \leq Mf(a),\end{aligned}$$

where the first inequality makes us of f decreasing (so that $f' \leq 0$) and the second inequality makes use of $f(b) \geq 0$. The proof of $mf(a) \leq \int_a^b f(x)g(x)dx$ is similar. Then by the intermediate value theorem, we have $\int_a^b f(x)g(x)dx = f(a)G(c) = f(a)\int_a^c g(x)dx$ for some $c \in (a, b)$.

Exercise 3.2.59

Suppose f is decreasing. Then by Exercise 3.2.58, we have $\int_a^b (f(x) - f(b))g(x)dx = (f(a) - f(b))\int_a^c g(x)dx$ for some $c \in (a, b)$. This is the same as

$$\begin{aligned}\int_a^b f(x)g(x)dx =&f(b)\int_a^b g(x)dx + (f(a) - f(b))\int_a^c g(x)dx\\ =&f(a)(f(a) - f(b))\int_a^c g(x)dx + f(b)\int_c^b g(x)dx.\end{aligned}$$

For the case f is increasing, we may consider $-f$, which is decreasing.

Exercise 3.2.60

If $f(a) \leq b$, then $a \leq f^{-1}(b)$, and

$$\begin{aligned}\int_0^a f(x)dx + \int_0^b f^{-1}(y)dy = &\int_0^a f(x)dx + \int_0^{f(a)} f^{-1}(y)dy + \int_{f(a)}^b f^{-1}(y)dy\\ \geq &\int_0^a f(x)dx + \int_0^a f^{-1}(f(x))df(x) + f^{-1}(f(a))(b - f(a))\\ = &\int_0^a (fdx + xdf) + a(b - f(a)) = af(a) + +a(b - f(a)) = ab.\end{aligned}$$

If $f(a) \geq b$, then $a \geq f^{-1}(b)$, and

$$\begin{aligned}
\int_0^a f(x)dx + \int_0^b f^{-1}(y)dy &= \int_{f^{-1}(b)}^a f(x)dx + \int_0^{f^{-1}(b)} f(x)dx + \int_0^b f^{-1}(y)dy \\
&\geq f(f^{-1}(b))(a - f^{-1}(b)) + \int_0^{f^{-1}(b)} f(x)dx + \int_0^{f^{-1}(b)} f^{-1}(f(x))df(x) \\
&= b(a - f^{-1}(b)) + \int_0^{f^{-1}(b)} (fdx + xdf) \\
&= b(a - f^{-1}(b)) + f^{-1}(b)f(f^{-1}(b)) = ab.
\end{aligned}$$