OLD MATH203 FINAL EXAMS
(*Answers not absolutely guaranteed to be correct*)

Math 203 Final, Spring 2004

(1) (20 points)

1. Prove that if $f(x)$ is uniformly continuous on a bounded interval (a, b), then the function is bounded on (a, b).
2. Explain why the uniform condition is necessary in the first part.
3. Explain why the function $f(x) = \dfrac{e^x \sin x + e^{-x} \cos x}{\cos(\sin x) - 1}$ is not uniformly continuous on $(0, 1]$.
4. Explain why the function $g(x) = \sqrt{x}$ is uniformly continuous on $[0, \infty)$.

(2) (10 points) How much of the corners of a square with side length 1 should be cut so that the tray made from it has the biggest volume?
(3) (24 points) Compute the derivatives.

1. Find $\dfrac{d^3y}{dx^3}$ for $x = \cos^3 t$, $y = \sin^3 t$.
2. Find $f^{(n)}(x)$ for $f(x) = \dfrac{1+x}{\sqrt{1-x}}$.
3. Find $f^{(n)}(0)$ for $f(x) = \arcsin x$. (Hint: use the Taylor expansion of $f'(x)$ at 0)

(4) (15 points) Find the second order Taylor expansion of $(1 + ax + bx^2)^{\frac{1}{x}}$ at 0. Then compute the limit $\lim_{x\to 0} \dfrac{(1 + 2x + x^2)^{\frac{1}{x}} - (1 + 2x - x^2)^{\frac{1}{x}}}{x}$.
(5) (15 points) Sketch the graph of the function $f(x) = \dfrac{e^x}{\sqrt{|x|}}$.
(6) (16 points) True or false. No reason needed.

1. Suppose $f(x)$ is differentiable. Then $f'(x) > 0$ for all $x \implies f(x)$ is strictly increasing.
2. Suppose $f(x)$ is differentiable. Then $f(x)$ is strictly increasing $\implies f'(x) > 0$ for all x.
3. $f(x)$ is uniformly continuous $\implies f(x)^2$ is uniformly continuous.
4. $f(x)^2$ is uniformly continuous and $f(x) \geq 0 \implies f(x)$ is uniformly continuous.
5. Suppose $\lim_{x\to 0} f(x) = 0$ and $f(x)$ is differentiable near 0. Then $\lim_{x\to 0} \dfrac{f(x)}{x} = 0$ $\implies \lim_{x\to 0} f'(x) = 0$.

6. Suppose $\lim_{x\to 0} f(x) = 0$ and $f(x)$ is differentiable near 0. Then $\lim_{x\to 0} f'(x) = 0$ $\implies \lim_{x\to 0} \dfrac{f(x)}{x} = 0$.

7. If $f(x)$ and $g(x)$ have local maximum at a, then $\min\{f(x), g(x)\}$ has local maximum at a.

8. If $f(x)$ and $g(x)$ have local maximum at a, then $f(x)g(x)$ has local maximum at a.

Answer to Math 203 Final, Spring 2004

(1) Suppose $f(x)$ is uniformly continuous on a bounded interval (a, b). For $\epsilon = 1 > 0$, there is $\delta > 0$, such that $a < x, y < b$ and $|x - y| < \delta$ implies $|f(x) - f(y)| < 1$. Fix x_0 in (a, b) and let $n > \dfrac{b-a}{\delta}$ be a fixed integer. Then for any $a < x < b$, we can find $x_1, x_2, \ldots, x_n = x$, such that $|x_1 - x_0| < \delta$, $|x_2 - x_1| < \delta$, ..., $|x_n - x_{n-1}| < \delta$. This implies $|f(x_1) - f(x_0)| < 1$, $|f(x_2) - f(x_1)| < 1$, ..., $|f(x_n) - f(x_{n-1})| < 1$, which further implies

$$|f(x) - f(x_0)| \le |f(x_1) - f(x_0)| + |f(x_2) - f(x_1)| + \ldots + |f(x_n) - f(x_{n-1})| \le n.$$

Thus $f(x_0) + n$ and $f(x_0) - n$ are respectively the upper and lower bound.

The uniform condition is necessary because of the following counterexample. The function $f(x) = \dfrac{1}{x}$ is continuous on $(0, 1)$. But it is not bounded (because it is not uniformly continuous.

The function $f(x) = \dfrac{e^x \sin x + e^{-x} \cos x}{\cos(\sin x) - 1}$ is obtained by the arithmetic operations and compositions of the continuous functions e^x, $\sin x$, $\cos x$. Moreover, the numerator $\cos(\sin x) - 1 \neq 0$ on $(0, 1]$. Thus $f(x)$ is continuous on $(0, 1]$. However, Since $\lim_{x\to 0^+}(e^x \sin x + e^{-x} \cos x) = 1$ and $\lim_{x\to 0^+}(\cos(\sin x) - 1) = 0$, we have $\lim_{x\to 0^+} f(x) = \infty$. In particular, the function is not bounded on $(0, 1)$. By the first part, the function is not uniformly continuous.

The function $g(x)$ is continuous and is therefore uniformly continuous on the bounded closed interval $[0, 1]$. On the other hand, on the interval $[1, \infty)$, we have $|g'(x)| \le 1$. Then by the mean value theorem, for $x, y \ge 1$ we have $|g(x) - g(y)| = |g'(c)(x - y)| \le |x - y|$. Thus $x, y \ge 1, |x - y| < \epsilon \implies |g(x) - g(y)| < \epsilon$. This proves $g(x)$ is uniformly continuous on $[1, \infty)$. Combined with uniform continuity on $[0, 1]$, we find $g(x)$ is uniformly continuous on $[0, \infty)$.

(2) Let x be the side length of the squares cut. Then $0 \le x \le \dfrac{1}{2}$ and the volume is $V(x) = x(1 - 2x)^2$. By $V'(x) = 1 - 8x + 12x^2 = (1 - 2x)(1 - 6x)$, we get the possible local extremes $x = 0$, $x = \dfrac{1}{6}$ and $x = \dfrac{1}{2}$. The volumes at the three places are $V(0) = 0$, $V(\dfrac{1}{6}) = \dfrac{2}{27}$, $V(\dfrac{1}{2}) = 0$. Thus the biggest volume $\dfrac{2}{27}$ is obtained when $x = \dfrac{1}{6}$.

(3.1) $\dfrac{dx}{dt} = -3\cos^2 t \sin t$, $\dfrac{dy}{dt} = 3\sin^2 t \cos t$. $\dfrac{dy}{dx} = \dfrac{\frac{dy}{dt}}{\frac{dx}{dt}} = \dfrac{3\sin^2 t\cos t}{-3\cos^2 t \sin t} = -\tan t$. $\dfrac{d^2y}{dx^2} = \dfrac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} = \dfrac{-\sec^2 t}{-3\cos^2 t\sin t} = \dfrac{1}{3\cos^4 t \sin t}$. By $\left(\dfrac{1}{\cos^4 t \sin t}\right)' = \dfrac{4\cos^3 t \sin^2 t - \cos^5 t}{\cos^8 t \sin^2 t} = \dfrac{4\sin^2 t - \cos^2 t}{\cos^5 t \sin^2 t}$, we get $\dfrac{d^3 y}{dx^3} = \dfrac{\frac{d}{dt}\left(\frac{d^2y}{dx^2}\right)}{\frac{dx}{dt}} = \dfrac{\frac{4\sin^2 t - \cos^2 t}{3\cos^5 t \sin^2 t}}{-3\cos^2 t \sin t} = -\dfrac{4\sin^2 t - \cos^2 t}{9\cos^7 t \sin^3 t}$.

(3.2) $f(x) = \dfrac{2-(1-x)}{\sqrt{1-x}} = 2(1-x)^{-\frac{1}{2}} - (1-x)^{\frac{1}{2}}$.

$$\begin{aligned} f^{(n)}(x) &= 2\frac{1}{2}\frac{3}{2}\cdots\frac{2n-1}{2}(1-x)^{-\frac{2n+1}{2}} - \left(-\frac{1}{2}\right)\frac{1}{2}\cdots\frac{2n-3}{2}(1-x)^{-\frac{2n-1}{2}} \\ &= \frac{1\cdot 3\cdots(2n-1)}{2^n}(1-x)^{-\frac{2n+1}{2}}(3-x) = \frac{(2n)!}{2^n(2\cdot 4\cdots 2n)}(1-x)^{-\frac{2n+1}{2}}(3-x) \\ &= \frac{(2n)!}{2^{2n}n!}(1-x)^{-\frac{2n+1}{2}}(3-x). \end{aligned}$$

(3.3) $f'(x) = (1-x^2)^{\frac{-1}{2}}$. By (see the end of previous part)

$(1-x)^{-\frac{1}{2}} = 1 + \dfrac{1}{2}x + \cdots + \dfrac{1}{n!}\dfrac{1\cdot 3\cdots(2n-1)}{2^n}x^n + \cdots = 1 + \dfrac{1}{2}x + \cdots + \dfrac{(2n)!}{2^{2n}(n!)^2}x^n + \cdots$,

we get

$$f'(x) = (1-x^2)^{-\frac{1}{2}} = 1 + \frac{1}{2}x^2 + \cdots + \frac{(2n)!}{2^{2n}(n!)^2}x^{2n} + \cdots$$

Since $\dfrac{f^{(n+1)}(0)}{n!} = \dfrac{(f'(x))^{(n)}\big|_{x=0}}{n!}$ is the coefficient of x^n in the Taylor expansion above, we get $f^{(2k)}(0) = 0$ and $f^{(2k+1)}(0) = (2k)!\dfrac{(2k)!}{2^{2k}(k!)^2} = \left(\dfrac{(2k)!}{2^k k!}\right)^2$.

(4) $\dfrac{\log(1+ax+bx^2)}{x} = \dfrac{1}{x}\left((ax+bx^2) - \dfrac{(ax+bx^2)^2}{2} + \dfrac{(ax+bx^2)^3}{3} + o((ax+bx^2)^3)\right) = a + \left(b - \dfrac{a^2}{2}\right)x + \left(-ab + \dfrac{a^3}{3}\right)x^2 + o(x^2)$.

$$\begin{aligned} (1+ax+bx^2)^{\frac{1}{x}} &= e^{\frac{\log(1+ax+bx^2)}{x}} = e^a e^{\left(b-\frac{a^2}{2}\right)x + \left(-ab+\frac{a^3}{3}\right)x^2 + o(x^2)} \\ &= e^a\left(\left[\left(b-\frac{a^2}{2}\right)x + \left(-ab+\frac{a^3}{3}\right)x^2\right] + \frac{1}{2}\left[\left(b-\frac{a^2}{2}\right)x + o(x)\right]^2 + o(x^2)\right) \\ &= e^a\left(b-\frac{a^2}{2}\right)x + e^a\left(-ab+\frac{a^3}{3}+\frac{1}{2}\left(b-\frac{a^2}{2}\right)^2\right)x^2 + o(x^2) \end{aligned}$$

Then

$$\lim_{x\to 0}\frac{(1+2x+x^2)^{\frac{1}{x}}-(1+2x-x^2)^{\frac{1}{x}}}{x}=\lim_{x\to 0}\frac{(-e^2x+o(x))-(-3e^2x+o(x))}{x}=2e^2.$$

(5) The function is not defined at 0, and $\lim_{x\to 0} f(x)=+\infty$. Moreover, $\lim_{x\to+\infty} f(x) = +\infty$ and $\lim_{x\to-\infty} f(x)=0$. For $x>0$, we have

$$f'(x)=\left(-\frac{1}{2}x^{-\frac{3}{2}}+x^{-\frac{1}{2}}\right)e^x=\frac{(2x-1)e^x}{2x^{\frac{3}{2}}}$$
$$f''(x)=\left(\frac{3}{4}x^{-\frac{5}{2}}-\frac{1}{2}x^{-\frac{3}{2}}-\frac{1}{2}x^{-\frac{3}{2}}+x^{-\frac{1}{2}}\right)e^x=\frac{(4x^2-4x+3)e^x}{4x^{\frac{5}{2}}}$$

For $x<0$, we have

$$f'(x)=\left(\frac{1}{2}(-x)^{-\frac{3}{2}}+(-x)^{-\frac{1}{2}}\right)e^x=\frac{-(2x-1)e^x}{2(-x)^{\frac{3}{2}}}$$
$$f''(x)=\left(\frac{3}{4}(-x)^{-\frac{5}{2}}+\frac{1}{2}(-x)^{-\frac{3}{2}}+\frac{1}{2}(-x)^{-\frac{3}{2}}+(-x)^{-\frac{1}{2}}\right)e^x=\frac{(4x^2-4x+3)e^x}{4(-x)^{\frac{5}{2}}}$$

By the sign of $f'(x)$, we find $f(x)$ is increasing on $(-\infty,0)$, decreasing on $(0,\frac{1}{2})$, and increasing on $(\frac{1}{2},+\infty)$. Moreover, $\frac{1}{2}$ is a local minimum, with $f\left(\frac{1}{2}\right)=\sqrt{2e}$. Since $4x^2-4x+3=(2x+1)^2+2>0$, we always have $f''(x)>0$, so that the graph is always convex.

(6) T, F, F, T, F, T, T, F

Math 203 Final, Spring 2005

(1) (15 points) Suppose $f(x)$ is differentiable on a bounded interval $(a-\epsilon,b+\epsilon)$. Suppose $f(x)=0$ for infinitely many $x\in[a,b]$. Prove that there is $c\in[a,b]$, such that $f(c)=0$ and $f'(c)=0$.
(2) (15 points) Discuss the existence of the quadratic approximation and the existence of the second order derivative of the function $f(x)=|x^3(x-1)(x-2)^2|$ at $x_0=0,1,2$.
(3) (15 points) Suppose $f(x)$ is has second order derivative at 0. Suppose

$$\lim_{x\to 0}\left(1+x+\frac{f(x)}{x}\right)^{\frac{1}{x}}=e^\lambda.$$

1. Find $f(0)$, $f'(0)$ and $f''(0)$.
2. Find $\lim_{x\to 0}\left(1+\frac{f(x)}{x}\right)^{\frac{1}{x}}$.

(4) (20 points) By relating to the Riemann sum, use integration to compute the limits.

1. $\lim_{n\to\infty}\left(\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{2n}\right)$.

2. $\lim_{n\to\infty}\frac{1^\alpha+3^\alpha+\cdots+(2n+1)^\alpha}{n^{\alpha+1}}$, $\alpha>0$.

(5) (15 points) Suppose $f(x)$ is continuous on $[a,b]$ and differentiable on (a,b). Suppose $f(a)=0$, $1\ge f'(x)\ge 0$. Prove that $\left(\int_a^b f(x)dx\right)^2 \ge \int_a^b f(x)^3dx$.
(6) (20 points) Suppose $\phi(x)$ is continuous on $[a,b]$, differentiable on (a,b). Suppose $A<\phi'(x)<B$ for some constant $A,B>0$ and all $x\in(a,b)$.

1. Prove that for any $a\le x_1<x_2\le b$, we have $A(x_2-x_1)<\phi(x_2)-\phi(x_1)<B(x_2-x_1)$.

2. Prove that $\omega_{[x_1,x_2]}(f\circ\phi)=\omega_{[\phi(x_1),\phi(x_2)]}(f)$.

3. Prove that if $f(y)$ is integrable on $[\phi(a),\phi(b)]$, then $f(\phi(x))$ is integrable on $[a,b]$.

Answer to Math 203 Final, Autumn 2005

(1) Since $f(x)=0$ for infinitely many $x\in[a.b]$, we have $f(x)=0$ on a series in $[a,b]$ in which no two terms are the same. Since $[a,b]$ is bounded and closed, the series has a convergent subsequence. Denote the convergent subsequence by $\{x_n\}$ and denote $c=\lim_{n\to\infty}x_n$. Then we have $f(x_n)=0$ and by the continuity of $f(x)$, we also have $f(c)=\lim_{n\to\infty}f(x_n)=0$. Moreover, since $f(x)$ is differentiable at c, we have

$$f(c)=\lim_{x\to c}\frac{f(x)-f(c)}{x-c}=\lim_{n\to\infty}\frac{f(x_n)-f(c)}{x-c}=\lim_{n\to\infty}\frac{0-0}{x_n-c}=0.$$

(2) $|x^3|=\begin{cases}x^3 & \text{if } x<0\\ -x^3 & \text{if } x\ge 0\end{cases}$ has second order derivative at $x_0=0$. Near $x_0=0$, we have $|(x-1)(x-2)^2|=-(x-1)(x-2)^2$, which also has second order derivative at $x_0=0$. Thus $f(x)$ has second order derivative at $x_0=0$. Consequently, $f(x)$ has quadratic approximation at $x_0=0$.

$f(x)=-x^3(x-1)(x-2)^2$ near $x_0=2$. Therefore $f(x)$ has second order derivative and also quadratic approximation at $x_0=0$.

$f(x)=\begin{cases}-x^3(x-1)(x-2)^2 & \text{if } x<1\\ x^3(x-1)(x-2)^2 & \text{if } x\ge 1\end{cases}$ near $x_0=1$. Therefore $f'(1^-)=-1$ and $f'(1^+)=1$, so that $f(x)$ has no first order derivative at $x_0=1$. Equivalently, $f(x)$ also has no linear approximation at $x_0=1$. Consequently, $f(x)$ has neither second order derivative, nor quadratic approximation at $x_0=1$.

(3) If $f(0)\ne 0$, then $\lim_{x\to0}\left(1+x+\frac{f(x)}{x}\right)=\infty$ and $\lim_{x\to0}\left(1+x+\frac{f(x)}{x}\right)^{\frac{1}{x}}$ will diverge. Therefore $f(0)=0$. Since $f(x)$ has second order derivative at 0, we have

$f(x) = ax + bx^2 + o(x^2)$, where $a = f'(0)$, $b = \dfrac{f''(0)}{2}$. Then the limit of

$$\frac{1}{x}\log\left(1 + x + \frac{f(x)}{x}\right) = \frac{\log(1 + x + a + bx + o(x))}{x}$$

should be λ. In particular, $\lim_{x\to 0}\log(1+x+a+bx+o(x)) = \log(1+a) = 0$. Thus $a = 0$. Then

$$\lambda = \lim_{x\to 0}\frac{\log(1 + x + bx + o(x))}{x} = 1 + b.$$

Therefore $b = \lambda - 1$. We conclude that

$$f(0) = 0, \quad f'(0) = 0, \quad f''(0) = 2\lambda - 2.$$

Moreover, w ehave

$$\lim_{x\to 0}\frac{1}{x}\log\left(1 + \frac{f(x)}{x}\right) = \lim_{x\to 0}\frac{\log(1 + bx + o(x))}{x} = b = \lambda - 1,$$

so that $\lim_{x\to 0}\left(1 + \dfrac{f(x)}{x}\right)^{\frac{1}{x}} = e^{\lambda - 1}$.

(4.1) The function $\dfrac{1}{x}$ is integrable on $[1, 2]$. Consider the partition $P\colon 1 < \dfrac{n+1}{n} < \dfrac{n+2}{n} < \cdots < \dfrac{2n}{n} = 2$ and the right ends $\dfrac{n+1}{n}, \dfrac{n+2}{n}, \cdots, \dfrac{2n}{n}$ as x_i^*. By $x_i - x_{i-1} = \dfrac{n+i}{n} - \dfrac{n+i-1}{n} = \dfrac{1}{n}$, we have the Riemann sum

$$S\left(P, \frac{1}{x}\right) = \frac{1}{\frac{n+1}{n}}\frac{1}{n} + \frac{1}{\frac{n+2}{n}}\frac{1}{n} + \cdots + \frac{1}{\frac{2n}{n}}\frac{1}{n} = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n}.$$

Thus

$$\lim_{n\to\infty}\left(\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n}\right) = \lim_{\|P\|\to 0} S\left(P, \frac{1}{x}\right) = \int_1^2 \frac{1}{x}dx = \log 2.$$

(4.2) Consider the integrable function x^α on $[0, 2]$ and the partition $P\colon 0 < \dfrac{2}{n} < \dfrac{4}{n} < \cdots < \dfrac{2n}{n} = 2$ and the middle points $\dfrac{1}{n}, \dfrac{3}{n}, \cdots, \dfrac{2n-1}{n}$ as x_i^*. By $x_i - x_{i-1} = \dfrac{2}{n}$, we have the Riemann sum

$$S(P, x^\alpha) = \left(\frac{1}{n}\right)^\alpha\frac{2}{n} + \left(\frac{3}{n}\right)^\alpha\frac{2}{n} + \cdots + \left(\frac{2n-1}{n}\right)^\alpha\frac{2}{n} = 2\frac{1^\alpha + 3^\alpha + \cdots + (2n-1)^\alpha}{n^{\alpha+1}}.$$

Thus

$$\lim_{n\to\infty}\frac{1^\alpha + 3^\alpha + \cdots + (2n-1)^\alpha}{n^{\alpha+1}} = \frac{1}{2}\lim_{\|P\|\to 0} S(P, x^\alpha) = \int_0^2 x^\alpha dx = \frac{2^{\alpha+1} - 1}{2(\alpha+1)}.$$

Then by

$$\lim_{n\to\infty}\frac{(2n+1)^\alpha}{n^{\alpha+1}}=\lim_{n\to\infty}\frac{1}{n}\lim_{n\to\infty}\left(2+\frac{1}{n}\right)^\alpha=0\cdot 2^\alpha=0.$$

We get

$$\lim_{n\to\infty}\frac{1^\alpha+3^\alpha+\cdots+(2n+1)^\alpha}{n^{\alpha+1}}=\lim_{n\to\infty}\frac{1^\alpha+3^\alpha+\cdots+(2n-1)^\alpha}{n^{\alpha+1}}=\frac{2^{\alpha+1}-1}{2(\alpha+1)}.$$

(5) Consider $g(x)=\left(\int_a^x f(t)dt\right)^2-\int_a^x f(t)^3dt$. We will prove $g(x)$ is increasing for $x\ge a$. This will imply that $g(b)=\left(\int_a^b f(x)dx\right)^2-\int_a^b f(x)^3dx\ge g(a)=0$.

To see $g(x)$ is increasing, we take the derivative:

$$g'(x)=2f(x)\int_a^x f(t)dt-f(x)^3=f(x)\left(2\int_a^x f(t)dt-f(x)^2\right).$$

By $f'(x)\ge 0$, we have $f(x)$ increasing and $f(x)\ge f(a)=0$ for $x\ge a$. Thus to show $g(x)\ge 0$, it suffices to show

$$h(x)=2\int_a^x f(t)dt-f(x)^2>0.$$

Taking the derivative of $h(x)$, we get

$$h'(x)=2f(x)-2f'(x)f(x)=2f(x)(1-f'(x)).$$

We have shown $f(x)\ge 0$. Moreover, it was assumed that $f'(x)\le 1$. Therefore $h'(x)\ge 0$ and $h(x)$ is increasing. Therefore for $x\ge a$, we have $h(x)\ge h(a)=0$. This completes the proof.

(6) For $x_1<x_2$, by the mean value theorem, we have $\phi(x_2)-\phi(x_1)=\phi'(c)(x_2-x_1)$. Since $A<\phi'(c)<B$, we have $A(x_2-x_1)<\phi(x_2)-\phi(x_1)<B(x_2-x_1)$.

In particular, ϕ is strictly increasing and $y=\phi(x)$ is a change of variable between $x_1\le x\le x_2$ and $\phi(x_1)\le y\le\phi(x_2)$. Thus

$$\sup_{x_1\le x\le x_2}f(\phi(x))=\sup_{\phi(x_1)\le y\le\phi(x_2)}f(y),\quad \inf_{x_1\le x\le x_2}f(\phi(x))=\inf_{\phi(x_1)\le y\le\phi(x_2)}f(y).$$

As the difference between the supremum and infimum, the oscillations are equal:

$$\omega_{[x_1,x_2]}(f\circ\phi)=\omega_{[\phi(x_1),\phi(x_2)]}(f).$$

Let P be a partition of $[a,b]$. We have $\omega_{[x_{i-1},x_i]}(f\circ\phi)=\omega_{[\phi(x_{i-1}),\phi(x_i)]}(f)$ from the second part. Then

$$\sum\omega_{[x_{i-1},x_i]}(f\circ\phi)(x_i-x_{i-1})\le\frac{1}{A}\sum\omega_{[\phi(x_{i-1}),\phi(x_i)]}(f)(\phi(x_i)-\phi(x_{i-1}))$$

Now $\phi(P)$ is a partition of $[\phi(a), \phi(b)]$. By $\phi(x_2) - \phi(x_1) < B(x_2 - x_1)$, we also know $\|P\| < B\|P\|$.

Since $f(y)$ is integrable on $[\phi(a), \phi(b)]$, for any $\epsilon > 0$, there is $\delta > 0$, such that $\|\phi(P)\| < \delta$ implies $\sum \omega_{[\phi(x_{i-1}),\phi(x_i)]}(f)(\phi(x_i) - \phi(x_{i-1})) \leq A\epsilon$. Thus

$$\|P\| < \frac{\delta}{B} \implies \|\phi(P)\| < B\|P\| < \delta \implies \sum \omega_{[x_{i-1},x_i]}(f \circ \phi)(x_i - x_{i-1}) < \frac{1}{A}A\epsilon = \epsilon.$$

This completes the proof of the integrability of $f(\phi(x))$.

Math 203 Final, Autumn 2006

(1) Suppose $f(x)$ is continuous on an open interval containing $[a, b]$. Prove that

$$\lim_{h\to 0} \int_a^b \frac{f(x+h) - f(x)}{h} dx = f(b) - f(a).$$

Can you weaken the continuity condition to merely integrable?

(2) Suppose $f(x)$ is integrable on $[a, b]$. For any partition P of $[a, b]$ and choices x_i^*, define the "Riemann product"

$$\Pi(P, f) = (1 + f(x_1^*)\Delta x_1)(1 + f(x_2^*)\Delta x_2) \cdots (1 + f(x_n^*)\Delta x_n).$$

Prove that $\lim_{\|P\|\to 0} \Pi(P, f) = e^{\int_a^b f(x)dx}$.

(3) Prove that $\displaystyle\int_0^1 x^{tx} dx = \sum_{n=1}^{\infty} \frac{(-t)^{n-1}}{n^n}$. Justify each step.

[Hint: expand $x^{tx} = e^{tx \log x}$.]

(4) Suppose $a_n > 0$ is increasing. Prove that $\sum \frac{1}{a_n}$ converges if and only if $\sum \frac{n}{a_1 + a_2 + \cdots + a_n}$ converges.

(5) Suppose $a_n > 0$ and $r = \overline{\lim}_{n\to\infty} \frac{\log a_n}{\log n}$. Consider the series $f(x) = \sum_{n=1}^{\infty} \frac{a_n}{n^x}$.

1. Prove that for any $r' > r$, we have $a_n < n^{r'}$ for sufficiently big n.
2. Prove that for any $R > r + 1$, the series uniformly converges on $[R, +\infty)$.
3. Prove that $f(x)$ has derivatives of any order on $(r + 1, +\infty)$.
4. Prove that the series diverges on $(-\infty, r)$. On the other hand, show that the series may converge for some $x < r + 1$ by constructing an example.

(6) True or false. No explanation needed.

1. If $f(x)$ and $g(x)$ are integrable on $[a, b]$, then $f(x)^{g(x)}$ is integrable on $[a, b]$.
2. If $f(x)$ is integrable on $[a, b]$ and $g(x)$ is integrable on $[b, c]$, then $\begin{cases} f(x) & \text{on } [a, b] \\ g(x) & \text{on } (b, c] \end{cases}$ is integrable on $[a, c]$.

3. If $f(x) + g(x)$ is integrable on $[a, b]$, then $f(x)$ and $g(x)$ are integrable on $[a, b]$.
4. If $f(x)g(x)$ is integrable on $[a, b]$ and $0 < c_1 \le f(x), g(x) \le c_2$ for some constants c_1 and c_2, then $f(x)$ and $g(x)$ are integrable on $[a, b]$.
5. If $\sum(a_n + b_n)$ converges, then $\sum a_n$ and $\sum b_n$ converge.
6. If $\sum a_n$ and $\sum b_n$ absolutely converge, then $\sum(a_n + b_n)$ absolutely converges.
7. If $\sum a_n$ and $\sum b_n$ conditionally converge, then $\sum(a_n + b_n)$ conditionally converges.
8. If $\sum a_n$ and $\sum b_n$ diverge, then $\sum(a_n + b_n)$ diverges.
9. If $f_n(x) + g_n(x)$ does not uniformly converge, then $f_n(x)$ and $g_n(x)$ do not uniformly converge.
10. If $f_n(x)$ and $g_n(x)$ uniformly converge, then $f_n(x) + g_n(x)$ uniformly converges.
11. If $f_n(x)$ uniformly converges, then any subsequence $f_{n_k}(x)$ uniformly converges.
12. If the even subsequence $f_{2n}(x)$ and the odd subsequence $f_{2n+1}(x)$ uniformly converge, then $f_n(x)$ uniformly converge.
13. If $\sum |u_n(x)|$ uniformly converges, then $\sum u_n(x)$ uniformly converges.
14. If $\sum u_n(x)$ and $\sum v_n(x)$ uniformly converge, then $\sum u_n(x)v_n(x)$ uniformly converge.
15. If $u_n(x) \ge 0$ and $\sum u_n(x)$ uniformly converge, then $\sum u_n(x)^2$ uniformly converges.

Answer to Math 203 Final, Autumn 2006

(1) We have

$$\begin{aligned}\int_a^b \frac{f(x+h) - f(x)}{h} dx &= \frac{1}{h}\left(\int_a^b f(x+h)dx - \int_a^b f(x)dx\right) \\ &= \frac{1}{h}\left(\int_{a+h}^{b+h} f(x)dx - \int_a^b f(x)dx\right) \\ &= \frac{1}{h}\int_b^{b+h} f(x)dx - \frac{1}{h}\int_a^{a+h} f(x)dx.\end{aligned}$$

If $f(x)$ is continuous at a and b (only integrability is needed at other places), then by the fundamental theorem of calculus, we have

$$\lim_{h\to 0}\int_a^b \frac{f(x+h) - f(x)}{h} dx = \lim_{h\to 0}\frac{1}{h}\int_b^{b+h} f(x)dx - \lim_{h\to 0}\frac{1}{h}\int_a^{a+h} f(x)dx = f(b) - f(a).$$

If the continuity is not assumed, then it is possible that the equality fails. Any counterexample to the fundamental theorem can becomes a counterexample of the equality.

(2) Since $f(x)$ is integrable, it is bounded. Therefore for sufficiently small $\|P\|$, we have $|f(x_i^*)\Delta x_i| < \dfrac{1}{2}$. Then by the second order Taylor expansion of $\log(1+x)$,

$$\log(1+f(x_i^*)\Delta x_i) = f(x_i^*)\Delta x_i - \frac{1}{2(1+c_i)^2}(f(x_i^*)\Delta x_i)^2,$$

where $|c_i| \le |f(x_i^*)\Delta x_i| < \dfrac{1}{2}$. Thus

$$0 \le \sum \frac{1}{2(1+c_i)^2}(f(x_i^*)\Delta x_i)^2 \le 2\left(\sum f(x_i^*)^2\Delta x_i\right)\|P\|.$$

Since $f(x)$ is integrable, $f(x)^2$ is also integrable. Therefore $\lim_{\|P\|\to 0}\sum f(x_i^*)^2\Delta x_i$ converges. This implies that $\sum f(x_i^*)^2\Delta x_i$ is bounded as $\|P\|\to 0$ and

$$\lim_{\|P\|\to 0}\sum \frac{1}{2(1+c_i)^2}(f(x_i^*)\Delta x_i)^2 = 0.$$

This further implies

$$\lim_{\|P\|\to 0}\log\Pi(P,f) = \lim_{\|P\|\to 0}\sum\log(1+f(x_i^*)\Delta x_i) = \lim_{\|P\|\to 0}\sum f(x_i^*)\Delta x_i = \int_a^b f(x)dx.$$

By the continuity of the exponential function, we get $\lim_{\|P\|\to 0}\Pi(P,f) = e^{\lim_{\|P\|\to 0}\log\Pi(P,f)} = e^{\int_a^b f(x)dx}$.

(3) By the Taylor expansion of e^x, we get

$$x^{tx} = e^{tx\log x} = \sum_{n=0}^{\infty}\frac{t^n x^n(\log x)^n}{n!}.$$

Since $x\log x$ is continuous on $[0,1]$, it is bounded on the interval (in fact, $\max_{[0,1]}|x\log x| = e^{-1}$). Therefore $|t^n x^n(\log x)^n| \le |t|^n B^n$ for some B and all $x\in[0,1]$. Then the convergence of $\sum\dfrac{|t|^n B^n}{n!}$ implies that the series for x^{tx} uniformly converges on $[0,1]$. Thus

$$\int_0^1 x^{tx}dx = \sum_{n=0}^{\infty}\frac{t^n}{n!}\int_0^1 x^n(\log x)^n dx.$$

The integral $\displaystyle\int_0^1 x^n(\log x)^n dx$ may be computed by the recursive formula in Exercise 3.2.23, p152. Specifically,

$$\begin{aligned}\int_0^1 x^m(\log x)^n dx &= \frac{1}{m+1}\int_0^1(\log x)^n dx^{m+1} = x^m(\log x)^n|_{x=0}^{x=1} - \frac{n}{m+1}\int_0^1 x^{m+1}(\log x)^{n-1}\frac{1}{x}dx\\ &= -\frac{n}{m+1}\int_0^1 x^m(\log x)^{n-1}dx = \cdots = (-1)^n\frac{n(n-1)\cdots 1}{(m+1)^n}\int_0^1 x^m dx = (-1)^n\frac{n!}{(m+1)^{n+1}}.\end{aligned}$$

Thus

$$\int_0^1 x^{tx}dx = \sum_{n=0}^{\infty} \frac{t^n}{n!}(-1)^n \frac{n!}{(n+1)^{n+1}} = \sum_{n=0}^{\infty} \frac{(-t)^n}{(n+1)^{n+1}} = \sum_{n=1}^{\infty} \frac{(-t)^{n-1}}{n^n}.$$

(4) Since $a_n > 0$ is increasing, we have

$$\frac{n}{a_1 + a_2 + \cdots + a_n} \geq \frac{n}{a_n + a_n + \cdots + a_n} = \frac{1}{a_n} > 0.$$

By the comparison test, the convergence of $\sum \dfrac{n}{a_1 + a_2 + \cdots + a_n}$ implies the convergence of $\sum \dfrac{1}{a_n}$. On the other hand, we also have

$$0 < \frac{2n}{a_1 + a_2 + \cdots + a_{2n}} \leq \frac{2n}{a_1 + a_2 + \cdots + a_n} \leq \frac{2n}{a_n + a_n + \cdots + a_n} = \frac{2}{a_n},$$

and

$$0 < \frac{2n-1}{a_1 + a_2 + \cdots + a_{2n-1}} \leq \frac{2n-1}{a_1 + a_2 + \cdots + a_n} \leq \frac{2n-1}{a_n + a_n + \cdots + a_n} = \frac{2n-1}{na_n} < \frac{2}{a_n}.$$

The convergence of $\sum \dfrac{1}{a_n}$ implies the convergence of $\sum \left(\dfrac{2}{a_n} + \dfrac{2}{a_n}\right)$, which by the comparison test further implies the convergence of $\sum \dfrac{n}{a_1 + a_2 + \cdots + a_n}$.

(5) If $r' > r = \overline{\lim}_{n\to\infty} \dfrac{\log a_n}{\log n}$, we have $r' > \dfrac{\log a_n}{\log n}$ for sufficiently big n. This means $\log n^{r'} = r' \log n > \log a_n$, or equivalently $n^{r'} > a_n$.

Suppose $R > r + 1$. We find R' satisfying $R > R' > r + 1$. By $R' - 1 > r$, there is N, such that $a_n < n^{R'-1}$ for $n > N$. Then for $x \geq R$ and $n > N$, we have $\dfrac{a_n}{n^x} \leq \dfrac{a_n}{n^R} < \dfrac{n^{R'-1}}{n^R} = \dfrac{1}{n^{R-R'+1}}$. By $R - R' + 1 > 1$, the series $\sum \dfrac{1}{n^{R-R'+1}}$ converges. By the comparison test, we conclude that $\sum \dfrac{a_n}{n^x}$ uniformly converges on $[R, +\infty)$.

The derivative series is $-\sum \dfrac{a_n \log n}{n^x}$ and the k-th order derivative series is $(-1)^k \sum \dfrac{a_n(\log n)^k}{n^x}$. Since

$$\overline{\lim_{n\to\infty}} \frac{\log(a_n(\log n)^k)}{\log n} = \overline{\lim_{n\to\infty}} \frac{\log a_n + k \log\log n}{\log n} = \overline{\lim_{n\to\infty}} \frac{\log a_n}{\log n} = r,$$

all the derivative series uniformly converge on $[R, +\infty)$ for any $R > r$. Thus $f(x)$ has derivative of any order on $(R, +\infty)$ for any $R > r$. This means $f(x)$ has derivative of any order on $(r + 1, +\infty)$.

Suppose $x < r$. Then there are infinitely many n satisfying $x < \dfrac{\log a_n}{\log n}$. This means $\log n^x = x \log n < \log a_n$, or equivalently $n^x < a_n$. Therefore we have $\dfrac{a_n}{n^x} > 1$ for infinitely many n. In particular, the sequence $\dfrac{a_n}{n^x}$ does not converge to zero, so that the series diverges.

The series may converge for some $x < r+1$. For example, take $a_n = 1$ when $n = k^2$, $k \in \mathbb{N}$ and $a_n = 0$ when n is not a square. Then $r = \overline{\lim}_{n\to\infty} \dfrac{\log a_n}{\log n} = \lim_{k\to\infty} \dfrac{\log 1}{\log k^2} = 0$. However, if $r + 1 = 1 > x > \dfrac{1}{2}$, then $\sum \dfrac{a_n}{n^x} = \sum \dfrac{1}{k^{2x}}$ converges.
(6)

T, T, F, F, F; T, F, F, F, T; T, F, T, F, F

Math 203 Final, Autumn 2007

(1) Suppose $f(x)$ is continuous on an open interval containing $[a, b]$. Prove that the limit $\lim_{n\to\infty} f\left(x + \dfrac{1}{n}\right) = f(x)$ is uniform.
(2) Suppose $f(x)$ is continuous and positive on $[0, 1]$. Prove that

$$\lim_{n\to 0} \sqrt[n]{f\left(\frac{1}{n}\right) f\left(\frac{2}{n}\right) \cdots f\left(\frac{n-1}{n}\right)}$$

converges and find the limit.
(3) Let $[x]$ be the biggest integer $\le x$. Let $a > 0$. Determine the convergence of the improper integral $\int_0^1 \left(\left[\dfrac{a}{x}\right] - a\left[\dfrac{1}{x}\right]\right) dx$.
(4) Determine the intervals on which the series $\sum n^x x^n$ uniformly converge.
(5) Suppose $\sum \dfrac{1}{a_n}$ absolutely converges. Prove that $\sum \dfrac{1}{x - a_n}$ converges to a function that has derivatives of any order away from all a_n.
(6) True or false. No explanation needed.

1. If $\sum a_n$ and $\sum b_n$ converge, then $\sum a_n b_n$ converges.
2. If $\sum a_n$ and $\sum b_n$ absolutely converge, then $\sum a_n b_n$ absolutely converges.
3. If $\sum a_n$ and $\sum b_n$ conditionally converge, then $\sum a_n b_n$ conditionally converges.
4. If both $\sum a_n$ and one rearrangement $\sum a_{n_k}$ converge but have different sum, then $\sum a_n$ conditionally converge.
5. If all rearrangements of $\sum a_n$ converge, then $\sum a_n$ absolutely converges.
6. If $\sum a_n$ converges, then $\sum \max\{a_{2n}, a_{2n+1}\}$ and $\sum \min\{a_{2n}, a_{2n+1}\}$ converge.
7. If $\sum \max\{a_{2n}, a_{2n+1}\}$ and $\sum \min\{a_{2n}, a_{2n+1}\}$ converge, then $\sum a_n$ converges.
8. If $\sum u_n(x)$ uniformly converges on X and $g(Y) \subset X$, then $\sum u_n(g(y))$ uniformly converges on Y.
9. If $\sum u_n(x)$ uniformly converges on X and g is uniformly continuous on Y, then $\sum g(u_n(x))$ uniformly converges on X.

10. If $f_n(x)$ and $g_n(x)$ uniformly converge to $f(x)$ and $g(x)$, and $f_n(x) > 0$, $f(x) > 0$, then $f_n(x)^{g_n(x)}$ uniformly converges to $f(x)^{g(x)}$.

Answer to Math 203 Final, Autumn 2007

(1) By assumption, $f(x)$ is continuous on $[a-\mu, b+\mu]$ for some $\mu > 0$. Then $f(x)$ is uniformly continuous on $[a-\mu, b+\mu]$. For any $\epsilon > 0$, there is $\delta > 0$, such that

$$x, y \in [a-\mu, b+\mu], |x-y| < \delta \implies |f(x) - f(y)| < \epsilon.$$

Then

$$x \in [a,b], n > \frac{1}{\mu} \implies x + \frac{1}{n}, x \in [a-\mu, b+\mu], \frac{1}{n} < \mu \implies \left| f\left(x + \frac{1}{n}\right) - f(x) \right| < \epsilon.$$

This shows that the limit $\lim_{n\to\infty} f\left(x + \frac{1}{n}\right) = f(x)$ is uniform.

(2) Let $a_n = \sqrt[n]{f\left(\frac{1}{n}\right) f\left(\frac{2}{n}\right) \cdots f\left(\frac{n-1}{n}\right)}$. Then

$$\log a_n = \frac{1}{n}\left(\log f\left(\frac{1}{n}\right) + \log f\left(\frac{2}{n}\right) + \cdots + \log f\left(\frac{n-1}{n}\right)\right) = S(P_n, f) - \frac{1}{n} f(0),$$

where $S(P_n, f)$ is the Riemann sum with respect to the special partition

$$P_n \colon 0 < \frac{1}{n} < \frac{2}{n} < \cdots < \frac{1}{n} < \frac{n-1}{n} < 1$$

of $[0,1]$ and $x_i^* = \dfrac{i-1}{n}$. The function $\log f$ is still continuous on the bounded interval $[0,1]$ and is therefore Riemann integrable. Therefore we have $\lim_{n\to\infty} S(P_n, f) = \int_0^1 \log f(x)dx$, so that

$$\lim_{n\to\infty} \log a_n = \lim_{n\to\infty} S(P_n, f) = \int_0^1 \log f(x) dx.$$

Finally, by the continuity of the exponential function, we get

$$\lim_{n\to\infty} a_n = e^{\int_0^1 \log f(x)dx}.$$

(3) The improperness only appears at 0^+, where the function is unbounded.

Let $f(x) = \left[\frac{1}{x}\right]$. Then

$$\int_c^1 \left(\left[\frac{a}{x}\right] - a\left[\frac{1}{x}\right]\right) dx = \int_c^1 \left(f\left(\frac{x}{a}\right) - af(x)\right) dx$$
$$= a\left(\int_{a^{-1}c}^{a^{-1}} - \int_c^1\right) f(x)dx = a\left(\int_1^{a^{-1}} - \int_c^{a^{-1}c}\right) f(x)dx.$$

Thus the problem is reduced to the convergence of $\lim_{c\to 0^+}\displaystyle\int_c^{a^{-1}c} f(x)dx$.

Let $t = \dfrac{1}{x}$ and $R = c^{-1}$. Then $-\displaystyle\int_c^{a^{-1}c} f(x)dx = \int_R^{aR} \frac{[t]}{t^2}dt$. Assume $a \geq 1$. Then by $x - 1 \leq [x] \leq x$, we get

$$\log a = \int_R^{aR} \frac{t-1}{t^2}dt \leq -\int_c^{a^{-1}c} f(x)dx \leq \int_R^{aR} \frac{t}{t^2}dt = \log a + \frac{1}{aR} - \frac{1}{R}.$$

Since both sides have limit $\log a$ as $R \to +\infty$, we conclude that $\lim_{c\to 0^+} -\displaystyle\int_c^{a^{-1}c} f(x)dx = \log a$. The case for $0 < a \leq 1$ is similar.

Thus we conclude that $\displaystyle\int_0^1 \left(\left[\frac{a}{x}\right] - a\left[\frac{1}{x}\right]\right) dx$ converges and has value $a\displaystyle\int_1^{a^{-1}} \left[\frac{1}{x}\right] dx + a\log a$.

(4) We have $\lim_{n\to\infty} \sqrt[n]{|n^x x^n|} = |x|$. By the root test, the series diverges for $|x| > 1$ and converges for $|x| < 1$. Moreover, for $x = 1$, the series $\sum n^1 1^n = \sum n$ diverges, and for $x = -1$, the series $\sum n^{-1}(-1)^n = \sum \dfrac{(-1)^n}{n}$ converges.

For any $|R| < 1$, we have

$$|x| \leq R \implies |n^x x^n| \leq nR^n.$$

The convergence of $\sum nR^n$ implies that $\sum n^x x^n$ uniformly converges on $[-R, R]$ for any $0 < R < 1$.

Now consider an interval containing -1. Studying $\sum n^x x^n$ for $x \in \left[-1, -\dfrac{1}{2}\right]$ is the same as studying $\sum (-1)^n \dfrac{t^n}{n^t}$ for $t \in \left[\dfrac{1}{2}, 1\right]$. For each fixed t in the interval, the series is alternating and $\dfrac{t^n}{n^t}$ is decreasing in n. Therefore by the Leibniz test, the series converges to a function $f(t)$ and satisfies

$$t \in \left[\frac{1}{2}, 1\right] \implies \left|\sum_{n\leq k} (-1)^n \frac{t^n}{n^t} - f(t)\right| \leq \frac{t^{k+1}}{(k+1)^t} \leq \frac{1}{\sqrt{k+1}}.$$

This implies that the series $\sum (-1)^n \dfrac{t^n}{n^t}$ uniformly converges to $f(t)$ on the interval.

Finally, for each fixed n, we have $\lim_{x\to 1^-} n^x x^n = n > 1$. This implies that the series $\sum n^x x^n$ does not uniformly converge on intervals of the form $[a, 1]$.

In conclusion, the series $\sum n^x x^n$ uniformly converges on $[-1, R]$ for any $R < 1$.

(5) Since $\sum \dfrac{1}{a_n}$ absolutely converges, we get $\lim_{n\to\infty} a_n = \infty$. Thus for any $R > 0$, there is N, such that $|a_n| > 2R$ for $n > N$. Then

$$|x| \leq R, n > N \implies |x| \leq \frac{|a_n|}{2} \implies |x - a_n| \geq \frac{|a_n|}{2}. \implies \left|\frac{1}{x - a_n}\right| \leq \frac{2}{|a_n|}.$$

The absolute convergence of $\sum \frac{1}{a_n}$ then implies that the series $\sum \frac{1}{x-a_n}$ uniformly converges on $[-R,R]-\{a_1,a_2,\dots\}$ for any R.

By taking k-th order derivative term by term, we get series $\sum \frac{(-1)^k k!}{(x-a_n)^{k+1}}$. By the same argument as above, we have

$$|x|\le R, n>N \implies \left|\frac{(-1)^k k!}{(x-a_n)^{k+1}}\right| \le \frac{2^{k+1}k!}{|a_n|^{k+1}}.$$

By $\lim_{n\to\infty} a_n = \infty$, the absolute convergence of $\sum \frac{1}{a_n}$ implies the absolute convergence of $\sum \frac{1}{a_n^{k+1}}$. Then the estimation above implies that the term by term derivative series also uniformly converges on $[-R,R]-\{a_1,a_2,\dots\}$ for any R. As a result, we have

$$\frac{d^k}{dx^k}\sum \frac{1}{x-a_n} = \sum \frac{(-1)^k k!}{(x-a_n)^{k+1}}.$$

(6)
F, T, F, T, T; F, T, T, F, F

Math 203 Final, Autumn 2008

(1) Prove that the integral $\int_0^\pi \frac{dx}{\sqrt[4]{1+\cos x}}$ converges.

(2) Prove that if $\sum_{n=0}^\infty a_n x^n$ is uniformly convergent in (a,b), then it is uniformly convergent in $[a,b]$.

(3) Let $f(x)=\sum_{n=1}^\infty \frac{x^n}{n^2}$.
(1) Show that f is continuous on $[-1,1]$.
(2) Show that $f'_+(-1)$ exists.

(4) Assume that
(1) f' is integrable on $[0,A]$ for any $A>0$;
(2) $f(x)$ is decreasing to 0 as $x\to+\infty$;
(3) $\int_0^{+\infty} f(x)\,dx$ converges.
Prove that $\int_0^{+\infty} x\,f'(x)\,dx$ converges.

(5)
(a) Suppose $f(x)$ is continuous on $[a,b]$ with $0\le f(x)<1$. Prove that the limit

$$\lim_{n\to\infty}\int_a^b [f(x)]^n\,dx = 0.$$

(b) Suppose $f(x)$ is continuous on (a,b) with $0 \le f(x) < 1$ and integrable on $[a,b]$. Prove that the limit

$$\lim_{n\to\infty} \int_a^b [f(x)]^n \, dx = 0.$$

(6) Find $\lim_{x\to 0} \frac{1}{x} \left(\sum_{n=1}^{\infty} (-1)^n \frac{1}{(n+x^2)} \right)'$.

Answer to Math 203 Final, Autumn 2008

(1) It is obvious that π is the only singular point for the integral. Since

$$\lim_{x\to\pi} \frac{1+\cos x}{(\pi-x)^2} = \lim_{x\to\pi} \frac{-\sin x}{2(-1)(\pi-x)} = \lim_{x\to\pi} \frac{-\cos x}{2} = 1/2,$$

we know that near $x=\pi$, $1+\cos x = O((\pi-x)^2)$. Thus, near $x=\pi$,

$$\frac{1}{\sqrt[4]{1+\cos x}} = O(\sqrt{\pi-x}).$$

Since

$$\int_{\pi-\delta_1}^{\pi-\delta_2} \frac{dx}{\sqrt{\pi-x}} = -2(\pi-x)^{1/2}\Big|_{x=\pi-\delta_1}^{x=\pi-\delta_2}$$

can be sufficiently small if δ_1, δ_2 are sufficiently small. This implies that

$$\int_{\pi-\delta}^{\pi} \frac{dx}{\sqrt{\pi-x}}$$

is convergent for small $\delta > 0$. By the Comparison Test, we know that that

$$\int_{\pi-\delta}^{\pi} \frac{dx}{\sqrt[4]{1+\cos x}}$$

is convergent. Since $1+\cos x > 0$ on $[0, \pi-\delta]$, the Riemann integral

$$\int_0^{\pi-\delta} \frac{dx}{\sqrt[4]{1+\cos x}}$$

exists for small $\delta > 0$. Hence

$$\int_0^{\pi} \frac{dx}{\sqrt[4]{1+\cos x}} = \int_0^{\pi-\delta} \frac{dx}{\sqrt[4]{1+\cos x}} + \int_{\pi-\delta}^{\pi} \frac{dx}{\sqrt[4]{1+\cos x}}$$

converges.

(2) Since the series is uniformly convergent in (a, b), so for any $\epsilon > 0$, there exists $N > 0$, such that

$$|a_m x^m + a_{m+1} x^{m+1} + \cdots + a_n x^n| < \epsilon$$

for $m, n \geq N$ and $x \in (a, b)$. Let $x \to a^+$, we have

$$|a_m a^m + a_{m+1} a^{m+1} + \cdots + a_n a^n| \leq \epsilon$$

for $m, n \geq N$. Similarly, we have

$$|a_m b^m + a_{m+1} b^{m+1} + \cdots + a_n b^n| \leq \epsilon$$

for $m, n \geq N$. Hence, we know that if $m, n \geq N$, then

$$|a_m x^m + a_{m+1} x^{m+1} + \cdots + a_n x^n| \leq \epsilon$$

for $x \in [a, b]$. Hence, by the Cauchy criterion, we know that the power series is uniformly convergent in $[a, b]$.
(3) (1) if $|x| \leq 1$, then

$$\left| \frac{x^n}{n^2} \right| \leq \frac{1}{n^2}.$$

Since $\sum 1/n^2$ is convergent, so $\sum \frac{x^n}{n^2}$ is uniformly convergent. By Proposition 4.2.5, f is continuous on $[-1, 1]$.

(2) For the series

$$\sum \left(\frac{x^n}{n^2} \right)' = \sum \frac{x^{n-1}}{n},$$

we know that it is convergent at $x = -1$. Since the radius of convergence for $\sum \frac{x^{n-1}}{n}$ is 1, by Proposition 4.2.10, we know that the series $\sum \frac{x^{n-1}}{n}$ is uniformly convergent on $[-1, 0]$. By Proposition 4.2.7, we can conclude that $f'_+(-1)$ exists.
(4) For any fixed $A > 0$, since

$$\begin{aligned} \int_0^A x\, f'(x)\, dx &= x f(x) \Big|_{x=0}^{x=A} - \int_0^A f(x)\, dx \\ &= A f(A) - \int_0^A f(x)\, dx, \end{aligned}$$

we only need to show that $\lim_{A \to +\infty} A f(A) = 0$.

Since $f(x)$ is decreasing to 0 as $x \to +\infty$, for sufficiently large x, we must have $f(x) \geq 0$. By the condition that $\int_0^{+\infty} f(x)\, dx$ converges, for any $\epsilon > 0$, there exists $M > 0$, such that if $x \geq M$, we have

$$0 \leq \int_x^{2x} f(x)\, dx < \epsilon.$$

Thus, if $A \geq M$, we have $0 \leq Af(A) \leq \int_A^{2A} f(x)\,dx < \epsilon$. This implies $\lim_{A\to+\infty} Af(A) = 0$.
(5)

(a) Since f is continuous on $[a,b]$, there exists a point $x_0 \in [a,b]$ such that $f(x_0) = \max_{x\in[a,b]}\{f(x)\}$. Thus

$$0 \leq \int_a^b [f(x)]^n\,dx \leq \int_a^b [f(x_0)]^n\,dx = [f(x_0)]^n \cdot (b-a).$$

Take $n \to +\infty$, since $0 \leq f(x_0) < 1$, by the Sandwich theorem, we have the limit.

(b) If we define

$$F(x) = \begin{cases} f(x), & \text{if } x \neq a, b; \\ 0, & \text{if } x = a \text{ or } x = b, \end{cases}$$

then, F and f may differ at most at two points $x=a$ and $x=b$. By Example 3.1.12, we know that

$$\int_{x_1}^{x_2} [F(x)]^n\,dx = \int_{x_1}^{x_2} [f(x)]^n\,dx$$

for any $x_1, x_2 \in [a,b]$. By the condition on f, we know that $0 \leq F(x) < 1$ on $[a,b]$. This implies that the sequence

$$\int_a^b [F(x)]^n\,dx = \int_a^b [f(x)]^n\,dx$$

is decreasing in n. Hence, the limit

$$\lim_{n\to\infty} \int_a^b [f(x)]^n\,dx$$

exists.

Since f is integrable on $[a,b]$, we know that f is bounded on $[a,b]$. For any $\epsilon > 0$ and for any fixed integer N, there is an $\delta > 0$ such that

$$\int_a^{a+\delta} [f(x)]^N\,dx < \epsilon, \qquad \int_{b-\delta}^b [f(x)]^N\,dx < \epsilon.$$

This gives that, for $n \geq N$,

$$0 \leq \int_a^{a+\delta} [f(x)]^n\,dx = \int_a^{a+\delta} [F(x)]^n\,dx \leq \int_a^{a+\delta} [F(x)]^N\,dx = \int_a^{a+\delta} [f(x)]^n\,dx < \epsilon,$$

and

$$0 \leq \int_{b-\delta}^b [f(x)]^n\,dx = \int_{b-\delta}^b [F(x)]^n\,dx \leq \int_{b-\delta}^b [F(x)]^N\,dx = \int_{b-\delta}^b [f(x)]^n\,dx < \epsilon.$$

Thus, for $n \geq N$, we have

$$\begin{aligned} \left|\int_a^b [f(x)]^n\,dx\right| &= \left|\int_a^{a+\delta} [f(x)]^n\,dx\right| + \left|\int_{b-\delta}^b [f(x)]^n\,dx\right| + \left|\int_{a+\delta}^{b-\delta} [f(x)]^n\,dx\right| \\ &\leq \epsilon + \epsilon + \left|\int_{a+\delta}^{b-\delta} [f(x)]^n\,dx\right|. \end{aligned}$$

If we let $n \to \infty$, by (a), we know that, for any fixed $\epsilon > 0$,

$$\lim_{n\to+\infty} \int_{a+\delta}^{b-\delta} [f(x)]^n \, dx = 0.$$

Thus,

$$0 \le \lim_{n\to+\infty} \int_{a}^{b} [f(x)]^n \, dx \le 2\epsilon.$$

Since ϵ is arbitrary, we conclude

$$\lim_{n\to+\infty} \int_{a}^{b} [f(x)]^n \, dx = 0.$$

(6) For the series

$$\sum_{n=1}^{\infty} (-1)^n \frac{1}{(n+x^2)},$$

it is obvious that:

(1) $\dfrac{1}{(n+x^2)}$ is monotone in n;

(2) $\dfrac{1}{(n+x^2)} \to 0$ uniformly as $n \to +\infty$, since

$$\frac{1}{(n+x^2)} \le \frac{1}{n}.$$

Thus, by Exercise 4.2.11, the series is uniformly convergent. In fact, it is also convergent everywhere. If we differentiate the series term-by-term, then we obtain the series

$$\sum_{n=1}^{\infty} (-1)^n \frac{-2x}{(n+x^2)^2}.$$

which is uniformly convergent. Hence, by Proposition 4.2.7,

$$\begin{aligned}
I &= \lim_{x\to 0} \frac{1}{x} \left(\sum_{n=1}^{\infty} (-1)^n \frac{1}{(n+x^2)} \right)' \\
&= \lim_{x\to 0} \frac{1}{x} \sum_{n=1}^{\infty} \left((-1)^n \frac{1}{(n+x^2)} \right)' \\
&= \lim_{x\to 0} \frac{1}{x} \sum_{n=1}^{\infty} (-1)^n \frac{-2x}{(n+x^2)^2}. \\
&= \lim_{x\to 0} \sum_{n=1}^{\infty} (-1)^n \frac{-2}{(n+x^2)^2}
\end{aligned}$$

By Proposition 4.2.6, the can interchange the limit with the summation in the last limit. Thus,

$$I = (-2) \sum_{n=1}^{\infty} (-1)^n \frac{1}{n^2}.$$

To find the value of the series $\sum_{n=1}^{\infty}(-1)^n\frac{1}{n^2}$, we use the result in Example 4.2.22, by expand the function x^2 as its Fourier series on $(0, 2\pi)$ to have

$$x^2 \sim \frac{4\pi^2}{3} + \sum_{n=1}^{\infty}\left(\frac{4}{n^2}\cos nx - \frac{4\pi}{n}\sin nx\right).$$

Putting $x = \pi$, by Theorem 4.2.13, we have

$$\pi^2 = \frac{4\pi^2}{3} + \sum_{n=1}^{\infty}\frac{(-1)^{n+1}4}{n^2},$$

which gives

$$\sum_{n=1}^{\infty}(-1)^n\frac{1}{n^2} = \frac{\pi^2}{12}.$$

Hence

$$I = -\pi^2/6.$$

Math 203 Final, Autumn 2009

(1) Suppose $f(x) = x + ax^n + o(x^n)$, $n \geq 2$, and $f(x)$ is invertible near $x = 0$.

1. Prove that the inverse of f is n-th order differentiable at 0, and find the n-th order approximation of the inverse.

2. Can you make a more general statement that include the function $f(x) = -2x + x^2 + 3x^4 + o(x^4)$. Moreover, find the 4th order approximation of the inverse function.

(2) Suppose $f(x)$ is integrable on $[0, 1]$ and continuous at 0. Prove that

$$\lim_{h\to 0^+}\int_0^1 \frac{h}{h^2 + x^2}f(x)dx = \frac{\pi}{2}f(0).$$

(3) Suppose $f(x)$ and $g(x)$ are integrable on $[a, b]$. Prove that for any $\epsilon > 0$, there is $\delta > 0$, such that for any partition P satisfying $\|P\| < \delta$ and choices $x_i^*, x_i^{**} \in [x_{i-1}, x_i]$, we have

$$\left|\sum f(x_i^*)g(x_i^{**})\Delta x_i - \int_a^b f(x)g(x)dx\right| < \epsilon.$$

(4) The function $f(x) = \sum \frac{x^n}{n^{1.5}}$ is defined on $[-1, 1]$. Study the differentiability of the function on $[-1, 1]$ (including one-side derivative at the end points).

(5) Consider the series $\sum \frac{\sin nx}{n^\alpha(\log n)^\beta}$.

1. Find all the combinations of x, α, β so that the series absolutely converges.

2. Find all the combinations of x, α, β so that the series conditionally converges.

3. Prove that if the series absolutely converges for all x, then the series uniformly converges on $\mathbb{R}$.

Answer to Math 203 Final, Autumn 2009

(1) Let $g(y)$ be the inverse of $f(x)$ near 0. By Proposition 2.1.5 and $f'(0) = 1 \neq 0$, we know $g(y)$ is differentiable at 0, with $g'(0) = \dfrac{1}{f'(0)} = 1$. Therefore $g(y) = y + o(y)$. This implies

$$\lim_{y\to 0} \frac{g(y)^n}{y^n} = \left(\lim_{y\to 0} \frac{g(y)}{y}\right)^n = 1^n = 1,$$

and

$$\lim_{y\to 0} \frac{o(g(y)^n)}{y^n} = \left(\lim_{y\to 0} \frac{o(g(y))}{g(y)} \frac{g(y)}{y}\right) = (0 \cdot 1)^n = 0.$$

In other words, we have

$$g(y)^n = y^n + o(y^n), \quad o(g(y)^n) = o(y^n).$$

Substituting $g(y) = y + o(y)$ into $y = f(g(y))$, we get

$$y = g(y) + ag(y)^n + o(g(y)^n) = g(y) + ay^n + o(y^n) + o(y^n).$$

This means $g(y) = y - ay^n + o(y^n)$, or $y - ay^n$ is the n-th order approximation of $g(y)$ at 0.

In general, if $f(x)$ is invertible near $x = 0$, f is n-th order differentiable at 0, and $f'(0) \neq 0$, then the inverse of f is also n-th order differentiable. The high order approximation of the inverse may be computed by substitution. For example, for $f(x) = -2x + x^2 + 3x^4 + o(x^4)$, we substitute $g(y) = ay + by^2 + cy^3 + dy^4 + o(y^4)$ into $y = f(g(y))$ to get

$$\begin{aligned} y &= -2(ay + by^2 + cy^3 + dy^4) + (ay + by^2 + cy^3)^2 + 3(ay)^4 + o(y^4) \\ &= -2(ay + by^2 + cy^3 + dy^4) + (a^2y^2 + b^2y^4 + 2aby^3 + 2acy^4) + 3a^4y^4 + o(y^4). \end{aligned}$$

Comparing the coefficients, we get

$$-2a = 1, \quad -2b + a^2 = 0, \quad -2c + 2ab = 0, \quad -2d + b^2 + 2ac + 3a^4 = 0.$$

Solving the equations, we get

$$a = -\frac{1}{2}, \quad b = -\frac{1}{2^3}, \quad c = \frac{1}{2^4}, \quad d = -\frac{7}{2^9}$$

and the 4-th order approximation of the inverse function

$$g(y) = -\frac{1}{2}y - \frac{1}{2^3}y^2 + \frac{1}{2^4}y^3 - \frac{7}{2^9}y^4 + o(y^4).$$

(2) By the continuity of $f(x)$, for any $\epsilon > 0$, there is $\delta > 0$, such that $0 \le x \le \delta$ implies $|f(x) - f(0)| \le \epsilon$. Moreover, since $f(x)$ is integrable, it is bounded, so that $|f(x)| < M$ for all x and a constant M. Then

$$\left|\int_0^1 \frac{h}{h^2+x^2} f(x)dx - \int_0^1 \frac{h}{h^2+x^2} f(0)dx\right| \le \int_0^\delta \frac{h}{h^2+x^2}|f(x)-f(0)|dx + \int_\delta^1 \frac{h}{h^2+x^2}|f(x)-f(0)|dx$$
$$\le \epsilon \int_0^\delta \frac{h}{h^2+x^2}dx + 2M\int_\delta^1 \frac{h}{x^2}dx \le \epsilon\int_0^{\frac{\delta}{h}} \frac{1}{1+y^2}dy + 2Mh\left(\frac{1}{\delta}-1\right)$$

Since

$$\int_0^{\frac{\delta}{h}} \frac{1}{1+y^2}dy = \arctan\frac{\delta}{h} < \frac{\pi}{2},$$

we get

$$\left|\int_0^1 \frac{h}{h^2+x^2} f(x)dx - \int_0^1 \frac{h}{h^2+x^2} f(0)dx\right| \le \frac{\pi}{2}\epsilon + 2Mh\left(\frac{1}{\delta}-1\right).$$

For the given ϵ (and the corresponding δ), we may choose small enough h, such that the right side is $\le 2\epsilon$. This implies that

$$\lim_{h\to 0^+}\left(\int_0^1 \frac{h}{h^2+x^2} f(x)dx - \int_0^1 \frac{h}{h^2+x^2} f(0)dx\right) = 0.$$

On the other hand,

$$\lim_{h\to 0^+}\int_0^1 \frac{h}{h^2+x^2} f(0)dx = f(0)\lim_{h\to 0^+}\int_0^{\frac{1}{h}} \frac{1}{1+y^2}dy = f(0)\lim_{h\to 0^+}\arctan\frac{1}{h} = \frac{\pi}{2}f(0).$$

Therefore we conclude that

$$\lim_{h\to 0^+}\int_0^1 \frac{h}{h^2+x^2} f(x)dx = \lim_{h\to 0^+}\int_0^1 \frac{h}{h^2+x^2} f(0)dx = \frac{\pi}{2}f(0).$$

(3) Since $f(x)$ is integrable, it is bounded: We have $|f(x)| < M$ for all $x \in [a,b]$ and a constant M.

Since g is integrable, for any $\epsilon > 0$, there is $\delta > 0$, such that $\|P\| < \delta$ implies $\sum \omega_{[x_{i-1},x_i]}(g)\Delta x_i < \epsilon$.

Since f and g are integrable, the product fg is also integrable. Therefore for any $\epsilon > 0$, there is $\delta' > 0$, such that $\|P\| < \delta'$ implies $\left|\sum f(x_i^*)g(x_i^*)\Delta x_i - \int_a^b f(x)g(x)dx\right| < \epsilon$.

Now for $\|P\| < \min\{\delta, \delta'\}$, we have

$$\left|\sum f(x_i^*)g(x_i^{**})\Delta x_i - \int_a^b f(x)g(x)dx\right|$$
$$\le \left|\sum f(x_i^*)g(x_i^{**})\Delta x_i - \sum f(x_i^*)g(x_i^*)\Delta x_i\right| + \left|\sum f(x_i^*)g(x_i^*)\Delta x_i - \int_a^b f(x)g(x)dx\right|$$
$$< \sum |f(x_i^*)(g(x_i^{**}) - g(x_i^*))|\Delta x_i + \epsilon \le M\sum \omega_{[x_{i-1},x_i]}(g)\Delta x_i + \epsilon < (M+1)\epsilon.$$

(4) If we take term wise differentiation, we get $g(x) = \sum \frac{x^{n-1}}{n^{0.5}}$. The power series $g(x)$ converges uniformly on $[-1, 1)$. The uniformity at 1 follows from Proposition 4.2.13 (Abel's Theorem). Then it follows from Proposition 4.2.12 that $f'(x) = g(x)$ on $[-1, 1)$ (including $f'_+(-1) = g(-1)$).

Although $g(x)$ diverges at 1, the fact does not necessarily imply that $f(x)$ is not left differentiable at 1 (the converse of Proposition 4.2.12 is not true). So we need to study the quotient

$$\frac{f(x) - f(1)}{x - 1} = \sum \frac{x^n - 1}{x - 1} \frac{1}{n^{1.5}}.$$

directly.

The divergence (to $+\infty$) of $\sum \frac{1}{k^{0.5}}$ implies that for any $b > 0$, there is n, such that

$$\sum_{k=1}^{n} \frac{1}{k^{0.5}} \geq b.$$

Then by $\lim_{x\to 1^-} \frac{x^k - 1}{x - 1} = k$, for the finitely many k between 1 and n, we may find $\delta > 0$, such that

$$1 - \delta < x < 1 \implies \frac{x^k - 1}{x - 1} > \frac{k}{2}, \quad k = 1, 2, \dots, n.$$

Then $1 - \delta < x < 1$ implies $\frac{x^k - 1}{x - 1} > 0$ for all k, and

$$\frac{f(x) - f(1)}{x - 1} \geq \sum_{k=1}^{n} \frac{x^k - 1}{x - 1} \frac{1}{k^{1.5}} > \sum_{k=1}^{n} \frac{k}{2} \frac{1}{k^{1.5}} = \frac{1}{2} \sum_{k=1}^{n} \frac{1}{k^{0.5}} \geq b.$$

Therefore the quotient is unbounded as $x \to 1^-$, and $\lim_{x\to 0^-} \frac{f(x) - f(1)}{x - 1}$ diverges.

(5.1 and 5.2) If the series $\sum \frac{1}{n^\alpha (\log n)^\beta}$ converges, then by the comparison test, the series $\sum \frac{\sin nx}{n^\alpha (\log n)^\beta}$ absolutely converges. This happens when $\alpha > 1$ or $\alpha = 1$ and $\beta > 1$.

In cases (i) $\alpha = 1$ and $\beta \leq 1$, (ii) $1 > \alpha > 0$, (iii) $\alpha = 0$ and $\beta > 0$, the sequence $\frac{1}{n^\alpha (\log n)^\beta}$ is decreasing and converges to 0. Moreover, by Example 4.1.12, the partial sum of $\sum \sin nx$ is bounded. By the Dirichlet test, the series $\sum \frac{\sin nx}{n^\alpha (\log n)^\beta}$ converges. We claim the convergence is conditional.

Following the argument in Example 4.1.10, for $0 < x \leq \frac{\pi}{2}$ and each k, we find n_k satisfying

$$\frac{4k + 1}{4}\pi \leq n_k x \leq \frac{4k + 3}{4}\pi.$$

Then $|\sin n_k x| \geq \frac{1}{\sqrt{2}}$, $\frac{1}{n_k} > \frac{x}{4k}$ and

$$\frac{|\sin n_k x|}{n_k^\alpha (\log n_k)^\beta} > \frac{1}{\sqrt{2}\left(\frac{x}{4k}\right)^\alpha \left(\log \frac{x}{4k}\right)^\beta} = \frac{1}{Ak^\alpha(\log k + B)^\beta} > \frac{1}{2Ak^\alpha(\log k)^\beta}$$

for sufficiently big k. The divergence of $\sum_{k=1}^\infty \frac{1}{k^\alpha(\log k)^\beta}$ in the three cases imply that the series $\sum \frac{\sin nx}{n^\alpha(\log n)^\beta}$ diverges in the three cases. The remark at the end of Example 4.1.10 also applies and extend the discussion to the case a is not a multiple of π.

If a is a multiple of π, then the series is constantly 0 and absolutely converges.

Finally, the remaining cases are $\alpha < 0$ or $\alpha = 0$ and $\beta \leq 0$. The terms $\frac{\sin nx}{n^\alpha(\log n)^\beta}$ will not converge to 0 and the series diverges.

(5.3) Now we consider the uniformity of the convergence. Of course this assumes that the combination of α and β should make the series converge on some interval.

If the series $\sum \frac{1}{n^\alpha(\log n)^\beta}$ converges, then by the comparison test, the series $\sum \frac{\sin nx}{n^\alpha(\log n)^\beta}$ uniformly converges on $\mathbb{R}$. This happens when $\alpha > 1$ or $\alpha = 1$ and $\beta > 1$, which is exactly the case the series absolutely converges for all x.

Further Discussion

In cases (i) $\alpha = 1$ and $\beta \leq 1$, (ii) $1 > \alpha > 0$, (iii) $\alpha = 0$ and $\beta > 0$, the sequence $\frac{1}{n^\alpha(\log n)^\beta}$ is decreasing and converges to 0. By the Dirichlet test, the series $\sum \frac{\sin nx}{n^\alpha(\log n)^\beta}$ converges uniformly on some interval if the partial sum of $\sum \sin nx$ is uniformly bounded on the interval. By Example 4.1.12, we know the partial sum is uniformly bounded on $[r, 2\pi - r]$, or more generally, on $[2k\pi + r, 2(k+1)\pi - r]$, for any $0 < r < \pi$. Thus under any of the three cases, the series uniformly converges on such intervals.

It remains to consider the uniformly of the convergence in the three cases on an interval $(-r, r)$ containing 0 (or $(2k\pi - \epsilon, 2k\pi + \epsilon)$ in general). We note that in case (ii) and (iii), and in case $\alpha = 1$ and $\beta \leq 0$ (part of case (i)), we have $\frac{1}{n^\alpha(\log n)^\beta} \geq \frac{1}{n}$ for sufficiently big n. Moreover, for sufficiently big N, we have $x = \frac{\pi}{4N} \in (-r, r)$. Then $\frac{\pi}{4} \leq nx \leq \frac{3\pi}{4}$ for $N \leq n \leq 3N$, and we have

$$\sum_{n=N}^{3N} \frac{\sin nx}{n^\alpha(\log n)^\beta} \geq \sum_{n=N}^{3N} \frac{1}{\sqrt{2}n} \geq (3N - N)\frac{1}{\sqrt{2}3N} = \frac{\sqrt{2}}{3}.$$

Therefore the Cauchy criterion for the uniform convergence is not satisfied for $\epsilon = \frac{\sqrt{2}}{3}$, and the series is not uniformly convergent on $(-r, r)$.

The remaining case is $\alpha = 1$ and $0 < \beta \leq 1$ (remaining part of case (i)). I believe the series converges uniformly on $\mathbb{R}$.

Math 2043 Final, Autumn 2011

(1) Show that $\lim_{n\to\infty}\dfrac{x}{n^x}=0$ uniformly on $[0,+\infty)$.

(Hint: Show that the maximum of $\dfrac{x}{n^x}$ on $[0,+\infty)$ converges to 0.)

(2) Show that for $p>1$, $\sum\left(e^x-\left(1+\dfrac{x}{n}\right)^n\right)^p$ converges uniformly on $[0,R]$ for any $R>0$ and not uniformly on $[0,+\infty)$.

(3) Suppose $f(x)$ is n-th order differentiable at 0 and has non-constant n-th order approximation. Prove that if 0 is a local maximum of $f(x)$, then $f(x)\le f(x^2)$ for x sufficiently close to 0.

(4) Rigorously explain that $\lim_{n\to\infty}\dfrac{\sqrt[n]{n!}}{n}=e^{\int_0^1\log x dx}=e^{-1}$. Note that $\displaystyle\int_0^1\log x dx$ is an improper integral.

(5) Suppose α has bounded variation on $[a,b]$ with variation function $v(x)=V_{[a,x]}(\alpha)$. Suppose f is Riemann-Stieltjes integrable with respect to v. Prove that f is Riemann-Stieltjes integrable with respect to α and $|f|$ is Riemann-Stieltjes integrable with respect to v. Moreover, prove that $\left|\displaystyle\int_a^b f d\alpha\right|\le\int_a^b|f|dv$.

Answer to Math 2043 Final, Autumn 2011

(1) For $f_n=\dfrac{x}{n^x}$, we have $f_n'=\dfrac{1-x\log n}{n^x}$. Since $f_n>0$ and $f_n(0)=0$, $\lim_{x\to+\infty}f_n=0$ (at least for $n>1$), we see that f_n reaches its maximum for some $x\in(0,+\infty)$ satisfying $f_n'(x)=0$. The only such x is $\dfrac{1}{\log n}$. Therefore

$$0\le f_n(x)\le\max_{[0,+\infty)}f_n(x)=f_n\left(\frac{1}{\log n}\right)=\frac{1}{n^{\frac{1}{\log n}}\log n}<\frac{1}{\log n}.$$

By $\lim_{n\to\infty}\dfrac{1}{\log n}=0$, we conclude that $f_n\to 0$ uniformly on $[0,+\infty)$.

Another Answer: For any $\epsilon>0$, we have $0\le\dfrac{x}{n^x}<2\epsilon$ on $[0,2\epsilon)$. It remains to find N, such that $0\le\dfrac{x}{n^x}<\epsilon$ for $n>N$ and $x\in[2\epsilon,+\infty)$.

For $x\ge 2\epsilon$ and $n\ge 2$, we have

$$0\le\frac{x}{n^x}\le\frac{x}{n^{\frac{x}{2}}}\frac{1}{n^{\frac{x}{2}}}\le\frac{x}{2^{\frac{x}{2}}}\frac{1}{n^\epsilon}.$$

Since $\dfrac{x}{2^{\frac{x}{2}}}$ is a continuous function on $[0,+\infty)$ with value 0 at 0 and limit 0 at $+\infty$, the function is bounded on $[0,+\infty)$. Let $0\le\dfrac{x}{2^{\frac{x}{2}}}\le B$ (B is the maximum of $\dfrac{x}{2^{\frac{x}{2}}}$ and is a specific constant). Then for $x\ge 2\epsilon$ and $n\ge 2$, we have

$$0\le\frac{x}{n^x}\le\frac{B}{n^\epsilon}.$$

Since $\lim_{n\to\infty} \dfrac{B}{n^\epsilon} = 0$, there is N (depending only on ϵ, B, and therefore on ϵ only), such that $\dfrac{B}{n^\epsilon} < \epsilon$. Then we have

$$x \ge 2\epsilon,\ n \ge N \implies 0 \le \frac{x}{n^x} < \epsilon.$$

Combined with $0 \le \dfrac{x}{n^x} < 2\epsilon$ on $[0, 2\epsilon)$, we have

$$x \ge 0,\ n \ge N \implies 0 \le \frac{x}{n^x} < 2\epsilon.$$

(2) For $0 \le x \le R$, by the remainder formula for Taylor expansion, we have (see the argument in Example 5.2.3)

$$\begin{aligned} e^x - \left(1+\frac{x}{n}\right)^n &= e^x - e^{n\log\left(1+\frac{x}{n}\right)} = e^x - e^{n\left(\frac{x}{n}-\frac{x^2}{2n^2(1+c)^2}\right)} \\ &= e^x - e^x e^{-\frac{x^2}{2n(1+c)^2}} = e^x\left(\frac{x^2}{2n(1+c)^2} + o\left(\frac{x^2}{2n(1+c)^2}\right)\right), \end{aligned}$$

where $0 < c < \dfrac{x}{n}$. Since $0 \le \dfrac{x^2}{2n(1+c)^2} \le \dfrac{x^2}{2n} \le \dfrac{R^2}{2n}$, which tends to 0 as $n \to \infty$, the above estimation tells us that $\left|e^x - \left(1+\dfrac{x}{n}\right)^n\right| \le \dfrac{A}{n}$ for sufficiently big n. By the convergence of $\sum\left(\dfrac{A}{n}\right)^p$ for $p > 1$, we see that $\sum\left(e^x - \left(1+\dfrac{x}{n}\right)^n\right)^p$ converges uniformly on $[0, R]$.

On the other hand, for fixed n, we have $\lim_{x\to+\infty}\left(e^x - \left(1+\dfrac{x}{n}\right)^n\right)^p = +\infty$. Therefore $\left(e^x - \left(1+\dfrac{x}{n}\right)^n\right)^p$ does not converge to 0 uniformly on $[0, +\infty)$. The series then does not converge uniformly on $[0, +\infty)$.

(3) Let the non-constant n-th order approximation be $f(x) = a_0 + a_1x + \cdots + a_nx^n + o(x^n)$. Since the approximation is not constant, at least one of $a_1, \ldots, a_n$ is nonzero. Let k be the smallest index such that $a_k \ne 0$. Then $f(x) = a_0 + a_kx^k + o(x^k)$.

By the first part of Proposition 2.2.5, since 0 is a local maximum of $f(x)$, k cannot be odd. Then by the (proof of) second part of the proposition, if $a_k > 0$, then 0 is a strict local minimum. Therefore we must have k even and $a_k < 0$. Then

$$f(x) - f(x^2) = a_0 + a_kx^k + o(x^k) - a_0 - a_kx^{2k} - o(x^{2k}) = a_kx^k + o(x^k).$$

By the third part of the proposition, we know 0 is a local maximum of $f(x) - f(x^2)$. Therefore for x sufficiently close to 0, we have

$$f(x) - f(x^2) \le f(0) - f(0^2) = 0.$$

(4) We have

$$\log\frac{\sqrt[n]{n!}}{n} = \log\sqrt[n]{\frac{1\cdot 2\cdots n}{n\cdot n\cdots n}} = \frac{1}{n}\left(\log\frac{1}{n} + \log\frac{2}{n} + \cdots + \log\frac{n}{n}\right),$$

and wish to show that the right side converges to $\int_0^1 \log x dx$. The right side appears to be the Riemann sum $S(P, \log x)$ for the partition P consisting of $x_i = \frac{i}{n}$ and sample points $x_i^* = x_i$. However, we cannot use the definition of Riemann integral because the integral here is improper.

Using the fact that $\log x$ is increasing and the idea of the proof of Proposition 5.1.3, we have

$$\begin{aligned}\frac{1}{n}\left(\log\frac{1}{n}+\log\frac{2}{n}+\cdots+\log\frac{n}{n}\right) &= \frac{1}{n}\left(\log\frac{1}{n}+\log\frac{2}{n}+\cdots+\log\frac{n-1}{n}\right)\\ &\le \left(\int_{\frac{1}{n}}^{\frac{2}{n}}+\int_{\frac{2}{n}}^{\frac{3}{n}}+\cdots+\int_{\frac{n-1}{n}}^{1}\right)\log x dx = \int_{\frac{1}{n}}^{1}\log x dx,\end{aligned}$$

and

$$\begin{aligned}\frac{1}{n}\left(\log\frac{1}{n}+\log\frac{2}{n}+\cdots+\log\frac{n}{n}\right) &= \frac{1}{n}\log\frac{1}{n}+\frac{1}{n}\left(\log\frac{2}{n}+\log\frac{3}{n}+\cdots+\log\frac{n}{n}\right)\\ &\ge \frac{1}{n}\log\frac{1}{n}+\left(\int_{\frac{1}{n}}^{\frac{2}{n}}+\int_{\frac{2}{n}}^{\frac{3}{n}}+\cdots+\int_{\frac{n-1}{n}}^{1}\right)\log x dx\\ &= \frac{1}{n}\log\frac{1}{n}+\int_{\frac{1}{n}}^{1}\log x dx,\end{aligned}$$

Therefore by the convergence of $\int_0^1 \log x dx$, we have

$$\frac{1}{n}\log\frac{1}{n}-\int_0^{\frac{1}{n}}\log x dx \le \frac{1}{n}\left(\log\frac{1}{n}+\log\frac{2}{n}+\cdots+\log\frac{n}{n}\right)-\int_0^1\log x dx \le -\int_0^{\frac{1}{n}}\log x dx.$$

The convergence of $\int_0^1 \log x dx$ tells us $\lim_{n\to\infty}\int_0^{\frac{1}{n}}\log x dx = 0$. Moreover, we also know $\lim_{n\to\infty}\frac{1}{n}\log\frac{1}{n} = 0$. Therefore

$$\lim_{n\to\infty}\frac{1}{n}\left(\log\frac{1}{n}+\log\frac{2}{n}+\cdots+\log\frac{n}{n}\right) = \int_0^1\log x dx = -1.$$

(5) The variation function v is increasing. Since f is Riemann-Stieltjes integrable with respect to v, by the first part of Theorem 4.2.8, for any $\epsilon > 0$, there is $\delta > 0$, such that

$$\|P\| < \delta \implies \sum \omega_{[x_{i-1},x_i]}(f)|\Delta v_i| = \sum \omega_{[x_{i-1},x_i]}(f)V_{[x_{i-1},x_i]}(\alpha) < \epsilon.$$

Then by the second part of the theorem, we see that f is Riemann-Stieltjes integrable with respect to α. Moreover, we have $\omega_{[x_{i-1},x_i]}(|f|) \le \omega_{[x_{i-1},x_i]}(f)$, and v increasing implies $|\Delta v_i| = V_{[x_{i-1},x_i]}(v)$. Therefore

$$\omega_{[x_{i-1},x_i]}(|f|)V_{[x_{i-1},x_i]}(v) \le \omega_{[x_{i-1},x_i]}(f)|\Delta v_i|,$$

so that

$$\|P\| < \delta \implies \sum \omega_{[x_{i-1},x_i]}(|f|)V_{[x_{i-1},x_i]}(v) = \sum \omega_{[x_{i-1},x_i]}(f)|\Delta v_i| < \epsilon.$$

By the second part of Theorem 4.2.8, we see that $|f|$ is Riemann-Stieltjes integrable with respect to v.

Now compare the Riemann-Stieltjes sums for $\left|\int_a^b f d\alpha\right|$ and $\int_a^b |f| dv$.

$$\begin{aligned}|S(P,f,\alpha)| &= \left|\sum f(x_i^*)\Delta\alpha_i\right| \le \sum |f(x_i^*)||\Delta\alpha_i| \le \sum |f(x_i^*)|V_{[x_{i-1},x_i]}(\alpha) \\ &= \sum |f(x_i^*)||\Delta v_i| = S(P,|f|,v).\end{aligned}$$

Taking limit as $\|P\| \to 0$, we get $\left|\int_a^b f d\alpha\right| \le \int_a^b |f| dv$.

Math 2043 Final, Spring 2014

(1) Prove that a continuous function $f(x)$ is periodic with period T if and only if $\int_0^T f(x+a)dx$ is independent of a.
(2) Let $a, b > 0$. Determine the convergence of $\int_0^{+\infty} a^x \sin b^x dx$. Also determine whether the convergence is absolute or conditional.
(3) Prove that if $\sqrt[n]{|a_n|} \le 1 - \dfrac{p \log n}{n}$ for some fixed $p > 1$ and sufficiently big n, then $\sum a_n$ converges. Also state the limit version of the test. Moreover, can you find a similar test for the divergence of $\sum a_n$?
(4) Suppose x_n is a non-repeating sequence in (a,b). Suppose $f(x)$ is a function on $[a,b]$ satisfying $f(x) = 0$ if x is not any x_n. Prove that $V_{[a,b]}f(x) = 2\sum |f(x_n)|$. In particular, the function has bounded variation if and only if $\sum f(x_n)$ absolutely converges.

Answer to Math 2043 Final, Spring 2014

(1) We have

$$\begin{aligned}\int_0^T f(x+b)dx - \int_0^T f(x+a)dx &= \int_b^{b+T} f(x)dx - \int_a^{a+T} f(x)dx = \int_{a+T}^{b+T} f(x)dx - \int_a^b f(x)dx \\ &= \int_a^b f(x+T)dx - \int_a^b f(x)dx = \int_a^b (f(x+T) - f(x))dx.\end{aligned}$$

Suppose $f(a+T) > f(a)$. Then by the (right) continuity of $f(x+T) - f(x)$ at a, there is $b > a$, such that $f(x+T) - f(x) > 0$ on $[a,b]$. This implies $\int_a^b (f(x+T) - f(x))dx > 0$, contradicting to the equality above. Similarly, we cannot have $f(a+T) < f(a)$. Therefore $f(a+T) = f(a)$.

(2) (25 pints) We first assume $b > 1$ and introduce $y = b^x$. Then

$$\int_0^B a^x \sin b^x dx = \int_1^{b^B} y^p \sin y \frac{dy}{y \log b} = \frac{1}{\log b} \int_1^{b^B} \frac{\sin y}{y^{1-p}} dy, \quad p = \frac{\log a}{\log b}.$$

We also have

$$\int_0^B |a^x \sin b^x| dx = \frac{1}{\log b} \int_1^{b^B} \frac{|\sin y|}{y^{1-p}} dy.$$

Since $B \to +\infty$ if and only if $e^B \to +\infty$, $\int_0^{+\infty} a^x \sin b^x dx$ (absolutely or conditionally) converges if and only if $\int_1^{+\infty} \frac{\sin y}{y^{1-p}} dy$ (absolutely or conditionally) converges. We know what happens to the second integral. Therefore for $b > 1$, $\int_0^{+\infty} a^x \sin b^x dx$ absolutely converges when $p = \frac{\log a}{\log b} < 0$, or $a < 1 < b$; conditionally converges when $0 \le p = \frac{\log a}{\log b} < 1$, or $1 \le a < b$; and diverges when $p = \frac{\log a}{\log b} \ge 1$, or $a > b > 1$.

Now assume $b < 1$. Then $\lim_{x\to+\infty} b^x = 0$, $a^x \sin b^x > 0$ for sufficiently big x, so that $\int_0^{+\infty} a^x \sin b^x dx$ cannot be conditionally convergent. Moreover, by $\lim_{x\to+\infty} \frac{a^x \sin b^x}{(ab)^x} = 1$ and the comparison test, $\int_0^{+\infty} a^x \sin b^x dx$ (absolutely) converges if and only if $\int_0^{+\infty} (ab)^x dx$ converges, which means $ab < 1$.

Finally, if $b = 1$, then $\int_0^{+\infty} a^x \sin b^x dx = \sin 1 \int_0^{+\infty} a^x dx$ cannot be conditionally convergent, and (absolutely) converges if and only if $a < 1$.

We conclude the following cases for absolute convergence: $a < 1 < b$, or $b \le 1$ and $ab < 1$; the following case for conditional convergence: $1 \le a < b$; the following case for divergence: $a > b > 1$, or $b \le 1$ and $ab \ge 1$.

(3) If $\sqrt[n]{|a_n|} \le 1 - \frac{p \log n}{n}$, then for sufficiently large n, we have

$$|a_n| \le \left(1 - \frac{p \log n}{n}\right)^n = e^{n \log\left(1 - \frac{p \log n}{n}\right)} = e^{-n\left(\frac{p \log n}{n} + o\left(\frac{(p \log n)^2}{n^2}\right)\right)} = n^{-p} e^{-o\left(\frac{(p \log n)^2}{n}\right)} \le \frac{2}{n^p}.$$

The last inequality is due to

$$\lim_{n\to\infty} e^{-o\left(\frac{(p \log n)^2}{n}\right)} = e^{-\lim_{n\to\infty} o\left(\frac{(p \log n)^2}{n}\right)} = e^0 = 1.$$

Since $p > 1$, $\sum \frac{2}{n^p}$ converges. By the comparison test, $\sum a_n$ converges.

The inequality $\sqrt[n]{|a_n|} \le 1 - \frac{p \log n}{n}$ is the same as $\frac{n}{\log n}(1 - \sqrt[n]{|a_n|}) \ge p$. If

$$\underline{\lim}_{n\to\infty} \frac{n}{\log n}(1 - \sqrt[n]{|a_n|}) > 1,$$

then the inequality is satisfied for sufficiently big n, and $\sum a_n$ converges.

If $\sqrt[n]{|a_n|} \geq 1 - \dfrac{\log n}{n}$ for sufficiently big n, then we have

$$|a_n| \geq \left(1 - \frac{\log n}{n}\right)^n = e^{n\log\left(1-\frac{\log n}{n}\right)} = e^{-n\left(\frac{\log n}{n}+o\left(\frac{(\log n)^2}{n^2}\right)\right)} = n^{-1}e^{-o\left(\frac{(\log n)^2}{n}\right)} \geq \frac{1}{2n}.$$

By the divergence of $\sum \dfrac{1}{2n}$ and the comparison test, we find $\sum |a_n|$ diverges.

The limit version of the divergence test is

$$\overline{\lim_{n\to\infty}} \frac{n}{\log n}(1 - \sqrt[n]{|a_n|}) < 1.$$

(4) Pick any finite number of terms from x_n and arrange them in increasing order

$$a < x_{n_1} < x_{n_2} < x_{n_3} < \cdots < x_{n_k} < b.$$

Then we can find a partition of $[a, b]$

$$P\colon a = y_0 < x_{n_1} < y_1 < x_{n_2} < y_2 < x_{n_3} < \cdots < x_{n_{k-1}} < y_{n-1} < x_{n_k} < y_k = b,$$

such that y_i is not any x_n. Then we have

$$\begin{aligned} V_P(f) &= \sum_{i=1}^{k}(|f(x_{n_i}) - f(y_{i-1})| + |f(y_i) - f(x_{n_i})|) \\ &= \sum_{i=1}^{k}(|f(x_{n_i}) - 0| + |0 - f(x_{n_i})|) = 2\sum_{i=1}^{k}|f(x_{n_i})|. \end{aligned}$$

Since the right side is twice of the sum of any finitely many terms in the series $\sum |f(x_n)|$, we take the supremum of the right side and get

$$V_{[a,b]}f(x) \geq 2\sum |f(x_n)|.$$

On the other hand, for any partition

$$P\colon a = y_0 < y_1 < \cdots < y_n = b,$$

we have

$$\begin{aligned} V_P(f) &= \sum_{i=0}^{n}|f(y_i) - f(y_{i-1})| \leq \sum_{i=0}^{n}(|f(y_i)| + |f(y_{i-1})|) \\ &= f(a) + 2\sum_{i=1}^{n-1}|f(y_i)| + f(b) = 2\sum_{i=1}^{n-1}|f(y_i)| \leq 2\sum |f(x_n)|. \end{aligned}$$

The last inequality is due to the fact that y_i are distinct, and $\sum |f(x_n)|$ is the sum of all nonzero values of $|f|$. By taking the supremum of the left side of the estimation, we get

$$V_{[a,b]}f(x) \leq 2\sum |f(x_n)|.$$

Math 2043 Final, Spring 2016

(1) Suppose f is positive and increasing on $[0,+\infty)$. Suppose $F(x)=\int_0^x f(t)dt$. Prove that $\int_0^{+\infty}\frac{dx}{f(x)}$ converges if and only if $\int_0^{+\infty}\frac{xdx}{F(x)}$ converges.

(2) Suppose f is bounded and β is continuous with bounded variation. Suppose g is Riemann-Stieltjes integrable with respect to β, and $\alpha(x)=\int_a^x gd\beta$. Prove that f is Riemann-Stieltjes integrable with respect to α if and only if fg is Riemann integrable with respect to β. Moreover, we have

$$\int_a^b fd\alpha=\int_a^b fgd\beta.$$

(3) A subset Y of X is dense if every element of X is the limit of a sequence in Y. Prove that if a sequence of continuous functions $f_n(x)$ on X converges uniformly on Y, then it converges uniformly on X. In particular, the limit function is continuous on X.

(4) Let $0<p\le 1$ and let $\sum\frac{(-1)^{k_n}}{k_n^p}$ be a rearrangement of $\sum\frac{(-1)^n}{n^p}$. Prove that if $\lim_{n\to\infty}\frac{k_n-n}{n^p}=0$, then $\sum\frac{(-1)^{k_n}}{k_n^p}$ converges and has the same sum as $\sum\frac{(-1)^n}{n^p}$.

Answer to Math 2043 Final, Spring 2016

(1) Suppose $f(x)$ is positive and increasing on $[0,+\infty)$. Suppose $F(x)=\int_0^x f(t)dt$. Prove that $\int_0^{+\infty}\frac{dx}{f(x)}$ converges if and only if $\int_0^{+\infty}\frac{xdx}{F(x)}$ converges.

By f positive and increasing, we have

$$\frac{x}{F(x)}=\frac{x}{\int_0^x f(t)dt}\le\frac{x}{\int_{\frac{x}{2}}^x f(t)dt}\le\frac{x}{\frac{x}{2}f\left(\frac{x}{2}\right)}=\frac{2}{f\left(\frac{x}{2}\right)}$$

By the comparison test, if $\int_0^{+\infty}\frac{2dx}{f\left(\frac{x}{2}\right)}=4\int_0^{+\infty}\frac{dx}{f(x)}$ converges, then $\int_0^{+\infty}\frac{xdx}{F(x)}$ converges.

By f positive and increasing, we also have

$$\frac{x}{F(x)}=\frac{x}{\int_0^x f(t)dt}\ge\frac{x}{xf(x)}=\frac{1}{f(x)}.$$

By the comparison test, if $\int_0^{+\infty}\frac{xdx}{F(x)}$ converges, then $\int_0^{+\infty}\frac{dx}{f(x)}$ converges.

(2) Suppose $|f|<B$ for a constant B. Since g is Riemann-Stieltjes integrable with respect to β, and β is continuous, by Theorem 4.6.5, for any $\epsilon>0$, there is $\delta>0$, such that

$$\|P\|<\delta\implies\sum\omega_{[x_{i-1},x_i]}(g)V_{[x_{i-1},x_i]}(\beta)<\epsilon.$$

For the same choice of P and x_i^*, the Riemann-Stieltjes sum of f with respect to α is

$$S(P,f,\alpha)=\sum f(x_i^*)(\alpha(x_i)-\alpha(x_{i-1}))=\sum f(x_i^*)\int_{x_{i-1}}^{x_i}g(x)d\beta.$$

The Riemann-Stieltjes sum of fg with respect to β is

$$S(P,fg,\beta)=\sum f(x_i^*)g(x_i^*)(\beta(x_i)-\beta(x_{i-1}))=\sum f(x_i^*)\int_{x_{i-1}}^{x_i}g(x_i^*)d\beta.$$

When $\|P\|<\delta$, we have

$$\begin{aligned}|S(P,f,\alpha)-S(P,fg,\beta)| &\le \sum |f(x_i^*)|\left|\int_{x_{i-1}}^{x_i}g(x)d\beta-\int_{x_{i-1}}^{x_i}g(x_i^*)d\beta\right| \\ &= \sum |f(x_i^*)|\left|\int_{x_{i-1}}^{x_i}(g(x)-g(x_i^*))d\beta\right| \\ &\le \sum B\omega_{[x_{i-1},x_i]}(g)V_{[x_{i-1},x_i]}(\beta)<B\epsilon.\end{aligned}$$

This implies that the Riemann-Stieltjes sum of f with respect to α converges if and only if the Riemann-Stieltjes sum of fg with respect to β converges. Moreover, the two limits are the same.

(3) By the Cauchy criterion, for any $\epsilon>0$, there is N (depending only on ϵ), such that

$$m,n>N,\ y\in Y \implies |f_m(y)-f_n(y)|<\epsilon.$$

Now for any $x\in X$, let $x=\lim y_i$ for a sequence $y_i\in Y$. Then

$$m,n>N \implies |f_m(y_i)-f_n(y_i)|<\epsilon \text{ for all } i.$$

Taking $i\to\infty$ on the right side, we get

$$m,n>N \implies |f_m(x)-f_n(x)|\le\epsilon.$$

Since this holds for all $x\in X$, and N depends only on ϵ, this verifies the Cauchy criterion for the uniform convergence of f_n on X.

(4) Let s_n be the partial sum of $\sum\dfrac{(-1)^n}{n^p}$. Let t_n be the partial sum of $\sum\dfrac{(-1)^{k_n}}{k_n^p}$.
For any $\epsilon>0$, there is N, such that

$$n>N \implies |k_n-n|<\epsilon n^p.$$

This means that, for $n>N$, the difference between the partial sums s_n and t_n is a sum of some terms $\dfrac{(-1)^i}{i^p}$ with $n-\epsilon n^p<i<n+\epsilon n^p$. Therefore

$$|s_n-t_n|\le\sum_{[n-\epsilon n^p]-1}^{[n+\epsilon n^p]+1}\frac{1}{i^p}\le\frac{1}{([n-\epsilon n^p]-1)^p}+\int_{n-\epsilon n^p}^{n+\epsilon n^p}\frac{dx}{x^p}+\frac{1}{([n-\epsilon n^p]+1)^p}$$

Here the estimation by the integral is inspired by the proof of the integral comparison test. We have

$$\lim_{n\to\infty} \frac{1}{([n-\epsilon n^p]-1)^p} = \lim_{n\to\infty} \frac{1}{([n-\epsilon n^p]+1)^p} = 0.$$

For $p=1$, we have

$$\int_{n-\epsilon n}^{n+\epsilon n} \frac{dx}{x} = \log \frac{n+\epsilon n}{n-\epsilon n} = \log \frac{1+\epsilon}{1-\epsilon}.$$

For $0<p<1$, we have

$$\begin{aligned}\int_{n-\epsilon n^p}^{n+\epsilon n^p} \frac{dx}{x^p} &= \frac{1}{1-p}[(n+\epsilon n^p)^{1-p} - (n-\epsilon n^p)^{1-p}] = \frac{n^{1-p}}{1-p}[(1+\epsilon n^{p-1})^{1-p} - (1-\epsilon n^{1-p})^{1-p}] \\ &= \frac{n^{1-p}}{1-p} 2(1-p)[\epsilon n^{p-1} + o(\epsilon n^{p-1})] = 2[\epsilon + o(\epsilon)].\end{aligned}$$

In both cases, we can have $\int_{n-\epsilon n^p}^{n+\epsilon n^p} \frac{dx}{x}$ as small as possible when ϵ is small enough. Therefore

$$\lim_{n\to\infty} \int_{n-\epsilon n^p}^{n+\epsilon n^p} \frac{dx}{x^p} = 0,$$

and we get $\lim_{n\to\infty} |s_n - t_n| = 0$. This implies that s_n converges if and only if t_n converges, and they have the same limit value.