Exercise 2.1

$\lim_{x\to+\infty} f(x) = l$ means that, for any $\epsilon > 0$, there is B, such that $x > B$ implies $|f(x)-l| < \epsilon$.

$\lim_{x\to a^+} f(x) = -\infty$ means that, for any b, there is $\delta > 0$, such that $0 < x - a < \delta$ implies $f(x) < b$.

$\lim_{x\to\infty} f(x) = +\infty$ means that, for any b, there is B, such that $|x| > B$ implies $f(x) > b$.

Exercise 2.2

For any $\epsilon > 0$, there is $\delta > 0$, such that

$$0 < |x - a| < \delta \implies |f(x) - g(x)| < \epsilon.$$

One the other hand, $f(x) \le l \le g(x)$ implies $|f(x) - l| \le |f(x) - g(x)|$ and $|g(x) - l| \le |f(x) - g(x)|$. Then

$$0 < |x - a| < \delta \implies |f(x) - l| \le |f(x) - l| < \epsilon, \quad |g(x) - l| \le |f(x) - l| < \epsilon.$$

This proves $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = l$.

Exercise 2.4

1. F, 2. F, 3. F, 4. T, 5. F, 6. T, 7. T, 8. T, 9. F, 10. F

Exercise 2.6

(1) $\lim_{x\to a^-} f(-x) = \lim_{x\to -a^+} f(x)$. The two sides are equivalent.

If $\lim_{x\to -a^+} f(x) = l$, then for $x = -y$, we have

$$y \to a, \ y < a \implies x \to -a, \ x > -a.$$

The first condition of the composition rule is satisfied, and we have $\lim_{y\to a^-} f(-y) = \lim_{x\to -a^+} f(x) = l$.

If $\lim_{x\to a^-} f(-x) = l$, then for $x = -y$, we have

$$y \to -a, \ y > -a \implies x \to a, \ x < a.$$

The first condition of the composition rule is satisfied, and we have $\lim_{y\to -a^+} f(y) = \lim_{x\to a^-} f(-x) = l$.

(2) $\lim_{x\to a^+} f(x+1) = \lim_{x\to (a+1)^+} f(x)$. The two sides are equivalent.

The invertible relation $x = y + 1$ and $y = x - 1$ satisfies

$$x \to a, \ x > a \iff y \to a + 1, \ y \ge a + 1.$$

So the first condition of the composition rule is satisfied in both ways, and we have the equivalence.

(3) For $b > 0$, we have $\lim_{x\to a^+} f(bx + c) = \lim_{x\to (ab+c)^+} f(x)$, and the two sides are equivalent. The reason is that the invertible relation $y = bx + c$ satisfies

$$x \to a, \ x > a \iff y \to ab + c, \ y \ge ab + c,$$

so the first condition of the composition rule is satisfied in both ways.

By the similar reason, for $b < 0$, we have $\lim_{x\to a^+} f(bx+c) = \lim_{x\to(ab+c)^-} f(x)$, and the two sides are equivalent.

For $b = 0$, we have $\lim_{x\to a^+} f(bx+c) = f(c)$.

(4) $\lim_{x\to 0^+} f(x^2) = \lim_{x\to 0^+} f(x)$. The two sides are equivalent, because the invertible relation $x = y^2$ and $y = \sqrt{x}$ satisfies

$$x \to 0,\ x > 0 \iff y \to 0,\ y \geq 0.$$

(5) $\lim_{x\to 0} f((x+1)^3) = \lim_{x\to 1} f(x)$. The two sides are equivalent because the invertible relation $y = (x+1)^3$ and $x = \sqrt[3]{y-1}$ satisfies

$$x \to 0,\ x \neq 0 \iff y \to 1,\ y \neq 1.$$

(6) $\lim_{x\to 0^+} f(\sqrt{x}) = \lim_{x\to 0^+} f(x)$. The two sides are equivalent. Argument omitted.

(7) $\lim_{x\to 0} f\left(\frac{1}{x}\right) = \lim_{x\to\infty} f(x)$. The two sides are equivalent. Argument omitted.

(8) $\lim_{x\to 2^+} f\left(\frac{1}{x}\right) = \lim_{x\to\frac{1}{2}^-} f(x)$. The two sides are equivalent. Argument omitted.

EXERCISE 2.7

(1) Suppose $|f - l| < \epsilon$ and $|g - k| < \epsilon$. Then

$$l - \epsilon < f < l + \epsilon, \quad k - \epsilon < g < k + \epsilon.$$

This implies

$$\max\{l,k\} - \epsilon = \max\{l-\epsilon, k-\epsilon\} \leq \max\{f,g\} \leq \max\{l+\epsilon, k+\epsilon\} = \max\{l,k\} + \epsilon,$$

which is the same as $|\max\{f,g\} - \max\{l,k\}| < \epsilon$.

By $\lim_{x\to a} f(x) = l$ and $\lim_{x\to a} g(x) = k$, for any $\epsilon > 0$, there is $\delta > 0$, such that

$$0 < |x-a| < \delta \implies |f(x) - l| < \epsilon, \quad |g(x) - k| < \epsilon.$$

By the discussion above, this further implies $|\max\{f(x), g(x)\} - \max\{l,k\}| < \epsilon$.

(2) By $\lim_{x\to a^+} f(x) = \infty$, for any $b > 0$, there is $\delta > 0$, such that $0 < x - a < \delta$ implies $f(x) > bc^{-1}$. By the assumption on g, we may further choose δ small enough so that $0 < x - a < \delta$ implies $g(x) > c$. Then $0 < x - a < \delta$ implies $f(x)g(x) > bc^{-1}c = b$. This proves $\lim_{x\to a^+} f(x)g(x) = \infty$.

(3) For any $\epsilon > 0$, by $\lim_{y\to+\infty} f(y) = c$, there is N, such that

$$y > N \implies |f(y) - c| < \epsilon.$$

For this N, by $\lim_{x\to a} g(x) = +\infty$, there is $\delta > 0$, such that

$$0 < |x-a| < \delta \implies g(x) > N.$$

Combining the two implications, we get

$$0 < |x - a| < \delta \implies |f(y) - c| < \epsilon.$$

This proves $\lim_{x\to a} f(g(x)) = c$.

(4) For any b, by $\lim_{x\to+\infty} g(x) = -\infty$, there is N, such that $x > N$ implies $g(x) < b$. Then $x > N$ further implies $f(x) \le g(x) < b$. This proves $\lim_{x\to+\infty} f(x) = -\infty$.

Exercise 2.8

Suppose $\lim_{x\to\infty} f(x) = +\infty$. Then for any B, there is b, such that

$$|x| > b \implies f(x) > B.$$

Now if $\lim x_n = \infty$, then for the b obtained above, there is N, such that

$$n > N \implies |x_n| > b.$$

Combining the two implications together, we get

$$n > N \implies |x_n| > b \implies f(x_n) > B.$$

This is $\lim_{n\to\infty} f(x_n) = +\infty$.

Suppose $\lim_{x\to\infty} f(x) \ne +\infty$. This means that there is B, such that for any b, there is x satisfying $|x| > b$ and $f(x) < B$. By taking b to be the natural number n, we get x_n satisfying $|x_n| > n$ and $f(x_n) < B$. Then $|x_n| > n$ implies $\lim x_n = \infty$, yet $f(x_n) < B$ implies $\lim_{n\to\infty} f(x_n) \ne +\infty$.

Exercise 2.9

The difference between the statement and Proposition 2.1.7 is that the value of the limit is not specified. In other words, we only need to show that the limit of different sequences $f(x_n)$, if always exit, must be the same.

Let x_n and y_n be two sequences satisfying $x_n \ne a$, $y_n \ne a$, $\lim x_n = \lim y_n = a$. Then $z_n = \begin{cases} x_k, & \text{if } n = 2k-1, \\ y_k, & \text{if } n = 2k, \end{cases}$ also satisfies $z_n \ne a$ and $\lim z_n = a$. Therefore $\lim f(z_n)$ also converges. In particular, the subsequences $f(x_n)$ and $f(y_n)$ of $f(z_n)$ have the same limit.

Exercise 2.10

Suppose $\lim_{x\to a^+} f(x) = l$. Then for any $\epsilon > 0$, there is $\delta > 0$, such that

$$0 < x - a < \delta \implies |f(x) - l| < \epsilon.$$

Suppose x_n is a strictly decreasing sequence satisfying $\lim_{n\to\infty} x_n = a$. Then for the $\delta > 0$ above, there is N, such that

$$n > N \implies |x_n - a| < \delta.$$

The fact that the sequence is strictly decreasing further shows the implication above is really

$$n > N \implies 0 < x_n - a < \delta.$$

Combining the two implications together, we get

$$n > N \implies |f(x_n) - l| < \epsilon.$$

This completes the proof that 1 implies 2.

Conversely, suppose $\lim_{x \to a^+} f(x) \neq l$, we will find a strictly decreasing sequence x_n satisfying $\lim_{n\to\infty} x_n = a$ but $\lim_{n\to\infty} f(x_n) = l$ does not hold.

Since $\lim_{x\to a^+} f(x) \neq l$, there is some $\epsilon > 0$ such that for any $\delta > 0$, we can find x satisfying $0 < x - a < \delta$ but $|f(x) - l| > \epsilon$. Now we construct a sequence step by step as follows.

1. For $\delta = 1$, there is x_1 satisfying $0 < x_1 - a < 1$, $|f(x_1) - l| > \epsilon$.
2. For $\delta = \dfrac{1}{2}$, there is x_2 satisfying $0 < x_2 - a < \min\left\{x_1 - a, \dfrac{1}{2}\right\}$, $|f(x_2) - l| > \epsilon$.
3. For $\delta = \dfrac{1}{3}$, there is x_2 satisfying $0 < x_3 - a < \min\left\{x_2 - a, \dfrac{1}{3}\right\}$, $|f(x_3) - l| > \epsilon$.
4. $\cdots$

Since $x_{n+1} - a < x_n - a$, the sequence is strictly decreasing. Since $0 < x_n - a < \dfrac{1}{n}$, the sequence converges to a. Since $|f(x_n) - l| > \epsilon$ for all n, the sequence does not converge to l.

Exercise 2.11

If $f(x)$ is bounded on $(a, b]$, then $l = \inf_{(a,b]} f(x)$ exists. For any $\epsilon > 0$, we have $f(a+\delta) < l+\epsilon$ for some $0 < \delta < b - a$ (i.e., $a + \delta \in (a, b)$). Then for

$$a < x < a + \delta \implies l \leq f(x) \leq f(a + \delta) < l + \epsilon,$$

where the first equality is because l is a lower bound for $f(x)$, the second is because $f(x)$ is increasing. The implication tells us $\lim_{x\to a^+} f(x) = l$.

Assume $f(x)$ is unbounded on $(a, b]$. By $f(x) \leq f(b)$ on $(a, b]$, $f(x)$ ha supper bound. Therefore $f(x)$ has no lower bound. In other words, for any B, there is $0 < \delta < B - a$, such that $f(a + \delta) < B$. Then by the increasing assumption, we get

$$a < x < a + \delta \implies f(x) \leq f(a + \delta) < B.$$

This proves $\lim_{x\to a^+} f(x) = -\infty$.

Exercise 2.12

Suppose $f(x)$ is increasing on $(a, +\infty)$.

1. If $f(x)$ is bounded, then $\lim_{x\to+\infty} f(x)$ converges to $\sup_{(a,+\infty)} f(x)$.
2. If $f(x)$ is unbounded, then $\lim_{x\to+\infty} f(x) = +\infty$.

Exercise 2.13

[Only statements are given]

Suppose $f(x)$ is defined on (a, b). Then $\lim_{x\to a^+} f(x)$ converges if and only if for any $\epsilon > 0$, there is $\delta > 0$, such that

$$a < x, y < a + \delta \implies |f(x) - f(y)| < \epsilon.$$

Suppose $f(x)$ is defined on $(a, +\infty)$. Then $\lim_{x\to+\infty} f(x)$ converges if and only if for any $\epsilon > 0$, there is N, such that

$$x, y > N \implies |f(x) - f(y)| < \epsilon.$$

Exercise 2.14

Suppose for any ϵ, we have δ as described. Then for $a - \delta < x, x' < a$, we fix some y satisfying $a < y < a + \delta$ and get

$$|f(x) - f(x')| \le |f(x) - f(y)| + |f(x') - f(y)| < 2\epsilon.$$

By the same argument, for $a < y, y' < a + \delta$, we also have $|f(y) - f(y')| < 2\epsilon$. Therefore conclude that for x, y satisfying $0 < |x - a| < \delta$, $0 < |y - a| < \delta$, we have $|f(x) - f(y)| < 2\epsilon$. This verifies the Cauchy criterion for the convergence of $\lim_{x\to a} f(x)$.

Exercise 2.15

By applying Proposition 2.1.7 to $\lim_{x\to l} x^p = l^p$ for $l > 0$, we get $\lim_{n\to\infty} x_n = l > 0$ implying $\lim_{n\to\infty} x_n^p = l^p$.

If $\lim_{n\to\infty} x_n = 0^+$, then $p > 0$ implies $\lim_{n\to\infty} x_n^p = 0$ and $p < 0$ implies $\lim_{n\to\infty} x_n^p = +\infty$.

If $\lim_{n\to\infty} x_n = +\infty$, then $p > 0$ implies $\lim_{n\to\infty} x_n^p = +\infty$ and $p < 0$ implies $\lim_{n\to\infty} x_n^p = 0$.

Exercise 2.16

By applying the composition rule to $\lim_{x\to l} x^p = l^p$ for $l > 0$, we get $\lim_{x\to a} f(x) = l > 0$ implying $\lim_{x\to a} f(x)^p = l^p$.

If $\lim_{x\to a} f(x) = 0^+$, then $p > 0$ implies $\lim_{x\to a} f(x)^p = 0$ and $p < 0$ implies $\lim_{x\to a} f(x)^p = +\infty$.

If $\lim_{x\to a} f(x) = +\infty$, then $p > 0$ implies $\lim_{x\to a} f(x)^p = +\infty$ and $p < 0$ implies $\lim_{x\to a} f(x)^p = 0$.

Exercise 2.17

Let $q = |p|$. For any large positve number x, we have $n - 1 \le x \le n$ for some natural number n. Then

$$0 < x^p c^x \le x^q c^x \le n^q c^{n-1} = \frac{1}{c} n^q c^n.$$

By the limit (1.1.5), we have $\lim n^q c^n = 0$. So for any $\epsilon > 0$, there is N, such that $n > N$ implies $n^q c^n < c\epsilon$. Then

$$x > N \implies n-1 \le x \le n,\ n > N \implies 0 < x^p c^x \le \frac{1}{c} n^q c^n < \epsilon.$$

This proves that $\lim_{x\to+\infty} x^p c^x = 0$.

If $c > 1$, then $0 < d = c^{-1} < 1$ and $\lim_{x\to+\infty} x^p c^x = \dfrac{1}{\lim_{x\to+\infty} x^{-p} d^x} = \dfrac{1}{0^+} = +\infty$. If $c = 1$, then $\lim_{x\to+\infty} x^p c^x = \lim_{x\to+\infty} x^p$. Thus we get

$$\lim_{x\to+\infty} x^p c^x = \begin{cases} 0, & \text{if } 0 < c < 1, \\ +\infty, & \text{if } c > 1, \\ 0, & \text{if } c = 1,\ p < 0, \\ +\infty, & \text{if } c = 1,\ p > 0, \\ 1, & \text{if } c = 1,\ p = 0. \end{cases}$$

Exercise 2.18

For $0 < x < 1$, we have $\dfrac{1}{n} \le x \le \dfrac{1}{n-1}$ for some natural number n. Then

$$1 < x^x \le \frac{1}{(n-1)^x} \le \frac{1}{(n-1)^{\frac{1}{n}}}.$$

By Example 1.2.3, we have $\lim \dfrac{1}{(n-1)^{\frac{1}{n}}} = \dfrac{1}{\lim (n-1)^{\frac{1}{n}}} = \dfrac{1}{1} = 1$. Thus for any $\epsilon > 0$, there is

N, such that $n > N$ implies $\left|\frac{1}{(n-1)^{\frac{1}{n}}} - 1\right| < \epsilon$. Then

$$0 < x < \frac{1}{N} \implies \frac{1}{n} \le x \le \frac{1}{n-1} \text{ for some natural number } n > N \implies 0 < x^x - 1 \le \frac{1}{(n-1)^{\frac{1}{n}}} - 1 < \epsilon.$$

This proves that $\lim_{x\to 0^+} x^x = 1$.

The nonzero polynomial $p(x) = x^k q(x)$, with $q(0) \neq 0$. Then $0 < A < |q(x)| < B$ for some constant A, B and x close to 0. For $x > 0$ and x close to 0, we have $0 < A^x < |q(x)|^x < B^x$. By $\lim_{x\to 0} A^x = \lim_{x\to 0} B^x = 1$ and the sandwich rule, we get $\lim_{x\to 0^+} |q(x)|^x = 1$. Then

$$\lim_{x\to 0^+} |p(x)|^x = (\lim_{x\to 0^+} x^x)^k \lim_{x\to 0^+} |q(x)|^x = 1^k \cdot 1 = 1.$$

Exercise 2.19

Since $\lim_{x\to 0^+} x = 0 < c < +\infty = \lim_{x\to 0^+} \frac{1}{x}$, by the order rule, we can find $\delta > 0$ such that $0 < x < \delta$ implies $x < c < \frac{1}{x}$. Then for $x > 0$ we further have $x^x < c^x < \frac{1}{x^x}$. By Exercise 2.17, we have $\lim_{x\to 0^+} x^x = 1$. Then $\lim_{x\to 0^+} \frac{1}{x^x} = \frac{1}{\lim_{x\to 0^+} x^x} = \frac{1}{1} = 1$. By the sandwich rule, we get $\lim_{x\to 0^+} c^x = 1$. This further implies $\lim_{x\to 0^-} c^x = \lim_{x\to 0^+} c^{-x} = \lim_{x\to 0^+} \frac{1}{c^x} = \frac{1}{\lim_{x\to 0^+} c^x} = \frac{1}{1} = 1$. Therefore $\lim_{x\to 0^+} c^x = \lim_{x\to 0^-} c^x = 1$. This implies $\lim_{x\to 0} c^x = 1$.

Exercise ??

(1) $\lim_{x\to 0^+} |x|^x = \lim_{x\to 0^+} x^x = 1$ and $\lim_{x\to 0^-} |x|^x = \lim_{x\to 0^+} |x|^{-x} = \frac{1}{\lim_{x\to 0^+} x^x} = 1$. Therefore $\lim_{x\to 0} |x|^x = 1$.

(2) $\lim_{x\to 0} |x|^|x| = \lim_{x\to 0^+} x^x = 1$.

(3) $\lim_{x\to 0} |x|^{x^2} = \lim_{x\to 0} x^{2\frac{1}{2}x^2} = \lim_{x\to 0^+} x^{\frac{1}{2}x} = \lim_{x\to 0^+} \sqrt{x^x} = \sqrt{1} = 1$.

(4) $\lim_{x\to 0)} x^{\sqrt{x}} = \lim_{x\to 0^+} (x^2)^x = \lim_{x\to 0^+} (x^x)^2 = 1^2 = 1$.

(5) For $x > 1$, we have $0 < x^{-x} < x^{-1}$. By $\lim_{x\to +\infty} x^{-1} = 0$ and the sandwich rule, we get $\lim_{x\to +\infty} x^{-x} = 0$.

(6) By (5), $\lim_{x\to \infty} |x|^{-|x|} = \lim_{x\to +\infty} x^{-x} = 0$

(7) By (1), $\lim_{x\to \infty} |x|^{\frac{1}{x}} = \lim_{x\to 0} \left(\frac{1}{|x|}\right)^x = \frac{1}{\lim_{x\to 0} |x|^x} = \frac{1}{1} = 1$.

(8) $\lim_{x\to +\infty} x^{\frac{1}{x}} = \lim_{x\to 0^+} \left(\frac{1}{x}\right)^x = \frac{1}{\lim_{x\to 0^+} x^x} = \frac{1}{1} = 1$.

(9) By Exercise 2.18, $\lim_{x\to 0^+} (2x + x^3)^{\sqrt{x}} = \lim_{x\to 0^+} (2x^2 + x^6)^x = \lim_{x\to 0^+} (x^x)^2 (2 + x^4)^x = 1^2 2^0 = 1$.

(10) By Exercise 2.18, $\lim_{x\to +\infty} (x + x^3)^{\frac{1}{\sqrt{x}}} = \lim_{x\to +\infty} (x^2 + x^6)^{\frac{1}{x}} = \lim_{x\to 0} \frac{(x^4+1)^x}{(x^x)^6} = 1$.

(11) For big x, we have $1 < x^{\frac{1}{\sqrt{x+x^3}}} < x^{\frac{1}{\sqrt{x}}}$. By $\lim_{x\to +\infty} x^{\frac{1}{\sqrt{x}}} = \lim_{x\to +\infty} (x^2)^{\frac{1}{x}} = \lim_{x\to 0^+} \frac{1}{(x^x)^2} = 1$ and the sandwich rule, we get $\lim_{x\to +\infty} x^{\frac{1}{\sqrt{x+x^3}}} = 1$.

Exercise 2.20

We apply Proposition 2.1.7 to the continuous exponential function $f(x) = c^x$ to get $\lim x_n = l$ implying $\lim c^{x_n} = c^l$.

Proposition 2.1.7 is similar to the composition rule and can be extended to infinity limits. For $c \neq 1$ and $l = \pm\infty$, we still have $\lim c^{x_n} = c^l$, with $c^{+\infty} = +\infty$, $c^{-\infty} = 0$ for $c > 1$ and $c^{+\infty} = 0$, $c^{-\infty} = +\infty$ for $0 < c < 1$.

Exercise 2.21

We apply the composition rule to $x \mapsto y = f(x) \mapsto z = c^y = c^{f(x)}$. The limit $\lim_{x\to a} c^x = c^a$ shows that the second condition of the composition rule is satisfied. Therefore we have $\lim_{x\to a} c^{f(x)} = c^l$.

Exercise 2.22

We may assume $0 < A < 1$ and $B > 1$. Then

$$A^{-|g(x)|} \le f(x)^{g(x)} \le B^{|g(x)|}.$$

Now $\lim_{x\to a} g(x) = 0$ implies $\lim_{x\to a} |g(x)| = 0$. By Exercise 2.21, we know $\lim_{x\to a} A^{-|g(x)|} = \lim_{x\to a} B^{|g(x)|} = 1$. Then by the sandwich rule, we get $\lim_{x\to a} f(x)^{g(x)} = 0$.

Exercise 2.23

By $\lim_{x\to a} f(x) = l > 0$, $f(x)$ satisfies the condition in Exercise 2.22. Moreover, we have $\lim_{x\to a}(g(x) - k) = 0$. By Exercise 2.22, we get $\lim_{x\to a} f(x)^{g(x)-k} = 1$. On the other hand, by Exercise 2.16, we also have $\lim_{x\to a} f(x)^k = l^k$. Therefore

$$\lim_{x\to a} f(x)^{g(x)} = \lim_{x\to a} f(x)^{g(x)-k} f(x)^k = 1 \cdot l^k = l^k.$$

Exercise 2.24

Exponential rule for sequence: If $\lim x_n = l > 0$ and $\lim y_n = k$, then $\lim x_n^{y_n} = l^k$.

By Exercises 2.15 and 2.20, we get $\lim x_n^k = l^k$ and $\lim l^{y_n} = l^k$. Then we find $0 < A < 1$ and $B > 0$, such that $A < x_n < B$ for big n. Then $A^{-|y_n-k|} \le x_n^{y_n-k} \le B^{|y_n-k|}$. By $\lim |y_n - k| = 0$ and what we just proved, we know $\lim A^{-|y_n-k|} = \lim B^{|y_n-k|} = 1$. Therefore by the sandwich rule, we have $\lim x_n^{y_n-k} = 1$. Multiplying the limit with $\lim x_n^k = l^k$, we get

$$\lim_{n\to\infty} x_n^{y_n} = \lim_{n\to\infty} x_n^{y_n-k} \lim_{n\to\infty} x_n^k = l^k.$$

Exercise 2.25

(1) $f(x)=\begin{cases}0 & \text{if } x<\dfrac{1}{2}\\ 1 & \text{if } \dfrac{1}{2}\le x<1\\ 2 & \text{if } 1\le x<\dfrac{3}{2}\\ 3 & \text{if } x\ge \dfrac{3}{2}\end{cases}.$

(2) $f(x)=(2x-1)(x-1)(2x-3)D(x)$, where $D(x)$ is the Dirichlet function.

(3) $f(x)=\dfrac{1}{n}$ for $\dfrac{1}{n+1}\le x<\dfrac{1}{n}$ and natural numbers n, and $f(x)=2$ for $x\ge 1$.

(4) $f(x)=1$ for $\dfrac{1}{2}\le x\le 1$ and $x=\dfrac{3}{2}$. $f(x)=0$ otherwise.

Exercise 2.26

We always have $|a-a|<\delta$. Therefore $|f(a)-l|<\epsilon$ for any $\epsilon>0$. This implies $l=f(a)$.

Exercise 2.28

Suppose f is continuous at a. Then for any $\epsilon>0$, there is $\delta>0$, such that

$$|x-a|<\delta \implies |f(x)-f(a)|<\frac{\epsilon}{2}.$$

Then

$$|x-a|<\delta, |y-a|<\delta \implies |f(x)-f(y)|\le |f(x)-f(a)|+|f(y)-f(a)|<\epsilon.$$

The converse (Cauchy criterion implies continuity) is trivial: taking $y=a$. Note that this is much simpler than the Cauchy criterion for $\lim_{x\to a} f(x)=l$, where you cannot simply take $y=a$ in the Cauchy criterion.

Exercise 2.30

If the function is continuous on $(a,b]$ and $[b,c)$, then by writing down the definition, the function is continuous everywhere on (a,b) and (b,c). It remains to consider the continuity at b. The continuity on $(a,b]$ implies $\lim_{x\to b^-} f(x)=f(b)$. The continuity on $[b,c)$ implies $\lim_{x\to b^+} f(x)=f(b)$. Combining the two, we get $\lim_{x\to b} f(x)=f(b)$, so that f is also continuous at b.

The continuity on (a,b) and $[b,c)$ does not imply the continuity on (a,c). For example, the function $f(x)=\begin{cases}0 & \text{if } x<0\\ 1 & \text{if } x\ge 0\end{cases}$ is continuous on $(-1,0)$ and $[0,1)$, but is not continuous on $(-1,1)$.

Exercise 2.31

We have $\max\{f,g\}=\dfrac{1}{2}(f+g+|f-g|)$. The continuity of f and g implies the continuity of $f+g$ and $f-g$. Since the absolute value function is continuous, by the composition rule, $|f-g|$ is also continuous. Therefore $\max\{f,g\}$ is also continuous.

Exercise 2.32

At any $c \in [a,b]$, we have $\lim_{x\to c} f(x) = f(c)$. Choosing a sequence of rational numbers $r_n \in [a,b]$ converging to c and restricting the limit to the sequence, we get $f(c) = \lim_{n\to\infty} f(r_n) = 0$.

Exercise 2.33

The function is continuous at a if and only if $f(a) = 0$.

If $f(a) \neq 0$, then for $\epsilon = \dfrac{1}{2}|f(a)| > 0$, there are only finitely may x satisfying $|f(x)| \geq \epsilon$. Let $x_1, x_2, \dots, x_n$ be all the places satisfying $x_i \neq a$ and $|f(x_i)| \geq \epsilon$. Then for $\delta = \min |x_i - a|$, we have

$$0 < |x-a| < \delta \implies x \neq x_i, \ , x \neq a \implies |f(x)| < \epsilon \implies |f(x) - f(a)| \geq |f(a)| - |f(x)| > \epsilon.$$

Therefore $f(x)$ does not converge to $f(a)$ as $x \to a$.

If $f(a) = 0$, the for any $\epsilon > 0$, there are only finitely may x satisfying $|f(x)| \geq \epsilon$. Let $x_1, x_2, \dots, x_n$ be all the places satisfying $|f(x_i)| \geq \epsilon$. Then for $\delta = \min |x_i - a|$, we have

$$|x-a| < \delta \implies x \neq x_i \implies |f(x) - f(a)| = |f(x)| < \epsilon.$$

This proves $\lim_{x\to a} f(x) = f(a)$.

Exercise 2.34

If $f(x)$ is the restriction of a continuous function $\tilde{f}(x)$ on $[a,b]$, then

$$\lim_{x\to b^-} f(x) = \lim_{x\to b^-} \tilde{f}(x) = \tilde{f}(b),$$

where the first equality is due to $f(x) = \tilde{f}(x)$ for $x < b$ and x close to b, and the second equality is due to the left continuity of $\tilde{f}$ at b. Thus we conclude that $\lim_{x\to b^-} f(x)$ converges. The other limit $\lim_{x\to a^+} f(x)$ also converges for the same reason.

Conversely, suppose $\lim_{x\to a^+} f(x)$ and $\lim_{x\to b^-} f(x)$ converge. Then define

$$\tilde{f}(x) = \begin{cases} f(x), & \text{if } x \in (a,b), \\ \lim_{x\to a^+} f(x), & \text{if } x = a, \\ \lim_{x\to b^-} f(x), & \text{if } x = b. \end{cases}$$

For any $c \in (a,b)$, we have $\tilde{f}(x) = f(x)$ for x close to c. Then the continuity of $f(x)$ at c implies the continuity of $\tilde{f}(x)$ at c. Moreover, we have

$$\lim_{x\to b^-} \tilde{f}(x) = \lim_{x\to b^-} f(x) = \tilde{f}(b),$$

where the first equality is due to $\tilde{f}(x) = f(x)$ for $x < b$ and x close to b, and the second equality is the definition of $\tilde{f}(b)$. Therefore $\tilde{f}(x)$ is left continuous at b. By the same reason, $\tilde{f}(x)$ is also right continuous at a.

Exercise 2.35

The function is continuous at a if and only if $f(a) = g(a)$.

Exercise 2.36

The left limit of f at a means that for any $\epsilon > 0$, there is $\delta > 0$, such that

$$a - \delta < x < a \implies |f(x) - f(a^-)| < \epsilon.$$

This further implies

$$a - \delta < y < x < a \implies |f(y) - f(a^-)| < \epsilon.$$

Fix x and take $y \to x^-$, we get

$$a - \delta < x < a \implies |f(x^-) - f(a^-)| \le \epsilon.$$

This proves that $f(a^-) = \lim_{x\to a^-} f(x^-) = \lim_{x\to a^-} g(x) = g(a^-)$.

Exercise 2.37

The left limit of f at a means that for any $\epsilon > 0$, there is $\delta > 0$, such that

$$a - \delta < x < a \implies |f(x) - f(a^-)| < \epsilon.$$

This further implies

$$a - \delta < x < y < a \implies |f(y) - f(a^-)| < \epsilon.$$

Fix x and take $y \to x^+$, we get

$$a - \delta < x < a \implies |f(x^+) - f(a^-)| \le \epsilon.$$

This proves that $f(a^-) = \lim_{x\to a^-} f(x^+) = \lim_{x\to a^-} h(x) = h(a^-)$.

Exercise 2.38

Suppose $c \in [a, b]$ satisfies $f(a) < f(c) < f(b)$. Then for sufficiently small $\epsilon > 0$ ($\epsilon < \max\{f(c) - f(a), f(b) - f(c)\}$ is enough), we have $f(a) < f(x) - \epsilon < f(c) < f(x) + \epsilon < f(b)$. By the assumption, we have $f(c_-) = f(c) - \epsilon$ and $f(c_+) = f(c) + \epsilon$ for some $c_-, c_+ \in [a, b]$. Since $f(x)$ is increasing, the relation $f(x) - \epsilon < f(c) < f(x) + \epsilon$ implies $c_- < c < c_+$. Then for $\delta = \min\{c - c_-, c_+ - c\} > 0$, we have

$$\begin{aligned} |x - c| < \delta &\implies c_- < x < c_+ \\ &\implies f(c) - \epsilon = f(c_-) \le f(x) \le f(c_+) = f(c) + \epsilon \\ &\iff |f(x) - f(c)| \le \epsilon. \end{aligned}$$

This proves that $f(x)$ is continuous at c.

If $c \in [a, b]$ satisfies $f(c) = f(b)$, then the increasing assumption on $f(x)$ implies $f(x) = f(c)$ for $x \in [c, b]$. By considering $0 < \epsilon < f(c) - f(a)$ and choosing c_- only, the same proof also leads to the continuity at c. The case $f(a) = f(c)$ is similar.

Exercise 2.39

Suppose $f(x)$ satisfies $f(c^+) - f(c^-) > \epsilon$ at $c_1, c_2, \dots, c_n$. By rearranging the order, we may assume $c_1 < c_2 < \cdots < c_n$. Then by the increasing assumption, we have $f(a) \le f(c_i^+) \le f(c_{i+1}^-) \le f(b)$. Thus

$$\begin{aligned} n\epsilon &< (f(c_1^+) - f(c_1^-)) + (f(c_2^+) - f(c_2^-)) + \cdots + (f(c_n^+) - f(c_n^-)) \\ &= -f(c_1^-) - (f(c_2^-) - f(c_1^+)) - (f(c_3^-) - f(c_2^+)) - \cdots - (f(c_n^-) - f(c_{n-1}^+)) + f(c_n^+) \\ &\le -f(c_1^-) + f(c_n^+) \le f(b) - f(a). \end{aligned}$$

This implies $n < \dfrac{b-a}{\epsilon}$, and there are only finitely many c satisfying $f(c^+) - f(c^-) > \epsilon$.

Now for any natural number n, there are finitely many c satisfying $f(c^+) - f(c^-) > \dfrac{1}{n}$. Since there are countably many n, we conclude that there are countably many c satisfying $f(c^+) - f(c^-) > 0$. If c is not one of these countably many points, then we have $f(c^+) = f(c^-)$. By the increasing property, we have $f(c^-) \le f(c) \le f(c^+)$. Thus we have $\lim_{x\to c^+} f(x) = f(c^+) = f(c) = f(c^-) = \lim_{x\to c^-} f(x)$. In other words, the function is continuous at c.

EXERCISE 2.40

(1) By Theorem 2.4.1, the function is uniformly continuous on $[0,3]$. Therefore it is also uniformly continuous on the smaller interval $(0,3)$.

(2) For any $\epsilon > 0$, we may take $\delta = \epsilon$ and get

$$1 \le x,\ y \le 3,\ |x-y| < \delta \implies \left|\frac{1}{x} - \frac{1}{y}\right| = \frac{|x-y|}{xy} \le |x-y| < \delta = \epsilon.$$

Therefore the function is uniformly continuous.

(3) For $\epsilon > 1$ and any $\delta > 0$, take a big $b > \frac{1}{\delta}$. Then for $x = \frac{1}{b}$ and $y = \frac{1}{b+1}$, we have $|x-y| = \frac{1}{b(b+1)} < \frac{1}{b} < \delta$ and $\left|\frac{1}{x} - \frac{1}{y}\right| = 1 = \epsilon$. Therefore the function is not uniformly continuous.

(4) For any $\epsilon > 0$, we may take $\delta = 3\epsilon$ and get

$$x,\ y \ge 1,\ |x-y| < \delta \implies |\sqrt[3]{x} - \sqrt[3]{y}| = \frac{|x-y|}{|\sqrt[3]{x^2} + \sqrt[3]{xy} + \sqrt[3]{y^2}|} \le \frac{|x-y|}{3} < \epsilon.$$

Therefore the function is uniformly continuous on $[1,\infty)$. By the similar argument, the function is also uniformly continuous on $(-\infty,-1]$. Moreover, by Theorem 2.4.1, the function is uniformly continuous on $[-1,1]$. Combining the three intervals (see Exercise 2.41), we see that the function is uniformly continuous on $(-\infty,\infty)$.

(5) For $x > 0$ and $\delta > 0$, we have

$$(x+\delta)^{\frac{3}{2}} - x^{\frac{3}{2}} = \frac{(x+\delta)^3 - x^3}{(x+\delta)^{\frac{3}{2}} + x^{\frac{3}{2}}} = \frac{3x^2\delta + 3x\delta^2 + \delta^3}{(x+\delta)^{\frac{3}{2}} + x^{\frac{3}{2}}} \ge \frac{3x^2\delta}{2x^{\frac{3}{2}}} = \frac{3}{2}x^{\frac{1}{2}}\delta.$$

This implies $\lim_{x\to+\infty}[(x+\delta)^{\frac{3}{2}} - x^{\frac{3}{2}}] = +\infty$ for any $\delta > 0$. In particular, for $\epsilon = 1$ and any $\delta > 0$, there is a big x, such that $y = x+\delta$ satisfies $|x-y| = \delta$ and $|f(y) - f(x)| > 1$. This shows that the function is not uniformly continuous.

(6) For any $\epsilon > 0$, we may take $\delta = \epsilon$ and get

$$|x-y| < \delta \implies |\sin x - \sin y| = 2\left|\sin\frac{x-y}{2}\cos\frac{x+y}{2}\right| \le 2\left|\sin\frac{x-y}{2}\right| = |x-y| < \delta = \epsilon.$$

Therefore the function is uniformly continuous.

(7) For $\epsilon = 1$ and any $\delta > 0$, we may take a natural number n satisfying $\delta > \frac{1}{(n-1)\pi}$. Then for $x = \frac{2}{(2n-1)\pi}$ and $y = \frac{2}{(2n+1)\pi}$, we have $|x-y| < x < \delta$ and $|f(x) - f(y)| = 2 > \epsilon$. Therefore the function is not uniformly continuous.

(8) By defining $f(0) = 0$, the function is extended to a continuous function on $[0,1]$. By Theorem 2.4.1, the extended function is uniformly continuous. Restricting to $(0,1]$ again, we know the function $f(x)$ is continuous on $(0,1]$.

(9) By defining $f(0) = 1$, the function is extended to a continuous function on $[0, 1]$. By Theorem 2.4.1, the extended function is uniformly continuous. Restricting to $(0, 1]$ again, we know the function $f(x)$ is continuous on $(0, 1]$.

(10) [The argument for this example is elaborated in Exercise 2.42]

We have $\lim_{x\to\infty} f(x) = e$ and $\lim_{x\to 0^+} f(x) = 1$. Thus for any $\epsilon > 0$, by Cauchy criterion, there are $N > \delta > 0$, such that

$$x, y > N \implies |f(x) - f(y)| < \epsilon,$$

and

$$0 < x, y < \delta \implies |f(x) - f(y)| < \epsilon.$$

Pick any $0 < \mu < \delta$ (say $\mu = \frac{\delta}{2}$). Then $f(x)$ is also continuous on $[\delta - \mu, N + \mu]$. By Theorem 2.4.1, the function is also uniformly continuous on the interval. Therefore there is $0 < \delta' < \mu$, such that

$$x, y \in [\delta - \mu, N + \mu],\ |x - y| < \delta' \implies |f(x) - f(y)| < \epsilon.$$

The interval $(0, +\infty)$ is covered by three intervals $(0, \delta)$, $[\delta - \mu, N + \mu]$ and $(N, +\infty)$. The two overlapings between the three intervals have length $\mu > \delta'$. Therefore any $x, y \in (0, +\infty)$ satisfying $|x - y| < \delta'$ will be both in one of the three intervals. In each of the three cases, we have arranged to have $|f(x) - f(y)| < \epsilon$. Therefore we have

$$x, y \in (0, +\infty),\ |x - y| < \delta' \implies |f(x) - f(y)| < \epsilon.$$

This shows that $f(x)$ is uniformly continuous.

Exercise 2.41

For any $\epsilon > 0$, there is $\delta > 0$, such that

$$x, y \in (a, b],\ |x - y| < \delta \implies |f(x) - f(y)| < \epsilon,$$
$$x, y \in [b, c),\ |x - y| < \delta \implies |f(x) - f(y)| < \epsilon.$$

Now consider $x, y \in (a, c)$ satisfying $|x - y| < \delta$. There are three possibilities.

1. $x, y \in (a, b]$: Then we have $|f(x) - f(y)| < \epsilon$.
2. $x, y \in [b, c)$: Then we have $|f(x) - f(y)| < \epsilon$.
3. $x \in (a, b]$, $y \in [b, c)$ (or the other way around): Then $x, b \in (a, b]$ implies $|f(x) - f(b)| < \epsilon$ and $b, y \in [b, c)$ implies $|f(b) - f(y)| < \epsilon$. Thus we have $|f(x) - f(y)| \le |f(x) - f(b)| + |f(b) - f(y)| < 2\epsilon$.

So we always conclude $|f(x) - f(y)| < 2\epsilon$. This completes the proof of the uniform continuity on (a, c).

The result still holds as long as there is overlapping among intervals. For non-overlapping counterexample, we note that the constant functions 0 and 1 are uniformly continuous on

$(-1,0)$ and $[0,1)$, respectively. However, the function is not continuous, let alone uniformly continuous, on $(-1,1)$.

EXERCISE 2.42

(1) If a and b are finite, then by defining $f(a)=\lim_{x\to a^+}f(x)$ and $f(b)=\lim_{x\to b^-}f(x)$, f becomes a continuous function on $[a,b]$. See Exercise 2.34. By Theorem 2.4.1, the extended function is uniformly continuous on $[a,b]$. Restricting back to (a,b), we find the original f is uniformly continuous on the open interval.

The case a is finite and $b=+\infty$ can be proved by the method similar Exercise 2.40(10). First by defining $f(a)=\lim_{x\to a^+}f(x)$, we may assume $f(x)$ is actually continuous on $[a,\infty)$.

For any $\epsilon>0$, by applying the Cauchy criterion to the convergence of $\lim_{x\to+\infty}f(x)$, there is $N>a$, such that

$$x,y>N \implies |f(x)-f(y)|<\epsilon.$$

By Theorem 2.4.1, the continuous function $f(x)$ on the bounded closed interval $[a,N+1]$ is uniformly continuous. Therefore there is $0<\delta<1$, such that

$$x,y\in[a,N+1],\ |x-y|<\delta \implies |f(x)-f(y)|<\epsilon.$$

Now assume $x,y\in[a,+\infty)$ satisfy $|x-y|<\delta$. Since $[a,+\infty)$ is covered by $[a,N+1]$ and $(N,+\infty)$, and the overlapping of the two intervals has length $1>\delta$, we see that either both $x,y\in[a,N+1]$ or both $x,y\in(N,+\infty)$. In the first case, we have $|f(x)-f(y)|<\epsilon$ by the uniform continuity. In the second case, we have $|f(x)-f(y)|<\epsilon$ by the Cauchy criterion. Combining both cases, we always have $|f(x)-f(y)|<\epsilon$. This completes the proof of uniform continuity.

(2) The uniform continuity tells us that for any $\epsilon>0$, there is $\delta>0$, such that

$$x,y\in(a,b),\ |x-y|<\delta \implies |f(x)-f(y)|<\epsilon.$$

Then we have

$$a<x,y<a+\delta \implies x,y\in(a,b),\ |x-y|<\delta \implies |f(x)-f(y)|<\epsilon.$$

Thus the Cauchy criterion for the convergence of $\lim_{x\to a^+}f(x)$ is verified.

(3) The function $\sqrt{x}$ is uniformly continuous on $(1,+\infty)$. However, the limit $\lim_{x\to b^-}f(x)=\lim_{x\to+\infty}f(x)$ diverges.

(4) If the function $\sin\frac{1}{x}$ were uniformly continuous on $(0,1)$, then by the second part, $\lim_{x\to 0^+}\sin\frac{1}{x}$ would be convergent. Since we know the limit diverges, the function is not uniformly continuous.

EXERCISE 2.43

For any $\epsilon>0$, take $\delta=\frac{\epsilon}{L}$. Then $|x-y|<\delta$ implies $|f(x)-f(y)|\le L|x-y|<L\delta=\epsilon$.

EXERCISE 2.44

By Theorem 2.4.1, $f(x)$ is continuous on $[0,2p]$. Therefore for any $\epsilon>0$, there is $p>\delta>0$, such that

$$x,y\in[0,2p],\ |x-y|<\delta \implies |f(x)-f(y)|<\epsilon.$$

Now for any x,y satisfying $|x-y|<\delta$, by $\delta<p$, we can find an integer n, such that $x'=x+np, y'=y+np\in[0,2p]$. Then by $|x'-y'|=|x-y|<\delta$ and the periodic property of $f(x)$, we get $|f(x)-f(y)|=|f(x')-f(y')|<\epsilon$. This means that f is uniformly continuous on the whole real line.

Exercise 2.45

If $f(x)$ and $g(x)$ are uniformly continuous, then $f(x)+g(x)$ is uniformly continuous. Specifically, for any $\epsilon>0$, there are $\delta,\delta'>0$, such that

$$|x-y|<\delta \implies |f(x)-f(y)|<\epsilon,\quad |x-y|<\delta' \implies |g(x)-g(y)|<\epsilon.$$

This implies that

$$|x-y|<\min\{\delta,\delta'\} \implies |(f(x)+g(x))-(f(y)+g(y))|\le|f(x)-f(y)|+|g(x)-g(y)|<2\epsilon.$$

The product of uniformly continuous functions may not be uniformly continuous. For example, x is uniformly continuous on $(-\infty,+\infty)$. However, x^2 is not uniformly continuous on $(-\infty,+\infty)$.

If $f(x)$ and $g(y)$ are uniformly continuous, then $g(f(x))$ is uniformly continuous. Specifically, for any $\epsilon>0$, there is $\mu>0$, such that

$$|y-y'|<\mu \implies |g(y)-g(y')|<\epsilon.$$

Then for this μ, there is $\delta>0$, such that

$$|x-x'|<\delta \implies |f(x)-f(x')|<\mu.$$

Combining the two implications, we get

$$|x-x'|<\delta \implies |g(f(x))-g(f(x'))|<\epsilon.$$

The sum, the difference (by the same argument) and the composition of uniformly continuous functions are still uniformly continuous. Moreover, $|x|$ is uniformly continuous. Therefore, for uniformly continuous f and g, $\max\{f,g\}=\dfrac{1}{2}(f+g+|f-g|)$ is also uniformly continuous.

Exercise 2.46

$g(x)$ is clearly increasing and satisfies $g(x)\ge f(x)$. By the continuity of $f(x)$, it is uniformly continuous on $[a,b]$. Thus for any $\epsilon>0$, there is $\delta>0$, such that

$$x,y\in[a,b],\ |x-y|<\delta \implies |f(x)-f(y)|<\epsilon.$$

This implies $\sup_{(x-\delta,x+\delta)} f\le f(x)+\epsilon$.

Now for $a \le x < y < x + \delta$, $y \le b$, we have

$$g(x) \le g(y) = \max\{\sup_{[a,x]} f, \sup_{[x,y]} f\} \le \max\{\sup_{[a,x]} f, \sup_{(x-\delta,x+\delta)} f\} \le \max\{g(x), f(x) + \epsilon\} \le g(x) + \epsilon.$$

In particular, this shows that $|x - y| < \delta$ implies (by exchanging x and y in case $x > y$) that $|g(x) - g(y)| < \epsilon$. Thus g is (uniformly) continuous.

Exercise 2.47

Suppose c_{n-1} is constructed. Then by the continuity of f at c_{n-1}, there is $\delta > 0$, such that $|f(x) - f(c_{n-1})| < \epsilon$ on $(c_{n-1} - \delta, c_{n-1} + \delta)$. This shows that $c_n \ge c_{n-1} + \delta > c_{n-1}$.

Suppose there are infinitely many stops. Then the strictly increasing sequence c_n has a limit $C = \lim c_n$ (which may or may not be b). By the continuity of f at C, there is $\delta > 0$, such that (there is no "right side" of C if $C = b$)

$$C - \delta < x \le C \implies |f(x) - f(C)| < \frac{\epsilon}{2}.$$

Since $C = \lim c_n$, we have some c_n satisfying $C - \delta < c_n < C$. Then for any $x \in [c_n, C]$, we have

$$|f(x) - f(c_n)| \le |f(x) - f(C)| + |f(c_{n-1}) - f(C)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

This shows that $c_{n+1} = C$, so that the process of constructing the sequence c_n stops at the $(n+1)$-st step. This proves that

$$[a, b] = [c_0, c_1] \cup [c_1, c_2] \cup \cdots \cup [c_{n-1}, c_n]$$

for some finitely many and strictly increasing c_n. We note that by the way the sequence is constructed, we have $|f(x) - f(y)| \le \epsilon$ on any $[c_{k-1}, c_k]$.

Let $\delta = \min\{c_1 - c_0, c_2 - c_1, \ldots, c_n - c_{n-1}\}$ (a choice independent of x or y). For any $x, y \in [a, b]$ satisfying $|x - y| < \delta$, we have either $x, y \in [c_{k-1}, c_k]$ for some k or $x \in [c_{k-1}, c_k]$ and $y \in [c_k, c_{k+1}]$ for some k. In the first case, we have

$$|f(x) - f(y)| \le \epsilon,$$

In the second case, we have

$$|f(x) - f(y)| \le |f(x) - f(c_k)| + |f(c_k) - f(y)| \le 2\epsilon.$$

This complete the proof that f is uniformly continuous on $[a, b]$.

[Note: For the case of continuous $f(x)$ on $[a, b)$ such that $\lim_{x \to b^-} f(x)$ converges, the proof above can be modified to prove the uniform continuity. You may need to add that when $C = b$, the argument is based on the Cauchy criterion for the existence of $\lim_{x \to b^-} f(x)$.]

Exercise 2.48

$\frac{1}{x}$ is continuous and not bounded on $(0, 1)$. x is bounded and continuous on $(0, 1)$ but has no maximum (and no minimum). $\sin 2\pi x$ is continuous and bounded on $(0, 1)$, and reaches its maximum and minimum at $\frac{1}{4}$ and $\frac{3}{4}$. $f(x) = \begin{cases} x^{-1} & \text{if } 0 < x \le 1 \\ 0 & \text{if } x = 0 \end{cases}$ is not continuous and not bounded on $[0,1]$. The Dirichlet function is not continuous on $[0,1]$ but reaches its maximum 1 and minimum 0. $\frac{1}{1+x^2}$ is continuous and bounded on $(-\infty,+\infty)$ but does not reach maximum and minimum. $\sin x$ is continuous and bounded on $(-\infty,+\infty)$, and reaches maximum 1 and minimum -1.

Exercise 2.49

Let c be any point inside (a,b). Since the limits are $-\infty$, there are $a < a' < b' < b$, such that

$$a < x < a' \text{ or } b' < x < b \implies f(x) < f(c).$$

This implies $c \in [a',b']$. By Theorem 1.4.5, the restriction of $f(x)$ on $[a',b']$ reaches its maximum β. Since $c \in [a',b']$, we have $\beta \ge f(c) > f(x)$ for any x in $(a,b) - [a',b'] = (a,a') \cup (b',b)$. Therefore β is also the maximum of $f(x)$ on (a,b).

Exercise 2.50

The function $|f(x)|$ is also continuous. Therefore it reaches its minimum at some $c \in [a,b]$. By the assumption, there is $d \in [a,b]$, such that $|f(d)| \le \frac{1}{2}|f(c)|$. If $f(c) \ne 0$, then we get $|f(d)| < |f(c)|$, contradicting to the assumption that $|f|$ reaches minimum at c. Therefore we conclude $|f(c)| = 0$, which is the same as $f(c) = 0$.

Exercise 2.51

For $\epsilon = 1 > 0$, there is δ, such that $x - y < 2\delta$ implies $|f(x) - f(y)| < \epsilon = 1$. This implies that any $y \in (x-\delta, x+\delta)$ satisfies $f(x) - 1 < f(y) < f(x) + 1$, so that f is bounded on any open interval $(x-\delta, x+\delta)$ of length 2δ. Since the bounded interval I is covered by finitely many open intervals of length 2δ, f is bound on I.