Exercise 2.52

Given any polynomial $p(x)$ of odd degree and any y, $p(x) - y$ is still a polynomial of odd degree. By Example 2.5.2, $p(x) - y$ has a root c, which means $p(c) = y$.

Exercise 2.53

The function $f(x) = 2^x - 3x$ is continuous and satisfies $f(0) = 1$ and $f(1) = -1$. Since $f(0) > 0 > f(1)$, by the intermediate value theorem, there is $x \in (0, 1)$ such that $f(x) = 2^x - 3x = 0$.

The function $f(x) = 3^x - x^2$ is continuous and satisfies $f(-1) = -\dfrac{2}{3}$ and $f(0) = 1$. Since $f(-1) < 0 < f(0)$, by the intermediate value theorem, there is $x \in (-1, 0)$ such that $f(x) = 3^x - x^2 = 0$.

Exercise 2.54

The assertion is the same as the following: If a continuous function $f(x)$ on an interval is neither always positive nor always negative, then $f(x)$ must be zero somewhere.

The assumption is that $f(x)$ is positive at x_1 and negative at x_2. Since 0 is between $f(x_1)$ and $f(x_2)$, by the intermediate value theorem, there is c between x_1 and x_2, such that $f(c) = 0$.

Exercise 2.55

If $f(x)$ was not constant, then there are $x_1, x_2 \in (a, b)$ satisfying $f(x_1) < f(x_2)$. Let γ be an irrational number satisfying $f(x_1) < \gamma < f(x_2)$. By the intermediate value theorem, there is c between x_1 and x_2 satisfying $f(c) = \gamma$, contradicting to the assumption that $f(x)$ only takes rational numbers as values. Thus we conclude that $f(x)$ is constant.

Exercise 2.56

Apply intermediate value theorem to $f(x) - g(x)$.

Exercise 2.57

Consider $g(x) = f(x) - x$. Since $0 \le f(x) \le 1$ for any $0 \le x \le 1$, we have $g(0) = f(0) - 0 \ge 0$ and $g(1) = f(1) - 1 \le 0$. Then by the intermediate value theorem, we have $g(c) = 0$, or $f(x) = x$, for some $0 \le c \le 1$.

Exercise 2.58

Suppose $f(x)$ is continuous. Then it reaches its minimum α at exactly two places x_1 and x_2. It also reaches its maximum β at exactly two places y_1 and y_2. We may assume $x_1 < x_2$ and $y_1 < y_2$.

Suppose $x_1 < x_2 < y_1$. Then the restriction of $f(x)$ on $[x_1, x_2]$ reaches its maximum at $c \in (x_1, x_2)$. Pick any $\alpha < \gamma < f(c)$. Then by applying the intermediate value theorem to intervals $[x_1, c]$, $[c, x_2]$, $[x_2, y_1]$, together with $f(x_1) = \alpha < \gamma < f(c)$, $f(c) > \gamma > \alpha = f(x_2)$, $f(x_2) = \alpha < \gamma < f(c) \le \beta = f(y_2)$, we get three distinct points in the three intervals with value all equal to c.

Suppose $x_1 < y_1 < x_2 < y_2$. Then taking $c = y_1$ and repeat the argument as above, we also get three distinct points where f has the same value.

All the other cases have the similar argument and end up with at least three points where f takes the same value. The contradiction shows that $f(x)$ cannot be continuous.

Exercise 2.59

By $\lim_{x\to a} f(g(x)) = l$, for any $\epsilon > 0$, there is $\delta > 0$, such that

$$0 < |x-a| < \delta \implies |f(g(x)) - l| < \epsilon.$$

Then by the assumption, we have $g(b) > g(a)$ for some $b \in (a-\delta, a+\delta)$.

Let $\mu = g(b) - g(a) > 0$. Then

$$\begin{aligned} 0 < y - g(a) < \mu &\implies g(a) < y < g(a) + \mu = g(b) \\ &\implies y = g(c) \text{ for some } c \in (a,b) \\ &\implies y = g(c) \text{ for some } c \text{ satisfying } 0 < |c-a| < |b-a| < \delta \\ &\implies |f(y) - l| = |f(g(c)) - l| < \epsilon. \end{aligned}$$

The second $\implies$ is obtained by applying the intermediate value theorem to the continuous function $g(x)$ on $[a,b]$. The implication above means $\lim_{y\to g(a)^+} f(y) = l$.

Exercise 2.62

Since $f(x)$ is strictly decreasing, it is one-to-one. To show f is invertible, it is sufficient to show $f\colon [a,b) \to (\beta, \alpha]$ is onto.

For any $\epsilon > 0$, by $\beta + \epsilon > \lim_{x\to b^-} f(x)$, we have $\beta + \epsilon > f(c)$ for some $c \in [a,b)$. By the intermediate value theorem, any number in $[\beta+\epsilon, \alpha] \subset [f(c), f(a)]$ is reached by $f(x)$ somewhere on $[a,c] \subset [a,b)$. Since ϵ is arbitrary, any number in $(\beta, \alpha] = \cup_{\epsilon>0}[\beta+\epsilon, \alpha]$ is reaches by $f(x)$ somewhere on $[a,b)$. Thus $f(x)$ is onto.

The inverse f^{-1} is strictly decreasing. Then by Proposition 2.1.8, $\lim_{y\to\beta^+} f^{-1}(y)$ converges. Let the limit be c. We actually have $\lim_{y\to\beta^+} f^{-1}(y) = c^+$ by the decreasing property. Then by $\lim_{x\to c^+} f(x) = \begin{cases} f(c), & \text{if } a \le c < b \\ \beta, & \text{if } c = b \end{cases}$ and the composition rule for the function limit, we get

$$\beta = \lim_{y\to\beta^+} y = \lim_{y\to\beta^+} f(f^{-1}(y)) = \begin{cases} f(c), & \text{if } a \le c < b \\ \beta, & \text{if } c = b \end{cases}.$$

Since the equality holds only in the second case, we conclude that $c = b$.

Exercise 2.65

Let $y_n = \log x_n$. Then $x_n = e^{y_n}$. By $\lim_{n\to\infty} x_n = l > 0$ and the continuity of log, we have $\lim_{n\to\infty} y_n = \log l$. Then by Exercise **??**, we have

$$\lim_{n\to\infty} \log \sqrt[n]{x_1 x_2 \cdots x_n} = \lim_{n\to\infty} \frac{\log(x_1 x_2 \cdots x_n)}{n} = \lim_{n\to\infty} \frac{y_1 + y_2 + \cdots + y_n}{n} = \log l.$$

Since e^x is continuous, we have

$$\lim_{n\to\infty} \sqrt[n]{x_1 x_2 \cdots x_n} = e^{\lim_{n\to\infty} \log \sqrt[n]{x_1 x_2 \cdots x_n}} = e^{\log l} = l.$$

Exercise 2.66

By invertible change of variable $x = e^y, y = \log x$, we have

$$\lim_{x \to +\infty} \frac{(\log x)^p}{x} = \lim_{y \to +\infty} \frac{y^p}{e^y} = \lim_{y \to +\infty} y^p (e^{-1})^y = 0.$$

Exercise 2.17 is used in the last step.

By invertible change of variable $x = e^{-y}, y = -\log x$, we also have

$$\lim_{x \to 0^+} x |\log x|^p = \lim_{y \to +\infty} e^{-y} y^p = \lim_{y \to +\infty} y^p (e^{-1})^y = 0.$$

Exercise 2.67

(1) Let $\lim_{x\to a} f(x) = l > c > 1$. Then there is $\delta > 0$, such that $0 < |x-a| < \delta$ implies $f(x) > c$. Since $\lim_{x\to a} g(x) = +\infty$, for any natural number n, there is $\delta' > 0$, such that $0 < |x-a| < \delta'$ implies $g(x) > n$. Then $0 < |x-a| < \min\{\delta, \delta'\}$ implies $f(x)^{g(x)} > c^n$. Writing $c = 1+\gamma$. Then $c > 1$ implies $\gamma > 0$, and

$$c^n = 1 + n\gamma + \frac{n(n-1)}{2}\gamma^2 + \cdots > n\gamma.$$

The analysis shows that for any $b > 0$, if we take n to be a natural number satisfying $n > \dfrac{b}{\gamma}$ (and then find δ'), then $0 < |x-a| < \min\{\delta, \delta'\}$ implies $f(x)^{g(x)} > b$. This completes the proof of $\lim_{x\to a} f(x)^{g(x)} = +\infty$.

(2) Let $\lim_{x\to a} g(x) = k > c > 0$. Then there is $\delta > 0$, such that $0 < |x-a| < \delta$ implies $g(x) > c$. By $f(x) > 0$, $\lim_{x\to a} f(x) = 0$, for any $1 > \epsilon > 0$, there is $\delta' > 0$, such that $0 < |x-a| < \delta'$ implies $0 < f(x) < \epsilon^{c^{-1}}$. Then for $0 < |x-a| < \min\{\delta, \delta'\}$, we have $0 < f(x)^{g(x)} < f(x)^c < \left(\epsilon^{c^{-1}}\right)^c = \epsilon$. This completes the proof of $\lim_{x\to a} f(x)^{g(x)} = 0$.

[Further Exponential Rules]

(1) Suppose $\lim_{x\to a} f(x) = l$ with $0 < l < 1$ and $\lim_{x\to a} g(x) = +\infty$. Then $\lim_{x\to a} \dfrac{1}{f(x)} = \dfrac{1}{l}$ with $\dfrac{1}{l} > 1$. By $l^{+\infty} = +\infty$ for $l > 1$, we get $\lim_{x\to a} \dfrac{1}{f(x)^{g(x)}} = \lim_{x\to a} \left(\dfrac{1}{f(x)}\right)^{g(x)} = +\infty$. This implies $\lim_{x\to a} f(x)^{g(x)} = \dfrac{1}{+\infty} = 0$.

(2) Suppose $\lim_{x\to a} f(x) = l$ with $l > 1$ and $\lim_{x\to a} g(x) = -\infty$. Then $\lim_{x\to a}(-g(x)) = +\infty$.By $l^{+\infty} = +\infty$ for $l > 1$, we get $\lim_{x\to a} f(x)^{-g(x)} = +\infty$. This implies $\lim_{x\to a} f(x)^{g(x)} = \dfrac{1}{\lim_{x\to a} f(x)^{-g(x)}} = \dfrac{1}{+\infty} = 0$.

(3) Suppose $f(x) > 0$, $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = k < 0$. Then $\lim_{x\to a}(-g(x)) = -k > 0$. By $(0^+)^k = 0$ for $k > 0$, we get $\lim_{x\to a} f(x)^{-g(x)} = 0$. Taking reciprocal and using $f(x)^{-g(x)} > 0$, we get $\lim_{x\to a} f(x)^{g(x)} = +\infty$.

(4) Suppose $\lim_{x\to a} f(x) = +\infty$ and $\lim_{x\to a} g(x) = k > 0$. Then $f(x) > 0$ for x near a and $\lim_{x\to a} \dfrac{1}{f(x)} = 0$. By $(0^+)^k = 0$ for $k > 0$, we get $\lim_{x\to a} \left(\dfrac{1}{f(x)}\right)^{g(x)} = 0$. Taking reciprocal, we get $\lim_{x\to a} f(x)^{g(x)} = 0$.

Exercise 2.68

$\lim_{x\to+\infty} 2^x = +\infty$, $\lim_{x\to+\infty} \dfrac{1}{x} = 0$, $\lim_{x\to+\infty} (2^x)^{\frac{1}{x}} = 2 \neq 1$.

$\lim_{x\to 0^+} x^x = 1$, $\lim_{x\to 0^+} \dfrac{1}{x} = +\infty$, $\lim_{x\to 0^+} (x^x)^{\frac{1}{x}} = 0 \neq 1$.

$\lim_{x\to+\infty} \dfrac{1}{2^x} = 0$, $\lim_{x\to+\infty} \dfrac{1}{x} = 0$, $\lim_{x\to+\infty} \left(\dfrac{1}{2^x}\right)^{\frac{1}{x}} = \dfrac{1}{2} \neq 0$ and 1.

Exercise 2.69

For any $\epsilon > 0$, there is $\delta > 0$, such that $0 < x < \delta$ implies $\left|\frac{f(x)}{x} - 1\right| < \epsilon$. The inequality is the same as $|f(x) - x| < \epsilon x$. Then for $n > \frac{1}{\delta}$, we have

$$\left|f\left(\frac{k}{n^2}\right) - \frac{k}{n^2}\right| < \epsilon\frac{k}{n^2}, \quad 1 \le k \le n.$$

Adding all the inequalities together, we get

$$\begin{aligned}&\left|\left(f\left(\frac{1}{n^2}\right) + f\left(\frac{2}{n^2}\right) + f\left(\frac{3}{n^2}\right) + \cdots + f\left(\frac{n}{n^2}\right)\right) - \left(\frac{1}{n^2} + \frac{2}{n^2} + \frac{3}{n^2} + \cdots + \frac{n}{n^2}\right)\right| \\ &\le \epsilon\left(\frac{1}{n^2} + \frac{2}{n^2} + \frac{3}{n^2} + \cdots + \frac{n}{n^2}\right) = \epsilon\frac{n+1}{2n} \le \epsilon.\end{aligned}$$

Since $\lim_{n\to\infty}\left(\frac{1}{n^2} + \frac{2}{n^2} + \frac{3}{n^2} + \cdots + \frac{n}{n^2}\right) = \lim_{n\to\infty}\frac{n+1}{2n} = \frac{1}{2}$, the estimation above implies that

$$\lim_{n\to\infty}\left(f\left(\frac{1}{n^2}\right) + f\left(\frac{2}{n^2}\right) + f\left(\frac{3}{n^2}\right) + \cdots + f\left(\frac{n}{n^2}\right)\right) = \lim_{n\to\infty}\left(\frac{1}{n^2} + \frac{2}{n^2} + \frac{3}{n^2} + \cdots + \frac{n}{n^2}\right) = \frac{1}{2}$$

Exercise 2.70

For any $\epsilon > 0$, there is $\delta > 0$, such that $0 < x < \delta$ implies $\left|\frac{f(x)}{g(x)} - 1\right| < \epsilon$. The inequality is the same as $|f(x) - g(x)| < \epsilon g(x)$. Then for $|x_{n,k}| < \epsilon$, by $g(x) > 0$, we have

$$\begin{aligned}&|(f(x_{n,1}) + f(x_{n,2}) + \cdots + f(x_{n,k_n})) - (g(x_{n,1}) + g(x_{n,2}) + \cdots + g(x_{n,k_n}))| \\ <&\epsilon(g(x_{n,1}) + g(x_{n,2}) + \cdots + g(x_{n,k_n})).\end{aligned}$$

Since $\lim_{n\to\infty}(g(x_{n,1}) + g(x_{n,2}) + \cdots + g(x_{n,k_n}))$ converges, the sequence $g(x_{n,1}) + g(x_{n,2}) + \cdots + g(x_{n,k_n})$ is bounded. Then the estimation above implies

$$\lim_{n\to\infty}[(f(x_{n,1}) + f(x_{n,2}) + \cdots + f(x_{n,k_n})) - (g(x_{n,1}) + g(x_{n,2}) + \cdots + g(x_{n,k_n}))] = 0.$$

Moreover, the convergence of $\lim_{n\to\infty}(g(x_{n,1}) + g(x_{n,2}) + \cdots + g(x_{n,k_n}))$ further implies that

$$\lim_{n\to\infty}(f(x_{n,1}) + f(x_{n,2}) + \cdots + f(x_{n,k_n})) = \lim_{n\to\infty}(g(x_{n,1}) + g(x_{n,2}) + \cdots + g(x_{n,k_n})).$$

Exercise 2.78

By taking $x = y = 0$ in $f(x+y) = f(x) + f(y)$, we get $f(0) = f(0+0) = f(0) + f(0)$, which implies $f(0) = 0$. Then by taking $y = -x$, we get $0 = f(0) = f(x) + f(-x)$, which means $f(-x) = -f(x)$.

By repeatedly using $f(x+y) = f(x) + f(y)$, for any natural number n, we have

$$f(nx) = f(x + x + \cdots + x) = f(x) + f(x) + \cdots + f(x) = nf(x).$$

For the special case $x = 1$, we get

$$f(n) = nf(1) = an, \quad a = f(1),\ n \in \mathbb{N}.$$

Then for any positive rational number $x = \dfrac{m}{n}$, $m, n \in \mathbb{N}$, we have $nf(x) = f(nx) = f(m) = am$, which implies $f(x) = \dfrac{m}{n} = ax$. We also have $f(-x) = -f(x) = -ax$. This shows that we have $f(x) = ax$ for all rational x, no matter whether x is positive or negative.

Now we have $g(x) = f(x) - ax$ continuous and satisfy $g(x) = 0$ for all rational x. By Exercise 2.32, we conclude that $g(x) = 0$ for all x. In other words, $f(x) = x$ for all x.

Exercise 2.79

Similar to Exercise 2.78, the equality $f(x + y) = f(x)f(y)$ tells us $f(x) = a^x$ for rational x. Then the continuity of $g(x) = f(x) - a^x$ and the fact that $g(x) = 0$ for all rational x imply that $g(x) = a^x$ for all x.

Exercise 2.80

If f has a left inverse g, then

$$f(x_1) = f(x_2) \implies x_1 = g(f(x_1)) = g(f(x_2)) = x_2.$$

This means that f is one-to-one. Since f is increasing, we have

$$x_1 > x_2 \implies f(x_1) \geq f(x_2).$$

The one-to-one property implies that $f(x_1) \neq f(x_2)$. Therefore f is strictly increasing.

Now for a strictly increasing $f(x)$ on $[a, b]$, we try to construct an increasing $g(y)$ on $[f(a), f(b)]$, such that $g(f(x)) = x$. Of course if $y = f(x)$ for some $x \in [a, b]$, then we have to define $g(y) = x$. Now for any $y \in [f(a), f(b)]$, y may not be the value of f. Let

$$x_- = \sup\{x \colon f(x) \leq y\}, \quad x_+ = \inf\{x \colon f(x) \geq y\}.$$

We claim that $x_- = x_+$. First we note that since f is strictly increasing, any one in the set $\{x \colon f(x) \geq y\}$ is no less than any one in the set $\{x \colon f(x) \leq y\}$. Therefore $x_+ \geq x_-$. Moreover, for any $\delta > 0$, the definition of x_- implies $f(x_- + \delta) > y$. Therefore $x_- + \delta \geq x_+$ by the definition of x_+. The inequality $x_- + \delta \geq x_+ \geq x_-$ for any $\delta > 0$ implies that $x_- = x_+$.

Denote

$$x = \sup\{x \colon f(x) \leq y\} = \inf\{x \colon f(x) \geq y\}.$$

For any $\epsilon > 0$, we have $f(x - \epsilon) \leq y \leq f(x + \epsilon)$. If g is an increasing left inverse, then

$$x - \epsilon = g(f(x - \epsilon)) \leq g(y) \leq g(f(x + \epsilon)) = x + \epsilon.$$

Since this is true for all ϵ, we see that $g(y)$ must be equal to x. In particular, this shows the uniqueness of the increasing left inverse. It remains to show the existence.

Therefore for a strictly increasing f, we (must, if we want g to be increasing) define

$$g(x) = \sup\{x \colon f(x) \leq y\} = \inf\{x \colon f(x) \geq y\}.$$

In case $y = f(x')$, it is easy to see (by the increasing property of f) that $x = x'$, so that $g(y) = x'$. This shows that g is a left inverse of f. Moreover, if $y < y'$, then $\{x\colon f(x) \geq y\}$ is bigger than $\{x\colon f(x) \geq y'\}$, so that

$$g(y) = \inf\{x\colon f(x) \geq y\} \leq \{x\colon f(x) \geq y'\} = g(y').$$

In other words, g is increasing.

EXERCISE 2.81

If f has a right inverse, then $f(g(y)) = y$ for any $y \in [f(a), f(b)]$. In particular, any value in $[f(a), f(b)]$ is reached by f. By Exercise 2.38, f is continuous.

Conversely, if f is continuous, then by the intermediate value theorem, for any $y \in [f(a), f(b)]$, we have $y = f(x)$ for some $x \in [a, b]$. Although such x may not be unique, we pick one such x and define $g(y) = x$. Then g is a right inverse of f (although g may not be continuous).

For an increasing f, we have

$$x_1 \geq x_2 \implies f(x_1) \geq f(x_2).$$

This is the same as

$$f(x_1) < f(x_2) \implies x_1 < x_2.$$

Now for a right inverse g of an increasing f, we have

$$y_1 < y_2 \implies f(g(y_1)) = y_1 < y_2 = f(g(y_2)).$$

Taking $x_1 = g(y_1)$ and $x_2 = g(y_2)$ in the implication for f, we get

$$y_1 < y_2 \implies f(g(y_1)) < f(g(y_2)) \implies g(y_1) < g(y_2).$$

This proves that g is strictly increasing.

If $f(x)$ is continuous and strictly increasing, then Theorem 2.5.3 says that f is invertible, so that the right inverse must be the unique inverse function of f.

If $f(x)$ is continuous and not strictly increasing, then we have $f(x_1) = f(x_2)$ for some $x_1 < x_2$. Let $y_0 = f(x_1) = f(x_2)$. By the increasing property of f, we have $f(x) = y_0$ for any $x_1 \leq y \leq x_2$. Now for any right inverse g of f, we may change the value $g(y_0)$ to be any number between x_1 and x_2 and still get $f(g(y_0)) = y_0$. The result is still a right inverse of f. This shows that the right inverse is not unique if the increasing property of f is not strict.

?????????????????

Exercise **??**

[compare the proof of Exercise 2.14]

The continuity means $\lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x) = f(a)$. For this to be wrong, we have the following possibilities.

(1) $\lim_{x\to a^+} f(x)$ diverges. Then there is $\epsilon > 0$, such that for any $\delta > 0$, there are $x_1, x_2 \in (a-\delta, a)$ satisfying $|f(x_1) - f(x_2)| \geq 2\epsilon$. Then for any $y \in (a, a+\delta)$ (actually the condition on y is not necessary), we must either have $|f(x_1) - f(y)| \geq \epsilon$ or have $|f(x_2) - f(y)| \geq \epsilon$. This is the second case.

(2) $\lim_{x\to a^-} f(x)$ diverges. This is similar to (1), and we are in the second case.

(3) $\lim_{x\to a^-} f(x) = l$ and $\lim_{x\to a^+} f(x) = k$ converge, but $l \neq k$. Pick any $\epsilon < \dfrac{1}{3}|l-k|$. Then there is $\delta > 0$, such that $|f(x) - l| < \epsilon$ for $x \in (a-\delta, a)$ and $|f(y) - k| < \epsilon$ for $y \in (a, a+\delta)$. This further implies

$$|f(x) - f(y)| \geq |l - k| - |f(x) - l| - |f(y) - k| > \epsilon.$$

So we are in the second case.

(4) $\lim_{x\to a} f(x)$ converges but $\lim_{x\to a} f(x) \neq f(a)$. It is easy to see this leads to the first case.

Exercise

Let $\gamma \in (\alpha, \beta)$. Since $\lim_{x\to a^+} f(x) = \alpha < \gamma < \beta = \lim_{x\to b^-} f(x)$, there is $\delta > 0$, such that

$$a < x < a + \delta \implies f(x) < \gamma, \quad b - \delta < x < b \implies f(x) > \gamma.$$

Fix x_1 and x_2 satisfying $a < x_1 < a + \delta$ and $b - \delta < x_2 < b$. Then $f(x_1) < \gamma < f(x_2)$. By applying the intermediate value theorem to $f(x)$ on $[x_1, x_2]$, we find $c \in [x_1, x_2] \in (a, b)$ satisfying $f(c) = \gamma$.

Exercise 3.1

This is a consequence of the known fact that if $\lim_{x\to a} f(x) > 0$, then there is $\delta > 0$, such that $f(x) > 0$ for $0 < |x-a| < \delta$.

More specifically, the continuity tells us $\lim_{x\to x_0} f(x) = f(x_0)$. If $f(x_0) > 0$, then for $\epsilon = f(x_0)$, there is $\delta > 0$, such that $0 < |x-a| < \delta$ implies $|f(x) - f(x_0)| < f(x_0)$. Then $0 < |x-a| < \delta$ implies $f(x) - f(x_0) > -f(x_0)$, which means $f(x) > 0$.

Exercise 3.2

A function $f(x)$ is approximated by a constant a on the right of x_0 if for any $\epsilon > 0$, there is $\delta > 0$, such that

$$0 \le x - x_0 < \delta \implies |f(x) - a| \le \epsilon.$$

By taking $x = x_0$, we get $|f(x_0) - a| \le \epsilon$ for any $\epsilon > 0$. Therefore $a = f(x_0)$. Then for $x \ne x_0$, the definition becomes

$$0 < x - x_0 < \delta \implies |f(x) - f(x_0)| \le \epsilon.$$

This means $\lim_{x\to x_0^+} f(x) = f(x_0)$, or the right continuity of f at x_0.

Exercise 3.9

Let $\Delta x = x - x_0$. Then $x^2 = (x_0 + \Delta x)^2 = x_0^2 + 2x_0\Delta x + \Delta x^2 = x_0^2 + 2x_0\Delta x + o(\Delta x)$. Therefore $x_0^2 + 2x_0\Delta x$ is the linear approximation of x^2 at x_0.

By $x^3 = (x_0+\Delta x)^3 = x_0^3+3x_0^2\Delta x+3x_0\Delta x^2+\Delta x^3 = x_0^3+3x_0^2\Delta x+o(\Delta x)$, we find $x_0^3+3x_0^2\Delta x$ is the linear approximation of x^3 at x_0.

Exercise 3.10

Suppose for any $\epsilon > 0$, there are $\delta, \delta' > 0$, such that

$$\begin{aligned} |\Delta x| < \delta &\implies |f(x) - a - b\Delta x| \le \epsilon|\Delta x|, \\ |\Delta x| < \delta' &\implies |f(x) - a' - b'\Delta x| \le \epsilon|\Delta x|. \end{aligned}$$

Then

$$|\Delta x| < \min\{\delta, \delta'\} \implies |(a + b\Delta x) - (a' + b'\Delta x)| \le 2\epsilon|\Delta x|.$$

Taking $\Delta x = 0$, we get $|a - a'| \le 0$. Therefore $a = a'$, and the implication becomes

$$|\Delta x| < \min\{\delta, \delta'\} \implies |b\Delta x - b'\Delta x| \le 2\epsilon|\Delta x| \iff |b - b'| \le 2\epsilon.$$

Since this holds for any $\epsilon > 0$, we also get $b = b'$.

Exercise 3.11

Suppose for any $\epsilon > 0$, there are $\delta, \delta' > 0$, such that

$$\begin{aligned} |\Delta x| < \delta &\implies |f(x) - a - b\Delta x| \le \epsilon|\Delta x|, \\ |\Delta x| < \delta' &\implies |h(x) - a - b\Delta x| \le \epsilon|\Delta x|. \end{aligned}$$

Then

$$\begin{aligned} |\Delta x| < \delta &\implies f(x) \ge a + b\Delta x - \epsilon|\Delta x|, \\ |\Delta x| < \delta' &\implies h(x) \le a + b\Delta x + \epsilon|\Delta x|, \end{aligned}$$

and we get

$$\begin{aligned}|\Delta x| < \min\{\delta, \delta'\} &\implies a + b\Delta x - \epsilon|\Delta x| \le f(x) \le g(x) \le h(x) \le a + b\Delta x + \epsilon|\Delta x| \\ &\implies |g(x) - a - b\Delta x| \le \epsilon|\Delta x|.\end{aligned}$$

Exercise 3.12

$\Delta f = (x_0+\Delta x)^3 - x^3 = 3x_0^2\Delta x + 3x_0\Delta x^2 + \Delta x^3 = 3x_0^2\Delta x + o(\Delta x)$. The linear approximation is $3x_0^2\Delta x$. The differential is $df = 3x_0^2 dx$, and the derivative is $f'(x_0) = 3x_0^2$.

Exercise 3.13

(1) The limit $\lim_{x\to 0} \dfrac{|x|^p}{x} = \infty$ converges if and only if $p > 0$. So the function is differentiable at 0 if and only if $p > 0$.

(2) Since

$$\begin{aligned}\lim_{x\to a} \frac{\cos x - \cos a}{x - a} &= \lim_{x\to 0} \frac{\cos(x+a) - \cos a}{x} = \lim_{x\to 0} \frac{(\cos x - 1)\cos a - \sin x \sin a}{x} \\ &= \lim_{x\to 0} \frac{\cos x - 1}{x^2} x\cos a + \lim_{x\to 0} \frac{\sin x}{x} \sin a = \sin a,\end{aligned}$$

the function is differentiable at $x = 0$, with $f(a) + f'(a)x = \cos a - x\sin a$ as the linear approximation.

(3) By $\lim_{x\to 0} \dfrac{\sin^2 x}{x} = \lim_{x\to 0} \left(\dfrac{\sin x}{x}\right)^2 x = 0^2 \cdot 1 = 0$, the function is differentiable at 0, with 0 as the linear approximation.

(4) By $\lim_{\Delta x\to 0} \dfrac{e^{x_0+\Delta x} - e^{x_0}}{\Delta x} = \lim_{\Delta x\to 0} e^{x_0} \dfrac{e^{\Delta x} - 1}{\Delta x} = e^{x_0}$, the function is differentiable at x_0, with $e^{x_0} + e^{x_0}\Delta x$ as the linear approximation.

(5) Since $\lim_{x\to 0} \dfrac{\log(1+x) - \log 1}{x - 0} = \lim_{x\to 0} \dfrac{\log(1+x)}{x} = 1$, the function is differentiable at $x = 0$, with $f(0) + f'(0)x = x$ as the linear approximation.

(6) By $|\sin \pi x^3| \le |\pi x^3| = o(x)$, $|\sin \pi x^3|$ is differentiable at 0, with 0 as the linear approximation.

We have

$$\lim_{x\to 1^+} \frac{|\sin \pi x^3| - |\sin \pi 1^3|}{x - 1} = \lim_{x\to 1^+} \frac{\sin \pi x^3}{x - 1} = -\lim_{x\to 1^+} \frac{\sin \pi(x^3 - 1)}{x^3 - 1} \lim_{x\to 1^-} \frac{x^3 - 1}{x - 1} = -3\pi,$$

and

$$\lim_{x\to 1^-} \frac{|\sin \pi x^3| - |\sin \pi 1^3|}{x - 1} = -\lim_{x\to 1^-} \frac{|\sin \pi x^3|}{x - 1} = \lim_{x\to 1^-} \frac{\sin \pi(x^3 - 1)}{x^3 - 1} \lim_{x\to 1^-} \frac{x^3 - 1}{x - 1} = 3\pi.$$

Therefore $|\sin \pi x^3|$ has no derivative at 1, and is not differentiable at $x = 1$.

(7) We have

$$\lim_{x\to 0^+} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0^+} \frac{-x^p}{x} = \begin{cases} 0 & \text{if } p > 1 \\ -1 & \text{if } p = 1 \\ \infty & \text{if } 0 < p < 1 \end{cases},$$

and

$$\lim_{x\to 0^-}\frac{f(x)-f(0)}{x-0}=\lim_{x\to 0^+}\frac{x^q}{-x}=\begin{cases}0 & \text{if } q>1\\ -1 & \text{if } q=1\\ \infty & \text{if } 0<q<1\end{cases}.$$

The function is differentiable if and only if the two limits exist and are equal, which means $p,q>1$ (with linear approximation 0) or $p=q=1$ (with linear approximation $-x$).

(8) By the similar computation as in (7), the function is differentiable if and only if $p,q>1$ (with 0 as linear approximation).

Exercise 3.14

For any $x_0<0$ or $x_0>1$, we have $|x^3(x-1)(x-2)^2|=x^3(x-1)(x-2)^2$ near x_0. For $0<x_0<1$, we have $|x^3(x-1)(x-2)^2|=-x^3(x-1)(x-2)^2$ near x_0. Since both $x^3(x-1)(x-2)^2$ and $-x^3(x-1)(x-2)^2$ are differentiable, and the differentiability at x_0 only depends on the function near x_0, the function is differentiable away from 0 and 1.

Since $|(x-1)(x-2)^2|$ is bounded near 0 and $|x^3|=o(x)$ at 0, we get $|x^3(x-1)(x-2)^2|=o(x)$ at 0, so that the function is differentiable at 0 (with 0 as the linear approximation).

We have $f(x)=\begin{cases}-x^3(x-1)(x-2)^2 & \text{if } x<1\\ x^3(x-1)(x-2)^2 & \text{if } x\ge 1\end{cases}$ near $x_0=1$. Therefore $f'(1^-)=-1$ and $f'(1^+)=1$, so that $f(x)$ has no first order derivative at $x_0=1$. Equivalently, $f(x)$ is not differentiable at $x_0=1$.

We conclude the function is differentiable everywhere except 1.

Exercise 3.15

Since the function $R(x)$ is not continuous at rational numbers, it is not differentiable at rational numbers.

Let a be an irrational number. Then $R(a)=0$. For any prime number q, we have $\frac{p}{q}<a<\frac{p+1}{q}$ for some integer p. Then by considering the distances from a to $\frac{p}{q}$ and $\frac{p+1}{q}$, one of the two numbers, which we denote a_q, satisfies $|a_q-a|\le\frac{1}{2q}$. Then

$$\left|\frac{R(a_q)-R(a)}{a_q-a}\right|=\frac{1}{q|a_q-a|}\ge\frac{1}{2}.$$

On the other hand, we can also find irrational number b arbitrarily close to a, for which we have

$$\left|\frac{R(b)-R(a)}{b-a}\right|=\frac{0}{|b-a|}=0.$$

Therefore $\lim_{x\to a}\frac{R(x)-R(a)}{x-a}$ diverges, and $R(x)$ is not differentiable anywhere.

Exercise 3.16

Suppose $f\le g\le h$ near x_0, and $f(x_0)=g(x_0)=h(x_0)$. If f and h are differentiable at x_0, with $f'(x_0)=h'(x_0)$, then g is also differentiable at x_0, with $g'(x_0)=f'(x_0)=h'(x_0)$.

Exercise 3.17

Consider the quotient $\dfrac{f(x)-f(0)}{x-0}=\dfrac{f(x)}{x}$. For $x>0$, we have $f(x)\geq x$, so that $\dfrac{f(x)}{x}\geq 1$. For $x<0$, we have $f(x)\geq -x$, so that $\dfrac{f(x)}{x}\leq -1$. Thus either the left or the right limit does not exist, or both exist but are not equal. In particular, the limit $\lim_{x\to 0}\dfrac{f(x)}{x}$ does not exist.

The function $f(x)=x^2$ satisfies $f(x)\leq |x|$ for $|x|<1$, and $f(x)$ is differentiable at 0.

Exercise 3.18

Since $f(x)=0$ for infinitely many $x\in[a,b]$, there is a sequence $x_n\in[a,b]$, such that $x_m\neq x_n$ for $m\neq n$ and $f(x_n)=0$. By Bolzano-Weierstrass theorem, x_n has a convergent subsequence. Without loss of generality, we may still denote the subsequence by x_n. Moreover, we may assume $\lim_{n\to\infty}x_n=c\in[a,b]$ is different from any x_n.

By the continuity of $f(x)$ at c, we have $f(c)=\lim_{n\to\infty}f(x_n)=0$. Since $f(x)$ is also differentiable at c, the limit $\lim_{x\to c}\dfrac{f(x)-f(c)}{x-c}$ converges and the limit is

$$f'(c)=\lim_{x\to c}\frac{f(x)-f(c)}{x-c}=\lim_{n\to\infty}\frac{f(x_n)-f(c)}{x_n-c}=\lim_{n\to\infty}\frac{0-0}{x_n-c}=0.$$

Exercise 3.19

By $\log(1+t)=t+o(t)$, we have

$$\begin{aligned}&\log\left(\frac{f(x_0+t)}{f(x_0)}\right)^{\frac{1}{t}}=\frac{1}{t}\log\left(\frac{f(x_0)+f'(x_0)t+o(t)}{f(x_0)}\right)\\ =&\frac{1}{t}\left(\frac{f'(x_0)t+o(t)}{f(x_0)}+o\left(\frac{f'(x_0)t+o(t)}{f(x_0)}\right)\right)=\frac{f'(x_0)}{f(x_0)}+\frac{f'(x_0)}{f(x_0)}\frac{o(t)}{t}+\frac{1}{t}o\left(\frac{f'(x_0)t+o(t)}{f(x_0)}\right).\end{aligned}$$

Then by

$$\lim_{t\to 0}\frac{o(t)}{t}=0,\quad \lim_{t\to 0}\frac{1}{t}o\left(\frac{f'(x_0)t+o(t)}{f(x_0)}\right)=\lim_{t\to 0}\frac{f'(x_0)+\dfrac{o(t)}{t}}{f(x_0)}\frac{o\left(\dfrac{f'(x_0)t+o(t)}{f(x_0)}\right)}{\dfrac{f'(x_0)t+o(t)}{f(x_0)}}=\frac{f'(x_0)+0}{f(x_0)}0=0,$$

we get $\lim_{t\to 0}\log\left(\dfrac{f(x_0+t)}{f(x_0)}\right)^{\frac{1}{t}}=\dfrac{f'(x_0)}{f(x_0)}$. Then by the continuity of e^x, we get $\lim_{t\to 0}\left(\dfrac{f(x_0+t)}{f(x_0)}\right)^{\frac{1}{t}}=e^{\frac{f'(x_0)}{f(x_0)}}$.

Exercise 3.20

by $f(x_0)=g(x_0)=0$, we have $\lim_{x\to x_0}\dfrac{\dfrac{g(x)-g(x_0)}{x-x_0}}{\dfrac{f(x)-f(x_0)}{x-x_0}}=\lim_{x\to x_0}\dfrac{g(x)}{f(x)}=1$. Therefore $\lim_{x\to x_0}\dfrac{g(x)-g(x_0)}{x-x_0}=b$ if and only if $\lim_{x\to x_0}\dfrac{f(x)-f(x_0)}{x-x_0}=b$. In other words, $g(x)$ has linear approximation $0 + b\Delta x = g(x_0) + b\Delta x$ if and only if $f(x)$ has linear approximation $0 + b\Delta x = f(x_0) + b\Delta x$.

Exercise 3.21

Suppose $f(x)$ is differentiable at x_0. Then $f(x) = f(x_0) + (x - x_0)g(x)$, where

$$g(x) = \begin{cases} \dfrac{f(x) - f(x_0)}{x - x_0} & \text{if } x \neq x_0 \\ f'(x_0) & \text{if } x = x_0 \end{cases}$$

is continuous at x_0 because differentiability implies $\lim_{x \to x_0} \dfrac{f(x) - f(x_0)}{x - x_0} = f'(x_0)$.

Conversely, suppose $f(x) = f(x_0) + (x - x_0)g(x)$ with $g(x)$ continuous at x_0. Then $f(x) = f(x_0) + (x - x_0)g(x_0) + (x - x_0)(g(x) - g(x_0))$, with

$$\lim_{x \to x_0} \frac{(x - x_0)(g(x) - g(x_0))}{x - x_0} = \lim_{x \to x_0} (g(x) - g(x_0)) = 0.$$

Therefore $f(x_0) + (x - x_0)g(x_0)$ is a linear approximation of $f(x)$ at x_0. In particular, $f(x)$ is differentiable at x_0.

Exercise 3.22

We have $\lim_{x \to x_0, x \in \mathbb{Q}} g(x) = f(x_0)$ and $\lim_{x \to x_0, x \notin \mathbb{Q}} g(x) = -f(x_0)$. The function is continuous at x_0 if and only if the two limits are the same. This means $f(x_0) = 0$.

For the differentiability, since it implies the continuity, it is necessary for g to be continuous at x_0. This means $g(x_0) = 0$. So we assume $g(x_0) = 0$ and further study the differentiability.

Under the assumption $g(x_0) = 0$, the differentiability at x_0 is the same as the convergence of

$$h(x) = \frac{g(x) - g(x_0)}{x - x_0} = \begin{cases} \dfrac{f(x)}{x - x_0}, & \text{if } x \text{ is rational;} \\ -\dfrac{f(x)}{x - x_0}, & \text{if } x \text{ is irrational,} \end{cases}$$

as $x \to x_0$. By taking $\dfrac{f(x)}{x - x_0}$ in place of $f(x)$ in the first part, we see that the limit converges if and only if $\lim_{x \to x_0} \dfrac{f(x)}{x - x_0}$ converges and $\lim_{x \to x_0} \dfrac{f(x)}{x - x_0} = \lim_{x \to x_0} -\dfrac{f(x)}{x - x_0}$. This is equivalent to $f'(x_0) = \lim_{x \to x_0} \dfrac{f(x)}{x - x_0} = 0$.

We conclude that g is differentiable at x_0 if and only if $f(x_0) = 0$ and $f'(x_0) = 0$.