Exercise 5.46

Suppose $\lim_{x\to a} f(x,y) = g(y)$ uniformly on Y. Then for any $\epsilon > 0$, there is $\delta > 0$, such that

$$0 < |x-a| < \delta,\ y \in Y \implies |f(x,y) - g(y)| < \epsilon.$$

This is the same as

$$0 < |x-a| < \delta \implies |f(x,y) - g(y)| < \epsilon \text{ for all } y \in Y.$$

Taking the supremum of the right side, we get

$$0 < |x-a| < \delta \implies \sup_Y |f(x,y) - g(y)| \le \epsilon.$$

This proves $\lim_{x\to a} \sup_Y |f(x,y) - g(y)| = 0$.

Conversely, suppose $\lim_{x\to a} \sup_Y |f(x,y) - g(y)| = 0$. Then for any $\epsilon > 0$, there is $\delta > 0$, such that

$$0 < |x-a| < \delta \implies \sup_Y |f(x,y) - g(y)| \le \epsilon.$$

This further implies

$$0 < |x-a| < \delta \implies |f(x,y) - g(y)| \le \sup_Y |f(x,y) - g(y)| < \epsilon \text{ for all } y \in Y.$$

This proves that $\lim_{x\to a} f(x,y) = g(y)$ uniformly on Y.

Exercise 5.47

Suppose $\lim_{x\to a} f(x,y) = g(y)$ uniformly on Y_1 and Y_2. Then for any $\epsilon > 0$, there are δ_1, δ_2, such that

$$0 < |x-a| < \delta_1,\ y \in Y_1 \implies |f(x,y) - g(y)| < \epsilon,$$
$$0 < |x-a| < \delta_2,\ y \in Y_2 \implies |f(x,y) - g(y)| < \epsilon.$$

Then

$$0 < |x-a| < \min\{\delta_1, \delta_2\},\ y \in Y_1 \cup Y_2 \implies |f(x,y) - g(y)| < \epsilon.$$

Exercise 5.48

For any $\delta > 0$, we fix x satisfying $0 < |x-a| < \delta$. Then there is $y \in (b,c)$ sufficiently close to b, satisfying $|f(x,y) - g(y)| \ge C$. Thus we have

$$0 < |x-a| < \delta,\ y \in (b,c), \text{ but } |f(x,y) - g(y)| \ge C.$$

This shows that the uniformity on (b,c) fails for $\epsilon = C$.

Exercise 5.49 (1)

For any $R > 0$ and $\epsilon > 0$, we have

$$|x| \ge R,\ |y| > \frac{1}{R\epsilon} \implies \left|\frac{1}{xy}\right| < \frac{1}{R\frac{1}{R\epsilon}} = \epsilon.$$

So the convergence is uniform for $|y| \geq R$, for any fixed $R > 0$.

The convergence is not uniform near 0. For any $\delta > 0$ and $\epsilon = 1$, take $x = \dfrac{\delta}{2} \in (-\delta, \delta)$ and $t = \dfrac{2}{\delta}$. We have $\dfrac{1}{xy} = 1 = \epsilon$.

Exercise 5.49 (2)

For $R > 0$, we have (c is between 0 and tx)

$$|x| \leq R \implies \left|\frac{\sin xy}{x} - y\right| = \left|\frac{1}{x}\left(xy - \frac{-\sin c}{6}(xy)^3\right) - y\right| = \left|\frac{\sin c}{6}x^2y^3\right| \leq \frac{1}{6}x^2R^3.$$

This implies that $\lim_{x\to 0} \dfrac{\sin xy}{x} = y$ uniformly for $|y| \leq R$.

For fixed y, we have $\lim_{y\to+\infty}\left(\dfrac{\sin xy}{x} - y\right) = -\infty$. Like Example 5.3.8, this implies that $\lim_{x\to 0} \dfrac{\sin xy}{x} = y$ is not uniform for $y \geq R$ and for $y \leq -R$.

Exercise 5.49 (3)

By

$$|\sqrt{x+y} - \sqrt{y}| \leq \frac{x}{\sqrt{x+y} + \sqrt{y}} \leq \frac{x}{\sqrt{x}} = \sqrt{x},$$

we conclude that $\lim_{x\to 0^+} \sqrt{x+y} = \sqrt{y}$ uniformly for $y \geq 0$.

Exercise 5.49 (4)

By Examples 5.3.3 and 5.3.4, $\lim_{x\to 0^+}(1+xy)^{\frac{1}{x}} = e^y$ converges uniformly on $[-R, R]$ but not uniformly on $[R, +\infty)$. Moreover, $\lim_{x\to 0^-}(1+xy)^{\frac{1}{x}} = e^y$ converges uniformly on $(-\infty, R]$.

Exercise 5.49 (5)

For $R^{-1} \leq y \leq R$, by the Mean Value Theorem, we have (c is between 0 and x)

$$|y^x - 1| = |y^x - y^0| = |x|y^c \log y \leq |x|R^{|x|}\log R.$$

This implies that $\lim_{x\to 0} y^x = 1$ uniformly on $[R^{-1}, R]$ for any $R > 1$.

On the other hand, for any fixed $x > 0$, we have $\lim_{y\to 0^+}(y^x - 1) = -1$ and $\lim_{y\to+\infty}(y^x - 1) = +\infty$. Similar to the argument in Examples 5.3.1 and 5.3.2, the uniformity fails for $\epsilon = \frac{1}{2}$ on $(0, R^{-1}]$ and fails for $\epsilon = 1$ on $[R, +\infty)$.

Exercise 5.49 (6)

For $0 < R^{-1} \leq y \leq R$, we have (c is between 0 and x)

$$\left|\frac{y^x - 1}{x} - \log y\right| = \left|\frac{(y^x - x\log y) - (y^0 - 0\log y)}{x - 0}\right| = |y^c \log y - \log y| \leq |R^{|x|} - 1|\log R.$$

This implies that $\lim_{x\to 0} \dfrac{y^x - 1}{x} = \log y$ uniformly on $[R^{-1}, R]$ for any $R > 1$.

On the other hand, for fixed $t > 0$, we have

$$\lim_{x\to+\infty}\left(\frac{y^x-1}{x}-\log y\right)=+\infty,\quad \lim_{x\to 0^+}\left(\frac{y^x-1}{x}-\log y\right)=-\infty.$$

Like Example 5.3.8, this implies that $\lim_{x\to 0}\dfrac{y^x-1}{x}=\log y$ is not uniform on $(0,R^{-1}]$ or on $[R,+\infty)$. The same is true for $\lim_{x\to 0^-}$.

Exercise 5.50 (1)

By the Mean Value Theorem, we have

$$\left|\frac{\sin x-\sin y}{x-y}-\cos y\right|=|\cos c-\cos y|,\quad |c-y|<|x-y|.$$

By the uniform continuity of the cosine function, for any $\epsilon > 0$, there is $\delta > 0$, such that

$$|x-y|<\delta \implies |\cos x-\cos y|<\epsilon.$$

Then

$$|x-y|<\delta \implies |c-y|<\delta \implies \left|\frac{\sin x-\sin y}{x-y}-\cos y\right|<\epsilon.$$

This proves that $\lim_{x\to y}\dfrac{\sin x-\sin y}{x-y}=\cos y$ uniformly for all y.

Exercise 5.50 (2)

By the Mean Value Theorem, we have

$$\left|\frac{x^p-y^p}{x-y}-py^{p-1}\right|=|pc^{p-1}-py^{p-1}|,\quad |c-y|<|x-y|.$$

Similar to Exercise 5.50(1), the uniform continuity of x^{p-1} implies the uniform convergence of $\lim_{x\to y}\dfrac{x^p-y^p}{x-y}=py^{p-1}$. For example, for $1\le p\le 2$, we know x^{p-1} is uniformly continuous for $x\ge 0$. Therefore $\lim_{x\to y}\dfrac{x^p-y^p}{x-y}=py^{p-1}$ is uniform for $y>0$.

We also know that, for $p>2$ and any $R>0$, $\lim_{x\to y}\dfrac{x^p-y^p}{x-y}=py^{p-1}$ is uniform on $(0,R]$. If the convergence is uniform on $[R,+\infty)$, then for any $\epsilon>0$, there would be $\delta>0$, such that

$$0<|x-y|\le\delta,\ y\ge R \implies \left|\frac{x^p-y^p}{x-y}-py^{p-1}\right|<\epsilon.$$

This is the same as

$$0<|h|\le\delta,\ y\ge R \implies \left|\frac{(y+h)^p-y^p}{h}-py^{p-1}\right|<\epsilon.$$

Now for any δ, we fix $h = \delta$. For $y \geq R > \delta$, we have

$$\frac{(y+h)^p - y^p}{h} - py^{p-1} = \frac{y^p}{h}\left[\left(1+\frac{h}{y}\right)^p - 1 - p\frac{h}{y}\right] = \frac{y^p}{h}\left[\frac{p(p-1)}{2}\frac{h^2}{y^2} + o\left(\frac{h^2}{y^2}\right)\right]$$
$$= \frac{p(p-1)}{2}hy^{p-2} + o\left(hy^{p-2}\right).$$

Since $p > 2$, the estimation above implies $\lim_{y\to+\infty}\left(\frac{(y+h)^p - y^p}{h} - py^{p-1}\right) = +\infty$. Therefore for $\epsilon = 1 > 0$ and given $\delta, R > 0$, we can find $y > R$ and then choose $x = y + \delta$, such that

$$\frac{x^p - y^p}{x - y} - py^{p-1} = \frac{p(p-1)}{2}hy^{p-2} + o\left(hy^{p-2}\right) > 1.$$

This shows that for $p > 2$, $\lim_{x\to y}\frac{x^p - y^p}{x - y} = py^{p-1}$ is not uniform on $[R, +\infty)$.

By the similar argument, for $p < 1$, $\lim_{x\to y}\frac{x^p - y^p}{x - y} = py^{p-1}$ is uniform on $[R, +\infty)$ but not on $(0, R]$.

Exercise 5.51 (1)

For any $\epsilon > 0$, we have

$$n > \frac{1}{\epsilon} \implies 0 < \frac{1}{m+n} < \frac{1}{n} < \epsilon.$$

Therefore the convergence is uniform.

Exercise 5.51 (2)

For each fixed n, we have $\lim_{m\to\infty}\frac{m}{m+n} = 1$. This would imply that the limit $\lim_{n\to\infty}\frac{m}{m+n} = 0$ is not uniform.

Specifically, for $\epsilon = \frac{1}{2}$ and any n, by choosing $m = n$, we get $\left|\frac{m}{m+n} - 0\right| = \frac{1}{2} = \epsilon$. This shows the limit $\lim_{n\to\infty}\frac{m}{m+n} = 0$ is not uniform.

Exercise 5.51 (3)

For any $\epsilon > 0$, we have

$$n > \frac{1}{\epsilon} \implies 0 < \frac{1}{n^m} \leq \frac{1}{n} < \epsilon.$$

Therefore the convergence is uniform.

Exercise 5.51 (4)

For each fixed n, we have $\lim_{m\to\infty}\frac{m}{n} = \infty$. This would imply that the limit $\lim_{n\to\infty}\frac{m}{n} = 0$ is not uniform.

Specifically, for $\epsilon = 1$ and any n, by choosing $m = n$, we get $\left|\frac{m}{n} - 0\right| = 1 = \epsilon$. This shows the limit $\lim_{n\to\infty}\frac{m}{n} = 0$ is not uniform.

Exercise 5.51 (5)

For each fixed n, we have $\lim_{m\to\infty}\sqrt[n]{m}=\infty$. This would imply that the limit $\lim_{n\to\infty}\sqrt[n]{m}=1$ is not uniform.

Specifically, for $\epsilon=1$ and any n, by choosing $m=2^n$, we get $|\sqrt[n]{m}-1|=1=\epsilon$. This shows the limit $\lim_{n\to\infty}\sqrt[n]{m}=1$ is not uniform.

Exercise 5.52

Under the assumption, there is $r>0$, such that f is continuous on $[a-r,b+r]$. By Theorem 2.4.1, f is uniformly continuous on $[a-r,b+r]$. Therefore for any $\epsilon>0$, there is $0<\delta<r$, such that

$$|x-x'|<\delta,\ x,x'\in[a-r,b+r]\implies|f(x)-f(x')|<\epsilon.$$

Then

$$|x|<\delta,\ y\in[a,b]\implies|(x+y)-y|<\delta,\ x+y,y\in[a-r,b+r]\implies|f(x+y)-f(y)|<\epsilon.$$

This shows that $\lim_{x\to0}f(x+y)=f(y)$ is uniform on $[a,b]$.

Exercise 5.53

By the way f is extended, the continuity of f on $[a,b]$ is the same as the continuity of f on an open interval containing $[a,b]$. By Theorem 2.4.1, the continuity of f on an open interval containing $[a,b]$ is the same as the continuity of f on the interval.

By Exercise 5.52, the continuity of f on an open interval implies that $\lim_{x\to0}f(x+y)=f(y)$ is uniform on $[a,b]$. Conversely, it is easy to see that $\lim_{x\to0}f(x+y)=f(y)$ on $[a,b]$ (uniformity not needed) implies the continuity of f on $[a.b]$.

Exercise 5.54

Since $f'(x)$ is continuous on $[a-\delta,b+\delta]$, for any $\epsilon>0$, there is $\delta>\delta'>0$, such that

$$|x-y|<\delta',\ x,y\in[a-\delta,b+\delta]\implies|f'(x)-f'(y)|<\epsilon.$$

Then for $0<|x|<\delta'$ and $y\in[a,b]$, we have

$$\left|\frac{f(x+y)-f(y)}{x}-f'(y)\right|=|f'(c)-f'(y)|<\epsilon,$$

where c is between y and $x+y$, so that $|c-y|<\delta'$.

Exercise 5.55

Since $f(x)$ is continuous on $[a-\delta,b+\delta]$, for any $\epsilon>0$, there is $\delta>\delta'>0$, such that

$$|x-y|<\delta',\ x,y\in[a-\delta,b+\delta]\implies|f(x)-f(y)|<\epsilon.$$

Then for $0<|x|<\delta'$ and $y\in[a,b]$, we have

$$\left|\frac{1}{x}\int_y^{x+y}f(t)dt-f(y)\right|=\left|\frac{1}{x}\int_y^{x+y}(f(t)-f(y))dt\right|\le\epsilon,$$

because $|f(t)-f(y)|<\epsilon$ for t between y and $x+y$ in case $|x|<\delta'$.

Exercise 5.56

It is trivial to see that the uniform convergence of $\lim_{x\to a} f(x,y)$ implies the uniform convergence of $\lim_{n\to\infty} f(x_n, y)$.

Conversely, suppose $\lim_{x\to a} f(x,y)$ is not uniform. Then there is $\epsilon > 0$, such that for any δ, there is x satisfying $0 < |x-a| < \delta$ and y satisfying $|f(x,y) - g(y)| \geq \epsilon$. For $\delta = \dfrac{1}{n}$, suppose we have such x_n and y_n. Then we have $x_n \neq a$, $\lim x_n \to a$, and $|f(x_n, y_n) - g(y_n)| \geq \epsilon$. This shows that $f(x_n, y)$ does not converge to $g(y)$ uniformly.

Exercise 5.57

Suppose f_n and g_n uniformly converges to f and g. For any $\epsilon > 0$, there are N_1, N_2, such that

$$\begin{aligned} x \in X,\ n > N_1 &\implies |f_n(x) - f(x)| < \epsilon, \\ x \in X,\ n > N_2 &\implies |g_n(x) - g(x)| < \epsilon, \end{aligned}$$

Then

$$x \in X,\ n > \max\{N_1, N_2\} \implies |(af_n + bg_n)(x) - (af + bg)(x)| < (|a| + |b|)\epsilon.$$

Suppose f_n and g_n uniformly converges to f and g. If f and g are bounded, then $f_n g_n$ uniformly converges to fg. The bounded condition is necessary for the product. For example, $\lim_{n\to\infty} \left(x + \dfrac{1}{n}\right) = x$ is uniform, but $\lim_{n\to\infty} \left(x + \dfrac{1}{n}\right)^2 = x^2$ is not uniform.

Since the absolute value function is uniformly continuous, and linear combinations of uniformly convergent series are still uniform, we conclude that the maximum of uniformly convergent series is still uniform.

The composition of uniformly convergent sequences may not be uniformly convergent. For example, $f_n(x) = x + \dfrac{1}{n}$ uniformly converges to $f(x) = x$ on $\mathbb{R}$, and $g_n(y) = y^2$ uniformly converges to $g(y) = y^2$. However, $g_n(f_n(x)) = x^2 + \dfrac{2}{n}x + \dfrac{1}{n^2}$ does not converge uniformly on $\mathbb{R}$.

Exercise 5.58

For any N, we fix a natural number $n > N$. Then for any $\delta > 0$, there is x satisfying $0 < |x-a| < \delta$ and $|f_n(x) - f(x)| \geq c$. Thus we find x and n satisfying

$$n > N,\ 0 < |x-a| < \delta,\ \text{but } |f_n(x) - f(x)| \geq c.$$

This shows that the uniformity fails for $\epsilon = c$.

Exercise 5.59

For any $\epsilon < 0$, there is $\delta > 0$, such that $\sum \omega_{[x_{i-1},x_i]}(f)\Delta x_i < \epsilon$ for any partition P of $[a, b+1]$ satisfying $\|P\| < \delta$. Then for any $n > \dfrac{1}{\delta}$ and $a \leq x \leq b$, we consider a partition P of $[a, b+1]$ that extends the partition $x < x + \dfrac{1}{n} < x + \dfrac{2}{n} < \cdots < x + \dfrac{n}{n} = x + 1$ of $[x, x+1]$ and satisfy

$\|P\| < \delta$. Moreover, we choose x_i^* such that $x_j^* = x + \dfrac{i-1}{n}$ for the interval $\left[x + \dfrac{i-1}{n}, x + \dfrac{i}{n}\right]$. Then

$$\begin{aligned}
\left| \int_x^{x+1} f(t)dt - f_n(x) \right| &= \left| \sum_{i=0}^{n-1} \int_{x+\frac{i}{n}}^{x+\frac{i+1}{n}} \left(f(t) - f\left(x + \frac{i}{n}\right) \right) dt \right| \\
&\le \sum_{i=0}^{n-1} \int_{x+\frac{i}{n}}^{x+\frac{i+1}{n}} \left| f(t) - f\left(x + \frac{i}{n}\right) \right| dt \\
&< \sum_{i=0}^{n-1} \omega_{\left[x+\frac{i-1}{n}, x+\frac{i}{n}\right]}(f)\frac{1}{n} \le \sum_P \omega_{[x_{j-1},x_j]}(f)\Delta x_j < \epsilon.
\end{aligned}$$

We note that the proof actually does not depend on the particular choice of the partition of $[x, x+1]$.

Exercise 5.60 (1)

By Exercise 5.49 (5), $\lim x^{\frac{1}{n}} = 1$ is uniform on $[R^{-1}, R]$ for any $R > 1$. The similar argument as Exercise 5.49 (5) also shows that the convergence is not uniform on $(0, R^{-1}]$ and on $[R, +\infty)$. (The non-uniformity does not directly follow from Exercise 5.49 (5).)

Exercise 5.60 (2)

This is parallel to Exercise 5.49 (6). We have $\lim n(x^{\frac{1}{n}} - 1) = \log x$ uniformly on $[R^{-1}, R]$ for any $R > 1$, not uniformly $(0, R^{-1}]$, and not uniformly on $[R, +\infty)$.

Exercise 5.60 (3)

For any fixed $R > 0$, we have

$$n > \frac{1}{\epsilon} + R,\ x \ge -R \implies \left| \frac{1}{n+x} \right| \le \frac{1}{n-R} < \epsilon.$$

Therefore the sequence $\dfrac{1}{n+x}$ converges to 0 uniformly for $x \ge -R$.

The convergence to 0 is not uniform on $(-\infty, -R)$ for the following reason. For any N, we take any natural number $n > \max\{N, R\}$, and take $x = -n-1$. Then

$$n > N,\ x \in (-\infty, -R),\ \text{but} \ \left| \frac{1}{n+x} \right| = \left| \frac{1}{n + (-n-1)} \right| = 1.$$

Therefore the uniform convergence on $(-\infty, -R)$ fails for $\epsilon = 1$.

Exercise 5.60 (4)

For any fixed $r > 0$, we have

$$n > \frac{2}{\min\{r, \epsilon|x|\}},\ |x| \ge r \implies \left| \frac{1}{nx+1} \right| \le \frac{2}{|nx|} < \epsilon.$$

This shows that $\lim \frac{1}{nx+1} = 0$ (and $= 1$ for $x = 0$) is uniform for $|x| \ge r$.

The limit is not uniform on $(0, r]$ because for any $n > \frac{1}{r}$, we may choose $x = \frac{1}{n} \in (0, r]$ and get $\left|\frac{1}{nx+1}\right| = \frac{1}{2}$, so the uniformity fails for $\epsilon = \frac{1}{2}$. The limit is also not uniform on $[-r, 0)$ because for any $n > \frac{1}{r}$, we may choose $x = -\frac{1}{2n} \in [-r, 0)$ and get $\left|\frac{1}{nx+1}\right| = 2$, so the uniformity fails for $\epsilon = 2$.

Exercise 5.60 (5)

The sequence converges only for $x \geq 0$, and the limit is 0. For any $\epsilon > 0$, we have $\dfrac{x}{n^x} < \epsilon$ on $[0, \epsilon]$ for any n. Then there is $N > 0$ satisfying $N^{\frac{x}{2}} > x$ for $x \geq \epsilon$ and satisfying $N^{\frac{\epsilon}{2}} > \epsilon^{-1}$. Then for $n > N$ and $x \in [\epsilon, +\infty)$, we have $\dfrac{x}{n^x} < \dfrac{x}{N^{\frac{x}{2}}} \cdot \dfrac{1}{N^{\frac{\epsilon}{2}}} < \epsilon$. This proves the uniformity on $[0, +\infty)$.

Exercise 5.60 (6)

By $\left|\frac{\sin nx}{n}\right| \leq \frac{1}{n}$ and $\lim \frac{1}{n} = 0$, the convergence of $\lim \frac{\sin nx}{n} = 0$ is uniform.

Exercise 5.60 (7)

For any $R > 0$, we have

$$n > \frac{R}{\epsilon},\ |x| \leq R \implies \left|\sin \frac{x}{n}\right| \leq \frac{|x|}{n} < \epsilon.$$

Therefore $\lim \sin \frac{x}{n} = 0$ uniformly for $|x| \leq R$.

On the other hand, for any fixed n and any $R > 0$, we can find x satisfying $x > R$ and $\frac{x}{n} \in 2\mathbb{N}\pi + \frac{\pi}{2}$. Then $\sin \frac{x}{n} = 1$. This shows that the uniformity of $\lim \sin \frac{x}{n} = 0$ on $[R, +\infty)$ fails for $\epsilon = 1$. By the same reason, the limit is also not uniform on $(-\infty, -R]$.

Exercise 5.60 (8)

We have

$$\lim \sqrt[n]{1 + x^n} = g(x) = \begin{cases} 1, & \text{if } |x| < 1, \\ x, & \text{if } x \geq 1. \end{cases}$$

Let $0 < r < 1$. For $-r < x \leq 1$, we have

$$\sqrt[n]{1 + x^n} \leq \sqrt[n]{1 - r^n} \leq \sqrt[n]{2}.$$

By $\lim \sqrt[n]{1 - r^n} = \lim \sqrt[n]{2} = 1$ and sandwich type argument, we find that $\lim \sqrt[n]{1 + x^n} = g(x)$ is uniform on $[-r, 1]$.

For each fixed odd n, we have $\lim_{x \to (-1)^-} (\sqrt[n]{1 + x^n} - g(x)) = -1$. This implies that the uniformity on $(-1, -r)$ fails for $\epsilon = \frac{1}{2}$.

We have $\log(1 + z) < z$ for $z > 0$. Then for $x > 1$, we have

$$\begin{aligned} \sqrt[n]{1 + x^n} - x &= x \left(\sqrt[n]{1 + \left(\frac{1}{x}\right)^n} - 1 \right) = x \left(e^{\frac{1}{n} \log(1 + x^{-n})} - 1 \right) \leq x \left(e^{\frac{2}{n} x^{-n}} - 1 \right) \\ &= x \left(\frac{2}{n} x^{-n} + o\left(\frac{2}{n} x^{-n} \right) \right) < x \frac{3}{n} x^{-n} = \frac{3}{n}. \end{aligned}$$

By $\frac{x^{-n}}{n} \le \frac{1}{n}$, the second inequality holds for sufficiently big n. The estimation shows that $\lim \sqrt[n]{1+x^n} = g(x)$ is uniform on $(1, +\infty)$.

We conclude that $\lim \sqrt[n]{1+x^n} = g(x)$ uniformly on $(-r, +\infty)$, and not uniformly on $(-1, -r)$.

Exercise 5.60 (9)

We have $\lim \left(x + \frac{1}{n}\right)^p = x^p$ for $x > 0$. The limit is obtained by applying y^p to $\lim \left(x + \frac{1}{n}\right) = x$, which converges uniformly on $(0, +\infty)$. Since for $0 \le p \le 1$, the function y^p is uniformly continuous, we conclude that $\lim \left(x + \frac{1}{n}\right)^p = x^p$ uniformly for $x > 0$.

For $p > 1$, the function y^p is uniformly continuous on any bounded interval. Therefore for any $R > 0$, $\lim \left(x + \frac{1}{n}\right)^p = x^p$ is uniform on $(0, R]$. On the other hand, for $x \ge R$ and $n > \frac{1}{R}$, we have

$$\left(x + \frac{1}{n}\right)^p - x^p = x^p \left[\left(1 + \frac{1}{nx}\right)^p - 1\right] = x^p \left[\frac{p}{nx} + o\left(\frac{1}{nx}\right)\right] > x^p \frac{1}{nx} = \frac{x^{p-1}}{n},$$

where the inequality may require sufficiently big n (the bigness only depends on R). The estimation implies that, for each fixed big n, we have $\lim_{x \to +\infty} \left[\left(x + \frac{1}{n}\right)^p - x^p\right] = +\infty$. This implies that $\lim \left(x + \frac{1}{n}\right)^p = x^p$ is not uniform on $[R, +\infty)$.

For $p < 0$, the similar argument shows that $\lim \left(x + \frac{1}{n}\right)^p = x^p$ is uniform on $[R, +\infty)$ and not uniform on $(0, R]$.

Exercise 5.60 (10)

We have $\lim \left(1 + \frac{x}{n}\right)^p = 1$. The uniformity is the same as the uniformity of $\lim \frac{x}{n} = 0$, which is uniform for $|x| \le R$ and not uniform on $[R, +\infty)$ or $(-\infty, -R]$.

Exercise 5.60 (11)

We have $\lim \log \left(1 + \frac{x}{n}\right) = 1$. The uniformity is the same as the uniformity of $\lim \frac{x}{n} = 0$, which is uniform for $|x| \le R$ and not uniform on $[R, +\infty)$ or $(-\infty, -R]$.

Exercise 5.60 (12)

By the same argument as Example 5.3.8, the convergence is uniform for $|x| \le R$ and not uniform for $x \ge R$.

Exercise 5.61 (for Comparison Test)

Since $\sum v_n(x)$ uniformly converges, for any $\epsilon > 0$, there is N, such that

$$n \ge m > N,\ x \in X \implies |v_m(x) + v_{m+1}(x) + \cdots + v_n(x)| < \epsilon.$$

Then by $|u_n(x)| \le v_n(x)$, we have

$$\begin{aligned} n \ge m > N,\ x \in X \implies & |u_m(x) + u_{m+1}(x) + \cdots + u_n(x)| \\ & \le |u_m(x)| + |u_{m+1}(x)| + \cdots + |u_n(x)| \\ & \le v_m(x) + v_{m+1}(x) + \cdots + v_n(x) < \epsilon. \end{aligned}$$

This verifies the Cauchy criterion for the uniform convergence of $\sum u_n(x)$.

Exercise 5.62

For any $\epsilon > 0$, there is $\delta > 0$, such that

$$0 < |x - a| < \delta,\ 0 < |x' - a| < \delta,\ y \in Y \implies |f(x, y) - f(x', y)| < \epsilon.$$

Exercise 5.63

Suppose for each x, $u_n(x)$ is monotone in n. If $u_n(x)$ uniformly converges to 0, then $\sum(-1)^n u_n(x)$ uniformly converges to 0.

The statement can be proved by the uniform Dirichlet test (Proposition 5.3.3).

Direct Proof: For fixed x, if $u_n(x)$ is decreasing, then the partial sum satisfies

$$\begin{aligned} s_{2n}(x) &= s_{2n-2}(x) - u_{2n-1}(x) + u_{2n}(x) \le s_{2n-2}(x), \\ s_{2n+1}(x) &= s_{2n-1}(x) + u_{2n}(x) - u_{2n+1}(x) \ge s_{2n-2}(x). \end{aligned}$$

Therefore $s_{2n}(x)$ is decreasing and $s_{2n+1}(x)$ is increasing. Moreover, we have $s_{2n}(x) - s_{2n+1}(x) = u_{2n+1}(x) \ge 0$. Therefore the decreasing sequence is bigger than the increasing sequence, and the difference of the two sequences converges to 0. This implies both sequences converge and we have $\lim s_{2n}(x) = \lim s_{2n+1}(x) = g(x)$. Moreover, we also have $s_{2n}(x) \ge g(x) \ge s_{2n+1}(x)$, so that $s_{2n}(x) - g(x) \le s_{2n}(x) - s_{2n+1}(x) = u_{2n+1}(x)$. Similar argument also shows $g(x) - s_{2n+1}(x) \le u_{2n+2}(x)$. We conclude $|s_n(x) - g(x)| \le u_{n+1}(x)$.

If $u_n(x)$ is increasing, the same argument shows that $\sum u_n(x)$ converges and $|s_n(x) - g(x)| \le u_{n+1}(x)$. Then the uniformity of $\lim u_n(x) = 0$ implies the uniformity of $\lim s_n(x) = g(x)$.

Exercise 5.64

We have

$$\left|\frac{f_n(x)}{n^p}\right| \le \frac{1}{2}\left(f_n(x)^2 + \frac{1}{n^{2p}}\right).$$

By $2p > 1$, $\sum \dfrac{1}{n^{2p}}$ converges. By the uniform comparison test, the uniform convergence of $\sum f_n(x)^2$ implies the uniform convergence of $\sum \dfrac{f_n(x)}{n^p}$.

Exercise 5.65 (1)

The series is $\sum(xe^{-x})^n$. By the discussion in Example 5.3.7, for any $0 < R < 1$, the series converges uniformly for $|xe^{-x}| \le R$. The range is the same as $x \in [R, +\infty)$ for any (different) R. Therefore $\sum(xe^{-x})^n$ uniformly converges on $[R, +\infty)$ for any R.

Exercise 5.65 (2)

For any $0 < r < 1$, we have

$$|x| \le r \implies |n^x x^n| \le nr^n.$$

By the uniform comparison test, the convergence of $\sum nr^n$ implies the uniform convergence of $\sum n^x x^n$ for $|x| \le r$.

For each fixed n, we have $\lim_{x\to 1^-} n^x x^n = n > 1$. This implies that $n^x x^n$ does not uniformly converge to 0 for $r < x < 1$. Therefore $\sum n^x x^n$ does not uniformly converge for $r \le x < 1$.

For $-1 \le x < -r$, the series $\sum n^x x^n$ is alternating and converges to a function $g(x)$ by the Leibniz test. Moreover, the Leibniz test, the partial sum satisfies $|\sum_{k=1}^n k^x x^k - g(x)| \le |(n+1)^x x^{n+1}| \le (n+1)^{-r}$. By $\lim (n+1)^{-r} = 0$, we find that $\sum n^x x^n$ converges uniformly for $-1 < x < -r$.

We conclude that $\sum n^x x^n$ uniformly converges on $[-1, r]$ and not uniformly on $[r, 1)$.

Exercise 5.65 (3)

For any $0 < r < 1$, pick any r' satisfying $r < r' < 1$. We can further find $N > 0$ satisfying $\frac{1+N}{N} < \frac{r'}{r}$. Then

$$n > N,\ |x| \le r \implies \left|\frac{x(x+n)}{n}\right| \le |x|\frac{1+n}{n} \le r\frac{1+N}{N} < r' \implies \left|\left(\frac{x(x+n)}{n}\right)^n\right| < r'^n.$$

By the convergence of $\sum r'^n$, we find that $\sum \left(\frac{x(x+n)}{n}\right)^n$ converges uniformly for $|x| \le r$.

On the other hand, for any fixed and big n, we have

$$\lim_{x\to 1^-} \frac{x(x+n)}{n} = \frac{1+n}{n} > 1, \quad \lim_{x\to(-1)^+} \left|\frac{x(x+n)}{n}\right|^n = \left(1 - \frac{1}{n}\right)^n > \frac{e^{-1}}{2}.$$

The limits imply that the series does not uniformly converge on $(-1, -r]$ and $[r, 1)$.

Exercise 5.65 (4)

The series converges only when $p > 1$. Then for $x \ge 0$, we have $\frac{1}{n^p + x^p} \le \frac{1}{n^p}$. By the convergence of $\sum \frac{1}{n^p}$ and the comparison test, $\sum \frac{1}{n^p + x^p}$ uniformly converges for $x \ge 0$.

Exercise 5.65 (5)

The series converges only when $|a| > 1$. Fix arbitrary R. Then for $x \ge R$ and big n, we have $\left|\frac{1}{x+a^n}\right| \le \frac{1}{|a|^n - |R|}$. By the convergence of $\sum \frac{1}{|a|^n - |R|}$, the series $\sum \frac{1}{x+a^n}$ uniformly converges for $x \ge R$.

Exercise 5.65 (6)

The series converges only when $|x| < 1$. For any $0 < r < 1$ and $|x| \le r$, we have $\left|\frac{x^n}{1-x^n}\right| \le \frac{r^n}{1-r^n} \le \frac{r^n}{1-R}$. By the convergence of $\sum \frac{r^n}{1-r}$, the series uniformly converges for $|x| \le r$.

For any fixed n, we have $\lim_{x\to 1^-} \frac{x^n}{1-x^n} = \lim_{x\to(-1)^+} \frac{x^{2n}}{1-x^{2n}} = \infty$. This implies that the series does not uniformly converge on $(-1, -r]$ and $[r, 1)$.

Exercise 5.65 (7)

By the same reason as in Example 5.3.12, for $p > 1$, the series uniformly converges for all x. For $0 < p \le 1$, the series uniformly converges on $[r, \pi - r]$ for any $0 < r < \frac{\pi}{2}$ and does not uniformly converges on $(0, r]$ and $[\pi - r, r)$.

Exercise 5.65 (8)

We have $\sum \frac{\sin^3 nx}{n^p} = \frac{1}{4}\sum \frac{3\sin nx - \sin 3nx}{n^p}$.

For $p > 1$, both series $\sum \frac{\sin nx}{n^p}$ and $\sum \frac{\sin 3nx}{n^p}$ converges uniformly for all x. Therefore $\sum \frac{\sin^3 nx}{n^p}$ converges uniformly for all x.

For $0 < p \le 1$ and any $0 < r < \frac{\pi}{6}$, the series $\sum \frac{\sin nx}{n^p}$ uniformly converges on $[r, \pi - r]$. Then the uniformity of the convergence of the series $\sum \frac{\sin 3nx}{n^p}$ on $[r, \frac{\pi}{3} - r]$, $[\frac{\pi}{3} + r, \frac{2\pi}{3} - r]$, and $[\frac{2\pi}{3}+r,\pi-r]$ implies the uniformity of the convergence of $\sum\frac{\sin^3 nx}{n^p}$ on each of the three intervals. Moreover, the non-uniformity of the convergence of $\sum\frac{\sin 3nx}{n^p}$ on $[\frac{\pi}{3}-r,\frac{\pi}{3}+r]$ and $[\frac{2\pi}{3}-r,\frac{2\pi}{3}+r]$ implies the on-uniformity of the convergence of $\sum\frac{\sin^3 nx}{n^p}$ on the two intervals.

[The uniformity on $(0,r]$ is more complicated.]

Exercise 5.65 (9)

The series converges for all x when $p>0$, or $p=0$ and $q>0$.

We always have $\left|\frac{\sin nx}{n^p(\log n)^q}\right|\le\frac{1}{n^p(\log n)^q}$. If $p>1$, or $p=1$ and $q>1$, then $\sum\frac{1}{n^p(\log n)^q}$ converges. Then by the uniform comparison test, $\sum\frac{\sin nx}{n^p(\log n)^q}$ uniformly converges for all x.

If $0<p\le 1$, or $p=0$ and $q>0$, then $\frac{1}{n^p(\log n)^q}$ is decreasing and (uniformly) converges to 0. Moreover, the partial sum of $\sum nx$ is uniformly bounded on $[r,\pi-r]$ (and $[m\pi+r,(m+1)\pi-r]$) for any $0<r<\frac{\pi}{2}$. Then by the uniform Dirichlet test, $\sum\frac{\sin nx}{n^p(\log n)^q}$ uniformly converges on $[r,\pi-r]$.

For $p<1$, or $p=0$ and $q\le 0$, by the argument similar to Example 5.3.12, we get non-uniformity on $(0,r)$ (and on $(\pi-r,\pi)$). Specifically, suppose $\frac{\sin x}{x}>\frac{1}{2}$ on some interval $(0,b)$. For any N, fix any natural number satisfying $n>N$, $n>\frac{b}{2r}$ and $(\log k)^q\le k^{1-p}$ for all $k>n$ (the last property follows from $p<1$, or $p=0$ and $q\le 0$). Then $x=\frac{b}{2n}$ satisfies $x\in(0,r)$, $kx\in(0,b)$ for all $n<k\le 2n$, and

$$\sum_{k=n+1}^{2n}\frac{\sin kx}{k^p(\log k)^q}=x\sum_{k=n+1}^{2n}\frac{\sin kx}{kx}\frac{k^{1-p}}{(\log k)^q}>x\sum_{k=n+1}^{2n}\frac{1}{2}\frac{k^{1-p}}{(\log k)^q}>x\sum_{k=n+1}^{2n}\frac{1}{2}=\frac{nx}{2}=\frac{b}{4}.$$

This shows that the Cauchy criterion for the uniform convergence of $\sum\frac{\sin kx}{k^p(\log k)^q}$ on $(0,r)$ fails for $\epsilon=\dfrac{b}{4}$.

[It remains to consider $p=1$ and $0<q\le 1$. The uniformity on $(0,r]$ is more complicated.]

Exercise 5.65 (10)

In case $\frac{x^2}{n}$ is sufficiently small, we have

$$e^x-\left(1+\frac{x}{n}\right)^n=e^x-e^{n\log(1+\frac{x}{n})}=e^x-e^{n[\frac{x}{n}-\frac{x^2}{2n^2}+o(\frac{x^2}{n^2})]}=e^x-e^{x-\frac{x^2}{2n}+o(\frac{x^2}{n})}=e^x\left(\frac{x^2}{2n}+o\left(\frac{x^2}{n}\right)\right).$$

Let $R>0$. For $|x|\le R$, we have sufficiently small $\frac{x^2}{n}$ for sufficiently big n, and the estimation above can be applied to give

$$\frac{x^2e^x}{3n}\le e^x-\left(1+\frac{x}{n}\right)^n\le\frac{x^2e^x}{n}.$$

This implies that $\sum\left|e^x-\left(1+\dfrac{x}{n}\right)^n\right|^p$ converges for some $x\ne 0$ if and only if $\sum\frac{1}{n^p}$ converges, which means $p>1$. This is also the condition for the series to converge for *all* x.

So we assume $p>1$. For $|x|\le R$ and sufficiently big n, the estimation above implies

$$\left|e^x-\left(1+\frac{x}{n}\right)^n\right|^p\le\frac{(R^2e^R)^p}{n^p}.$$

By the comparison test, we conclude that $\sum \left| e^x - \left(1 + \frac{x}{n}\right)^n \right|^p$ uniformly converges for $|x| \le R$.

For each fixed n, we have $\lim_{x\to\infty} \left| e^x - \left(1 + \frac{x}{n}\right)^n \right|^p = +\infty$. This implies that the sequence $\left| e^x - \left(1 + \frac{x}{n}\right)^n \right|^p$ does not uniformly converge to 0, so that the convergence of the series is not uniform for $x \ge R$ and for $x \le -R$.

Exercise 5.66

The series $\sum(-1)^n x^n (1-x)$ has

$$s_n = \frac{1-x}{1+x}(1 + (-1)^n x^{n+1}), \quad \left| s_n - \frac{1-x}{1+x} \right| = \frac{1-x}{1+x} x^{n+1} \le (1-x)x^{n+1}.$$

For any $1 > \epsilon > 0$, let N be big enough to satisfy $(1-\epsilon)^N < \epsilon$. Then

$$\begin{aligned} x \in [1-\epsilon, 1] &\implies 1 - x \le \epsilon,\ x \le 1 \implies (1-x)x^{n+1} \le \epsilon, \\ n > N,\ x \in [0, 1-\epsilon] &\implies x^{n+1} \le (1-\epsilon)^N < \epsilon,\ 1 - x \le 1 \implies (1-x)x^{n+1} \le \epsilon. \end{aligned}$$

Combining the two implications, we get

$$n > N,\ x \in [0,1] \implies \left| s_n - \frac{1-x}{1+x} \right| < \epsilon.$$

Therefore the series uniformly converges on $[0,1]$.

If $0 \le x < 1$, then $\sum(-1)^n x^n$ absolutely converges, so that $\sum(-1)^n x^n(1-x)$ absolutely converges. On the other hand, when $x = 1$, the series $\sum(-1)^n x^n(1-x) = \sum 0$ also absolutely converges. Therefore the series absolutely converges at each $x \in [0,1]$.

Finally, for $x \in [0,1]$, the partial sum of $\sum |(-1)^n x^n (1-x)|$ is $1 - x^{n+1}$, which does not converge uniformly on $[0,1]$.

Exercise 5.67

For Exercise 5.49:

(1) The equality holds for $a \neq 0$.

$$\lim_{x\to\infty}\lim_{y\to a}\frac{1}{xy} = \lim_{x\to\infty}\lim_{y\to a}\frac{1}{xa} = 0,$$
$$\lim_{y\to a}\lim_{x\to\infty}\frac{1}{xy} = \lim_{y\to a} 0 = 0.$$

(2) The equality holds.

$$\lim_{x\to 0}\lim_{y\to a}\frac{\sin xy}{x} = \lim_{x\to 0}\frac{\sin xa}{x} = a,$$
$$\lim_{y\to a}\lim_{x\to 0}\frac{\sin xy}{x} = \lim_{y\to a} y = a.$$

(3) The equality holds.

$$\lim_{x\to 0^+}\lim_{y\to a}\sqrt{x+y} = \lim_{x\to 0^+}\sqrt{x+a} = \sqrt{a},$$
$$\lim_{y\to a}\lim_{x\to 0^+}\sqrt{x+y} = \lim_{y\to a}\sqrt{y} = \sqrt{a}.$$

(4) The equality holds.

$$\lim_{x\to 0}\lim_{y\to a}(1+xy)^{\frac{1}{x}} = \lim_{x\to 0}(1+xa)^{\frac{1}{x}} = e^a,$$
$$\lim_{y\to a}\lim_{x\to 0}(1+xy)^{\frac{1}{x}} = \lim_{y\to a} e^y = e^a.$$

(5) The equality holds for $a > 0$.

$$\lim_{x\to 0}\lim_{y\to a} y^x = \lim_{x\to 0} a^x = 1,$$
$$\lim_{y\to a}\lim_{x\to 0} y^x = \lim_{y\to a} 1 = 1.$$

However,

$$\lim_{x\to 0^+}\lim_{y\to 0^+} y^x = \lim_{x\to 0^+} 0 = 0, \qquad \lim_{x\to 0^-}\lim_{y\to 0^+} y^x = \lim_{x\to 0^-} +\infty = +\infty,$$
$$\lim_{y\to 0^+}\lim_{x\to 0^+} y^x = \lim_{y\to 0^+} 1 = 1, \qquad \lim_{y\to 0^+}\lim_{x\to 0^-} y^x = \lim_{y\to 0^+} 1 = 1.$$

(6) The equality holds for $a > 0$.

$$\lim_{x\to 0}\lim_{y\to a}\frac{y^x-1}{x} = \lim_{x\to 0}\frac{a^x-1}{x} = \log a,$$
$$\lim_{y\to a}\lim_{x\to 0}\frac{y^x-1}{x} = \lim_{y\to a}\log y = \log a.$$

Moreover,

$$\lim_{x\to 0^+}\lim_{y\to 0^+}\frac{y^x-1}{x}=\lim_{x\to 0^+}-\frac{1}{x}=-\infty, \qquad \lim_{x\to 0^-}\lim_{y\to 0^+}\frac{y^x-1}{x}=\lim_{x\to 0^-}-\infty=-\infty,$$
$$\lim_{y\to 0^+}\lim_{x\to 0^+}\frac{y^x-1}{x}=\lim_{y\to 0^+}\log y=-\infty, \qquad \lim_{y\to 0^+}\lim_{x\to 0^-}\frac{y^x-1}{x}=\lim_{y\to 0^+}\log y=-\infty.$$

For Exercise 5.50:
(1) The equality holds.

$$\lim_{x\to y}\lim_{y\to a}\frac{\sin x-\sin y}{x-y}=\lim_{x\to y}\frac{\sin x-\sin a}{x-a}=\cos a,$$
$$\lim_{y\to a}\lim_{x\to y}\frac{\sin x-\sin y}{x-y}=\lim_{y\to a}\cos y=\cos a.$$

(2) The equality holds for $a>0$.

$$\lim_{x\to y}\lim_{y\to a}\frac{x^p-y^p}{x-y}=\lim_{x\to y}\frac{x^p-a^p}{x-a}=pa^{p-1},$$
$$\lim_{y\to a}\lim_{x\to y}\frac{x^p-y^p}{x-y}=\lim_{y\to a}py^{p-1}=pa^{p-1}.$$

For Exercise 5.51:
(1) The equality holds.

$$\lim_{n\to\infty}\lim_{m\to\infty}\frac{1}{m+n}=\lim_{n\to\infty}0=0,$$
$$\lim_{m\to\infty}\lim_{n\to\infty}\frac{1}{m+n}=\lim_{m\to\infty}0=0.$$

(2) The equality fails.

$$\lim_{n\to\infty}\lim_{m\to\infty}\frac{m}{m+n}=\lim_{n\to\infty}1=1,$$
$$\lim_{m\to\infty}\lim_{n\to\infty}\frac{m}{m+n}=\lim_{m\to\infty}0=0.$$

(3) The equality holds.

$$\lim_{n\to\infty}\lim_{m\to\infty}\frac{1}{n^m}=\lim_{n\to\infty}0=0,$$
$$\lim_{m\to\infty}\lim_{n\to\infty}\frac{1}{n^m}=\lim_{m\to\infty}0=0.$$

(4) The equality fails.

$$\lim_{n\to\infty}\lim_{m\to\infty}\frac{m}{n}=\lim_{n\to\infty}+\infty=+\infty,$$
$$\lim_{m\to\infty}\lim_{n\to\infty}\frac{m}{n}=\lim_{m\to\infty}0=0.$$

(5) The equality fails.

$$\lim_{n\to\infty}\lim_{m\to\infty}\sqrt[n]{m} = \lim_{n\to\infty} +\infty = +\infty,$$
$$\lim_{m\to\infty}\lim_{n\to\infty}\sqrt[n]{m} = \lim_{m\to\infty} 1 = 1.$$

For Exercise 5.60:
(1) The equality does not hold for $a = 0^+$ and $a = +\infty$.

$$\lim_{n\to\infty}\lim_{x\to a} x^{\frac{1}{n}} = \lim_{n\to\infty} a^{\frac{1}{n}} = \begin{cases} 1, & \text{if } 0 < a < +\infty \\ +\infty, & \text{if } a = +\infty \\ 0, & \text{if } a = 0 \end{cases}$$
$$\lim_{x\to a}\lim_{n\to\infty} x^{\frac{1}{n}} = \lim_{x\to a} \begin{cases} 1, & \text{if } 0 < x < +\infty \\ 0, & \text{if } x = 0 \end{cases} = 1.$$

(2) The equality holds.

$$\lim_{n\to\infty}\lim_{x\to a} n(x^{\frac{1}{n}} - 1) = \lim_{n\to\infty} n(a^{\frac{1}{n}} - 1) = \log a$$
$$\lim_{x\to a}\lim_{n\to\infty} n(x^{\frac{1}{n}} - 1) = \lim_{x\to a} \log x = \log a.$$

(3) The equality holds.

$$\lim_{n\to\infty}\lim_{x\to a}\frac{1}{n+x} = \lim_{n\to\infty}\frac{1}{n+a} = 0,$$
$$\lim_{x\to a}\lim_{n\to\infty}\frac{1}{n+x} = \lim_{x\to a} 0 = 0.$$

(4) The equality does not hold for $a = 0$.

$$\lim_{n\to\infty}\lim_{x\to a}\frac{1}{nx+1} = \lim_{n\to\infty}\frac{1}{na+1} = \begin{cases} 1, & \text{if } a = 0 \\ 0, & \text{if } a \neq 0 \end{cases}$$
$$\lim_{x\to a}\lim_{n\to\infty}\frac{1}{nx+1} = \lim_{x\to a}\begin{cases} 1, & \text{if } x = 0 \\ 0, & \text{if } x \neq 0 \end{cases} = 0.$$

(5) The equality holds (what happens at $a = 0$ is a matter of opinion).

$$\lim_{n\to\infty}\lim_{x\to a}\frac{x}{n^x} = \lim_{n\to\infty}\frac{a}{n^a} = \begin{cases} -\infty, & \text{if } a < 0 \\ 0, & \text{if } a \geq 0 \end{cases},$$
$$\lim_{x\to a}\lim_{n\to\infty}\frac{x}{n^x} = \lim_{x\to a}\begin{cases} -\infty, & \text{if } x < 0 \\ 0, & \text{if } x \geq 0 \end{cases} = \begin{cases} -\infty, & \text{if } a < 0 \\ 0, & \text{if } a > 0 \end{cases}.$$

(6) The equality holds.

$$\lim_{n\to\infty}\lim_{x\to a}\frac{\sin nx}{n} = \lim_{n\to\infty}\frac{\sin na}{n} = 0,$$
$$\lim_{x\to a}\lim_{n\to\infty}\frac{\sin nx}{n} = \lim_{x\to a} 0 = 0.$$

(7) The equality holds.

$$\lim_{n\to\infty}\lim_{x\to a}\sin\frac{x}{n} = \lim_{n\to\infty}\sin\frac{a}{n} = 0,$$
$$\lim_{x\to a}\lim_{n\to\infty}\sin\frac{x}{n} = \lim_{x\to a} 0 = 0.$$

(8) The equality holds.

$$\lim_{n\to\infty}\lim_{x\to a}\sqrt[n]{1+x^n} = \lim_{n\to\infty}\sqrt[n]{1+a^n} = \max\{1,a\},$$
$$\lim_{x\to a}\lim_{n\to\infty}\sqrt[n]{1+x^n} = \lim_{x\to a}\max\{1,x\} = \max\{1,a\}.$$

(9) The equality holds.

$$\lim_{n\to\infty}\lim_{x\to a}\left(x+\frac{1}{n}\right)^p = \lim_{n\to\infty}\left(a+\frac{1}{n}\right)^p = a^p,$$
$$\lim_{x\to a}\lim_{n\to\infty}\left(x+\frac{1}{n}\right)^p = \lim_{x\to a} x^p = a^p.$$

(10) The equality holds.

$$\lim_{n\to\infty}\lim_{x\to a}\left(1+\frac{x}{n}\right)^p = \lim_{n\to\infty}\left(1+\frac{a}{n}\right)^p = 1,$$
$$\lim_{x\to a}\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^p = \lim_{x\to a} 1 = 1.$$

(11) The equality holds.

$$\lim_{n\to\infty}\lim_{x\to a}\log\left(1+\frac{x}{n}\right) = \lim_{n\to\infty}\log\left(1+\frac{a}{n}\right) = 0,$$
$$\lim_{x\to a}\lim_{n\to\infty}\log\left(1+\frac{x}{n}\right) = \lim_{x\to a} 0 = 0.$$

(12) The equality holds.

$$\lim_{n\to\infty}\lim_{x\to a}\left(1+\frac{x}{n}\right)^n = \lim_{n\to\infty}\left(1+\frac{a}{n}\right)^n = e^a,$$
$$\lim_{x\to a}\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = \lim_{x\to a} e^x = e^a.$$

For Exercise 5.65:

(1) The uniform convergence on $[R, +\infty)$ for any R implies that the equality holds on $[R, +\infty)$ for any R. Therefore the equality holds for all x.

(2) The uniform convergence on $[-1, r]$ for any $0 < r < 1$ implies that the equality holds on $[-1, r]$ for any $0 < r < 1$. Therefore the equality holds for all x satisfying $-1 \le x < 1$.

(3) The uniform convergence on $[-r, r]$ for any $0 < r < 1$ implies that the equality holds on $(-1, 1)$.

(4) The uniform convergence for $x \ge 0$ implies that the equality holds for $x \ge 0$.

(5) The uniform convergence for $x \ge R$ (R arbitrary) implies that the equality holds for all x.

(6) The uniform convergence on $[-r, r]$ for any $0 < r < 1$ implies that the equality holds on $(-1, 1)$.

(7) For $p > 1$, the uniform convergence for all x implies that the equality holds for all x. For $0 < p \le 1$, the uniform convergence on $[r, \pi - r]$ for any $0 < r < \frac{\pi}{2}$ implies that the equality holds on $(0, \pi)$. By trigonometric equality, the equality also holds for all x that are not integer multiples of π. The series diverges at integer multiples of π.

(8) For $p > 1$, the uniform convergence for all x implies that the equality holds for all x.

For $0 < p \le 1$, the uniform convergence on $[r, \frac{\pi}{3} - r]$, $[\frac{\pi}{3} + r, \frac{2\pi}{3} - r]$, and $[\frac{2\pi}{3} + r, \pi - r]$ for any $0 < r < \frac{\pi}{6}$ implies that the equality holds for x that are not integer multiples of $\frac{\pi}{3}$.

We consider $x = 0$ for the case $p = 1$. From the Fourier series, we have

$$\sum_{n=1}^{\infty} \frac{\sin nx}{n} = \begin{cases} -\frac{1}{2}(x - \pi), & \text{if } 0 < x < 2\pi, \\ 0, & \text{if } x = 0, \\ -\frac{1}{2}(x + \pi), & \text{if } -2\pi < x < 0. \end{cases}$$

Therefore $\lim_{x \to 0} \sum_{n=1}^{\infty} \frac{\sin nx}{n}$ diverges, while $\sum_{n=1}^{\infty} \lim_{x \to 0} \frac{\sin nx}{n} = \sum 0 = 0$. By $\sin^3 nx = \frac{1}{4}(3 \sin nx - \sin 3nx)$, this implies that $\lim_{x \to \frac{\pi}{3}} \sum_{n=1}^{\infty} \frac{\sin^3 nx}{n}$ and $\lim_{x \to \frac{2\pi}{3}} \sum_{n=1}^{\infty} \frac{\sin^3 nx}{n}$ diverge. We also have

$$\sum_{n=1}^{\infty} \frac{\sin^3 nx}{n} = \frac{1}{4} \sum_{n=1}^{\infty} \frac{3 \sin nx - \sin 3nx}{n} = \begin{cases} \frac{\pi}{4}, & \text{if } 0 < x < \frac{2}{3}\pi, \\ 0, & \text{if } x = 0, \\ -\frac{\pi}{4}, & \text{if } -\frac{2}{3}\pi < x < 0. \end{cases}$$

So $\lim_{x \to 0} \sum_{n=1}^{\infty} \frac{\sin^3 nx}{n}$ diverges, and we still have $\sum_{n=1}^{\infty} \lim_{x \to 0} \frac{\sin^3 nx}{n} = \sum 0 = 0$.

[For $0 < p < 1$, is it possible to estimate $\sum_{n=1}^{\infty} \frac{\sin nx}{n}$ for x close to 0^+?

If we can show that the sum is always $> \epsilon$ for some $\epsilon > 0$ and x close to 0^+, then (since the sum is odd) the sum is always $< -\epsilon$ and x close to 0^-. We may then conclude that $\lim_{x \to 0} \sum_{n=1}^{\infty} \frac{\sin^3 nx}{n^p}$ diverges.]

(9) For $p > 1$, or $p = 1$ and $q > 1$, the series uniformly converges for all x. Therefore the equality holds for all x.

For $0 < p \le 1$, or $p = 0$ and $q > 0$, the series uniformly converges on $[r, \pi - r]$ for any $0 < r < \frac{\pi}{2}$. Therefore the equality holds as long as x is not an integer multiple of π.

[For the equality at 0 in case $p \le 1$, see the discussion and remark in (8).]

(10) The uniform convergence for $|x| \le R$ and any R implies that the equality holds for all x.

Exercise 5.68

Suppose $f_n(x)$ converges to $f(x)$ near a. Moreover, we assume $f_n(x)$ is monotone in n, and both $f_n(x)$ and $f(x)$ are continuous at a. The question is whether the convergence is uniform near a. A counterexample can be obtained by stacking smaller and smaller version of x^n for $x \in [0,1]$ together. Specifically, the function $r + r\left(\frac{x}{r} - 1\right)^n$ "moves" the function x^n in the square $[0,1] \times [0,1]$ to the square $[r, 2r] \times [r, 2r]$. Then by taking $r = \frac{1}{2^k}$, we define $f_n(x) = \frac{1}{2^k} + \frac{1}{2^k}(2^k x - 1)^n$ for $x \in \left[\frac{1}{2^k}, \frac{1}{2^{k-1}}\right]$, $f(0) = 0$, and define $f(x)$ for $x \in [-1, 0)$ by symmetry. The sequence $f_n(x)$ is decreasing in n, with $f(x) = \lim_{n\to\infty} f_n(x) = \frac{1}{2^k}$ on $\left[\frac{1}{2^k}, \frac{1}{2^{k-1}}\right)$ and $f(0) = 0$. The function $f(x)$ is continuous at 0. However, any interval containing 0 must contain some interval $\left[\frac{1}{2^k}, \frac{1}{2^{k-1}}\right]$. On $\left[\frac{1}{2^k}, \frac{1}{2^{k-1}}\right]$, the sequence contains a smaller version the limit of x^n on $[0,1]$, which by Example 5.3.6 does not converge uniformly.

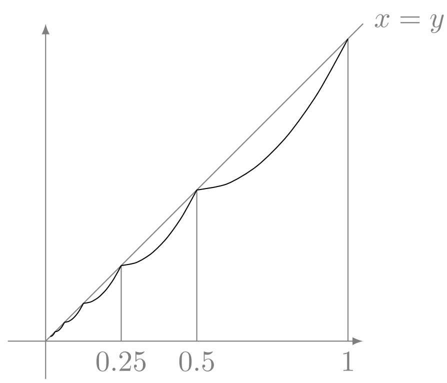

Figure 3: stacked sequence of x^n

Exercise 5.69

Suppose $\lim_{x\to a} f(x,y) = g(y)$ uniformly for $y \in Y$.

If for each x, $f(x,y)$ is continuous in y, then $g(y)$ is continuous.

(Dini) If Y is bounded closed interval, and for each fixed y, $f(x,y)$ is monotone in x, and $g(y)$ is continuous, then $\lim_{x\to a} f(x,y) = g(y)$ uniformly for $y \in Y$.

The proof of Dini's Theorem makes use of the fact that any sequence in Y has a subsequence converging to a limit in Y. The property means that Y is *compact*.

Exercise 5.70

The functions u_n are right continuous. Proposition 5.4.2 can also be applied to right continues: If f_n uniformly converges to f, and f_n are right continuous at x_0, then f is right continuous at x_0. See Exercise 5.68.

Exercise 5.71

For $|x| \le R$, $\frac{x}{n}$ is sufficiently small for suffciently big n (the bigness depends on R). Then we have

$$\frac{d}{dn}\log\left(1+\frac{x}{n}\right)^n = \log\left(1+\frac{x}{n}\right) - \frac{x}{n+x} = \log\left(1+\frac{x}{n}\right) - \frac{x}{n}\frac{1}{1+\dfrac{x}{n}}$$
$$= \frac{x}{n} - \frac{1}{2}\left(\frac{x}{n}\right)^2 - \frac{x}{n}\left(1-\frac{x}{n}\right) + o\left(\frac{x}{n}\right)^2 = \frac{1}{2}\left(\frac{x}{n}\right)^2 + o\left(\frac{x}{n}\right)^2 > 0.$$

Therefore the sequence is increasing for $|x| \le R$ and sufficiently big n. We also know the limit e^x is continuous. Therefore Dini's Theorem can be applied.

Exercise 5.72

By the Cauchy criterion, for any $\epsilon > 0$, there is N (depending only on ϵ), such that

$$m, n > N,\ y \in Y \implies |f_m(y) - f_n(y)| < \epsilon.$$

Now for any $x \in X$, let $x = \lim y_i$ for a sequence $y_i \in Y$. Then

$$m, n > N \implies |f_m(y_i) - f_n(y_i)| < \epsilon \text{ for all } i.$$

Taking $i \to \infty$ on the right side, we get

$$m, n > N \implies |f_m(x) - f_n(x)| \le \epsilon.$$

Since this holds for all $x \in X$, and N depends only on ϵ, this verifies the Cauchy criterion for the uniform convergence of f_n on X.

Exercise 5.73

For any $\epsilon > 0$, by the uniform convergence, there is n, such that $|f_n(x) - f(x)| < \epsilon$ for all x. By the uniform continuity of $f_n(x)$, there is $\delta > 0$, such that $|x - y| < \delta$ implies $|f_n(x) - f_n(y)| < \epsilon$. Then

$$|x - y| < \delta \implies |f(x) - f(y)| \le |f(x) - f_n(x)| + |f_n(x) - f_n(y)| + |f(y) - f_n(y)| < 3\epsilon.$$

Thus $f(x)$ is uniformly continuous.

Exercise 5.74

For any $\epsilon > 0$, there is N, such that

$$m, n > N,\ x \in [a, b] \implies |f_m(x) - f_n(x)| < \epsilon.$$

Then

$$m, n > N \implies \left|\int_a^b f_m(x)dx - \int_a^b f_n(x)dx\right| \le \epsilon(b-a).$$

This implies that $\displaystyle\int_a^b f_n(x)dx$ is a Cauchy sequence and converges to some limit I.

By the uniform convergence of f_n, for any $\epsilon > 0$, there is N, such that

$$n > N,\ x \in [a,b] \implies |f_n(x) - f(x)| < \epsilon \implies |S(P, f_n) - S(P, f)| < \epsilon(b-a),$$

where we choose the same sample points for f and f_n in the second implication. By $\lim_{n\to\infty} \int_a^b f_n(x)dx = I$, there is N', such that

$$n > N' \implies \left|\int_a^b f_n(x)dx - I\right| < \epsilon.$$

Now fix one $n > \max\{N, N'\}$. By the integrability of $f_n(x)$, for the given $\epsilon > 0$, there is $\delta > 0$, such that

$$\|P\| < \delta \implies \left|S(P, f_n) - \int_a^b f_n(x)dx\right| < \epsilon.$$

Then $\|P\| < \delta$ implies

$$|S(P,f) - I| \le |S(P, f_n) - S(P, f)| + \left|S(P, f_n) - \int_a^b f_n(x)dx\right| + \left|\int_a^b f_n(x)dx - I\right| \le \epsilon(b-a+2).$$

This proves that f is integrable and $\int_a^b f(x)dx = I$.

EXERCISE 5.75

Suppose $\lim_{x\to a} f(x,y) = g(y)$ uniformly for y in a bounded interval I.

If $f(x,y)$ is integrable on I for each x near x, then g is also integrable on I, and

$$\int_a^b g(y)dy = \lim_{x\to a} \int_a^b f(x,y)dy.$$

EXERCISE 5.76

By the uniform convergence, for any $\epsilon > 0$, there is N, such that $n > N$ implies $|f_n(x) - f(x)| < \epsilon$ for any $n > N$ and all x. Then

$$\left|\int_a^x f_n(t)dt - \int_a^x f(t)dt\right| \le \int_a^x |f_n(t) - f(t)|dt \le \epsilon(x-a) \le \epsilon(b-a)$$

for any $n > N$ and all $x \in [a,b]$. Therefore $\lim_{n\to\infty} \int_a^x f_n(t)dt = \int_a^x f(t)$ uniformly.

EXERCISE 5.77

Suppose f_n are Riemann-Stieltjes integrable with respect to α. Suppose $\lim_{n\to\infty} f_n = f$ uniformly on $[a,b]$ and α has bounded variation. Then f is Riemann-Stieltjes integrable with respect to α, and we have $\lim_{n\to\infty} \int_a^b f_n d\alpha = \int_a^b f d\alpha$.

The proof of Exercise 5.74 can be adopted. The key is that $|f_m - f_n| < \epsilon$ on $[a,b]$ implies $\left|\int_a^b f_m d\alpha - \int_a^b f_n d\alpha\right| \le \epsilon V_{[a,b]}(\alpha)$, and $|f_n - f| < \epsilon$ on $[a,b]$ implies $|S(P, f_n, \alpha) - S(P, f, \alpha)| \le \epsilon V_{[a,b]}(\alpha)$.

Exercise 5.78

By Exercise 5.77 and the integration by parts, if f has bounded variation, then we still have $\lim_{n\to\infty}\int_a^b f d\alpha_n = \int_a^b f d\left(\lim_{n\to\infty}\alpha_n\right)$.

Exercise 5.79

Suppose for each x near a, $f(x,y)$ is differentiable function of y in a bounded interval I. Suppose $\lim_{x\to a} f(x,y_0)$ converges for a point $y_0\in I$, and $\lim_{x\to a} f_y(x,y)$ uniformly converges for $y\in I$. Then $\lim_{x\to a} f(x,y)$ uniformly converges for $y\in I$, and the limit function is differentiable with derivative

$$\frac{d}{dy}\lim_{x\to a} f(x,y) = \lim_{x\to a}\frac{\partial f}{\partial y}(x,y).$$

Exercise 5.80 (1)

We have $x^{ax} = e^{ax\log x} = \sum_{n=0}^{\infty}\dfrac{(ax\log x)^n}{n!}$. Since $|ax\log x|\le |a|e^{-1}$ for $x\in[0,1]$ and $\sum_{n=0}^{\infty}\dfrac{y^n}{n!}$ uniformly converges for y in any fixed bounded interval, we see that $\sum_{n=0}^{\infty}\dfrac{(ax\log x)^n}{n!}$ uniformly converges on $[0,1]$, and we can compute the integral term by term

$$\int_0^1 x^{ax}dx = \sum_{n=0}^{\infty}\frac{a^n}{n!}\int_0^1 (x\log x)^n dx.$$

By Exercise 4.70(6), we have

$$\int_0^1 x^{\alpha}(\log x)^n dx = -\frac{n}{\alpha+1}\int_0^1 x^{\alpha}(\log x)^{n-1}dx.$$

From this we get

$$\begin{aligned}\int_0^1 (x\log x)^n dx &= -\frac{n}{n+1}\int_0^1 x^n(\log x)^{n-1}dx = \frac{n(n-1)}{(n+1)^2}\int_0^1 x^n(\log x)^{n-2}dx\\ &= \cdots = (-1)^n\frac{n!}{(n+1)^n}\int_0^1 x^n dx = (-1)^n\frac{n!}{(n+1)^{n+1}}.\end{aligned}$$

Therefore

$$\int_0^1 x^{ax}dx = \sum_{n=0}^{\infty}\frac{a^n}{n!}(-1)^n\frac{n!}{(n+1)^{n+1}} = \sum_{n=1}^{\infty}\frac{(-a)^{n-1}}{n^n}.$$

Exercise 5.80 (2)

For $\lambda a\le x\le a$, we have $|\lambda^n\tan\lambda^n x|\le|\lambda|^n|\tan\lambda^n a|\le|\lambda|^n|\tan a|$. The convergence of $\sum|\lambda|^n$ then implies that $\sum_{n=0}^{\infty}\lambda^n\tan\lambda^n x$ uniformly converges on $[\lambda a, a]$. Therefore the integration and the infinite sum can be exchanged, and we get

$$\int_{\lambda a}^{a}\left(\sum_{n=0}^{\infty}\lambda^n\tan\lambda^n x\right)dx = \sum_{n=0}^{\infty}\int_{\lambda a}^{a}\lambda^n\tan\lambda^n x dx = \sum_{n=0}^{\infty}\int_{\lambda^{n+1}a}^{\lambda^n a}\tan x dx.$$

The partial sum of the series on the right is $\displaystyle\int_{\lambda^{n+1}a}^{a} \tan x dx$. Therefore

$$\int_{\lambda a}^{a} \left(\sum_{n=0}^{\infty} \lambda^n \tan \lambda^n x \right) dx = \lim_{n\to\infty} \int_{\lambda^{n+1}a}^{a} \tan x dx = \int_{0}^{a} \tan x dx = -\log |\cos x|_{x=0}^{x=a} = -\log |\cos a|.$$

Exercise 5.80 (3)

The series $\zeta(t) - 1 = \sum_{n=2}^{\infty} \dfrac{1}{n^x}$ uniformly converges on any bounded interval $[x, b]$ with $x > 1$. Therefore

$$\int_{x}^{b} (\zeta(t) - 1) dt = \sum_{n=2}^{\infty} \int_{x}^{b} \frac{1}{n^t} dt = \sum_{n=2}^{\infty} \left(\frac{1}{n^x \log n} - \frac{1}{n^b \log n} \right).$$

Since $\dfrac{1}{n^b \log n} < \dfrac{1}{n^2}$ for $n \geq 2$ and $b \geq 2$, the series $\sum \dfrac{1}{n^b \log n}$ uniformly converges for $b \in [2, +\infty)$. Therefore $\lim_{b\to+\infty}$ commutes with the sum, and we get

$$\lim_{b\to+\infty} \sum_{n=2}^{\infty} \frac{1}{n^b \log n} = \sum_{n=2}^{\infty} \lim_{b\to+\infty} \frac{1}{n^b \log n} = 0.$$

We conclude that

$$\int_{x}^{+\infty} (\zeta(t) - 1) dt = \lim_{b\to+\infty} \int_{x}^{b} (\zeta(t) - 1) dt = \sum_{n=2}^{\infty} \frac{1}{n^x \log n} - \lim_{b\to+\infty} \sum_{n=2}^{\infty} \frac{1}{n^b \log n} = \sum_{n=2}^{\infty} \frac{1}{n^x \log n}.$$

Exercise 5.81 (1)

The k-th order derivative series is $\sum \dfrac{(-\log(\log n))^k}{n(\log n)^x}$. For $x \geq r > 1$, we have

$$\left| \frac{(-\log(\log n))^k}{n(\log n)^x} \right| \leq \frac{(\log(\log n))^k}{n(\log n)^r}.$$

The convergence of $\sum \dfrac{(\log(\log n))^k}{n(\log n)^r}$ implies that the series $\sum \dfrac{(-\log(\log n))^k}{n(\log n)^x}$ uniformly converges on $[r, +\infty)$ for any $r > 1$. This implies that $\dfrac{d^k}{dx^k} \sum \dfrac{1}{n(\log n)^x} = \sum \dfrac{(-\log(\log n))^k}{n(\log n)^x}$ on $(r, +\infty)$ for any $r > 1$. Thus the series has all the derivatives on $(1, +\infty)$.

Exercise 5.81 (2)

The k-th order derivative series is $\sum n(n-1)\cdots(n-k+1) \left(x + \dfrac{1}{n} \right)^{n-k}$. Then for $|x| < R < r < 1$ and $n > \dfrac{1}{r - R}$, we have

$$\left| n(n-1)\cdots(n-k+1) \left(x + \frac{1}{n} \right)^{n-k} \right| \leq n^k \left| x + \frac{1}{n} \right|^{n-k} \leq n^k r^{n-k}.$$

By the convergence of $\sum n^k r^{n-k}$, the k-th order derivative series uniformly converges on $[-R, R]$. This implies that $\frac{d^k}{dx^k}\sum\left(x+\frac{1}{n}\right)^n = \sum n(n-1)\cdots(n-k+1)\left(x+\frac{1}{n}\right)^{n-k}$ on $(-R, R)$ for any $0 < R < 1$. Thus the series has all the derivatives on $(-1, 1)$.

Exercise 5.81 (3)

The series is defined for $x \notin \mathbb{Z}$ and diverges when $p \leq 0$. Let $x \in (N, N+1)$, where N is an integer. Moreover, assume $p \geq 0$.

We have $\frac{1}{|n-N|^p} > \frac{1}{|n-x|^p} > \frac{1}{|n-N-1|^p}$ for $n < N$ and $\frac{1}{|n-N|^p} < \frac{1}{|n-x|^p} < \frac{1}{|n-N-1|^p}$ for $n > N+1$. Therefore $\sum_{n<N}\frac{1}{|n-x|^p}$ converges if and only if $\sum_{n<N}\frac{1}{|n-N|^p}$ converges (which is the same as the convergence of $\sum_{n<N}\frac{1}{|n-N-1|^p} = \sum_{n<N}\frac{1}{|n-N|^p} - 1$). Since the series $\sum_{n<N}\frac{1}{|n-N|^p} = \sum_{n=1}^{\infty}\frac{1}{n^p}$ converges if and only if $p > 1$. Therefore $\sum_{n<N}\frac{1}{|n-x|^p}$ converges if and only if $p > 1$. Similarly, $\sum_{n>N+1}\frac{1}{|n-x|^p}$ converges if and only if $p > 1$. Thus we conclude that

$$\sum_{n=-\infty}^{+\infty}\frac{1}{|n-x|^p} = \frac{1}{|N-x|^p} + \frac{1}{|N+1-x|^p} + \sum_{n<N}\frac{1}{|n-x|^p} + \sum_{n>N+1}\frac{1}{|n-x|^p}$$

converges if and only if $p > 1$.

In case $p > 1$, the estimation $\frac{1}{|n-N|^p} > \frac{1}{|n-x|^p}$ for $n < N$ and the convergence of $\sum_{n<N}\frac{1}{|n-N|^p}$ implies that $\sum_{n<N}\frac{1}{|n-x|^p}$ uniformly converges on $(N, N+1)$. Similarly, $\sum_{n>N+1}\frac{1}{|n-x|^p}$ uniformly converges on $(N, N+1)$. Therefore $\sum_{n=-\infty}^{+\infty}\frac{1}{|n-x|^p}$ converges uniformly on $(N, N+1)$ for any N, which means the series uniformly converges on $\mathbb{R} - \mathbb{Z}$.

Note that the k-th order derivative series is

$$p(p+1)\cdots(p+k-1)\left(\frac{1}{|N-x|^{p+k}} + \frac{(-1)^k}{|N+1-x|^{p+k}} + \sum_{n<N}\frac{1}{|n-x|^{p+k}} + \sum_{n>N+1}\frac{(-1)^k}{|n-x|^{p+k}}\right).$$

For $p > 1$, by the same reason, the series uniformly converges on $\mathbb{R} - \mathbb{Z}$. We conclude that $\sum_{n=-\infty}^{+\infty}\frac{1}{|n-x|^p}$ has derivative of all orders wherever it is defined.

Exercise 5.81 (4)

The k-th order derivative series is $\sum\frac{(-1)^{n+k}(\log n)^k}{n^x}$, an alternating series. Let $r > 0$. Then there is N, such that $\frac{(\log n)^k}{n^r}$ is decreasing for $n > N$. For $x \geq r$, we know $\frac{(\log n)^k}{n^x} =$

$\dfrac{1}{n^{x-r}}\dfrac{(\log n)^k}{n^r}$ is also decreasing for $n > N$. Thus we know the series $\sum \dfrac{(-1)^{n+k}(\log n)^k}{n^x}$ converges to a function $g(x)$. Moreover, for $x \geq r$ and $n > N$, we have

$$\left|\sum_{i=1}^{n} \frac{(-1)^{i+k}(\log i)^k}{i^x} - g(x)\right| \leq \frac{(\log(n+1))^k}{(n+1)^x} \leq \frac{(\log(n+1))^k}{(n+1)^r}.$$

For fixed k and r, the right side converges to 0 as $n \to \infty$. This implies that the convergence is uniform on $[r, +\infty)$ for any $r > 0$. We conclude that $\sum \dfrac{(-1)^n}{n^x}$ has derivative of all orders on $(0, +\infty)$.

Exercise 5.82

The series $\sum \frac{(-1)^n}{n+x^2}$ converges at $x = 0$. By taking the term wise derivative, the series $\sum \left(\frac{(-1)^n}{n+x^2}\right)' = \sum \frac{(-1)^{n+1}2x}{(n+x^2)^2}$ satisfies $\left|\frac{(-1)^{n+1}2x}{(n+x^2)^2}\right| \leq \frac{1}{n^2}$. By the convergence of $\sum \frac{1}{n^2}$, the derivative series uniformly converges. By Proposition 5.4.5, therefore, we know the original series also uniformly converges on bounded intervals and

$$\begin{aligned}
\lim_{x\to 0} \frac{1}{x}\left(\sum_{n=1}^{\infty}(-1)^n \frac{1}{n+x^2}\right)' &= \lim_{x\to 0} \frac{1}{x}\sum \frac{(-1)^{n+1}2x}{(n+x^2)^2} = 2\lim_{x\to 0}\sum \frac{(-1)^{n+1}}{(n+x^2)^2} = 2\sum \lim_{x\to 0}\frac{(-1)^{n+1}}{(n+x^2)^2} \\
&= 2\sum_{1}^{\infty} \frac{(-1)^{n+1}}{n^2} = 2\left(\sum \frac{1}{n^2} - 2\sum \frac{1}{(2n)^2}\right) \\
&= 2\left(1 - 2\frac{1}{4}\right)\sum \frac{1}{n^2} = \sum \frac{1}{n^2} = \frac{\pi^2}{6}.
\end{aligned}$$

Exercise 5.84

Since $\sum \dfrac{1}{a_n}$ absolutely converges, we get $\lim_{n\to\infty} a_n = \infty$. Thus for any $R > 0$, there is N, such that $|a_n| > 2R$ for $n > N$. Then

$$|x| \leq R, n > N \implies |x| \leq \frac{|a_n|}{2} \implies |x - a_n| \geq \frac{|a_n|}{2}. \implies \left|\frac{1}{x-a_n}\right| \leq \frac{2}{|a_n|}.$$

The absolute convergence of $\sum \dfrac{1}{a_n}$ then implies that the series $\sum \dfrac{1}{x-a_n}$ uniformly converges on $[-R, R] - \{a_1, a_2, \dots\}$ for any R.

By taking k-th order derivative term by term, we get series $\sum \dfrac{(-1)^k k!}{(x-a_n)^{k+1}}$. By the same argument as above, for $n > N$ and $|x| \leq R$, we have

$$\left|\frac{(-1)^k k!}{(x-a_n)^{k+1}}\right| \leq \frac{2^{k+1}k!}{|a_n|^{k+1}}.$$

By $\lim_{n\to\infty} a_n = \infty$, the absolute convergence of $\sum \frac{1}{a_n}$ implies the absolute convergence of $\sum \frac{1}{a_n^{k+1}}$. Then the estimation above implies that the term by term derivative series also uniformly converges on $[-R, R] - \{a_1, a_2, \dots\}$ for any R. As a result, we have

$$\frac{d^k}{dx^k} \sum \frac{1}{x - a_n} = \sum \frac{(-1)^k k!}{(x - a_n)^{k+1}}.$$

Exercise 5.85

Suppose $|x| > \overline{\lim}_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right|$. Then there is a natural number N such that $|x| > \left|\frac{a_n}{a_{n+1}}\right|$ for $n \geq N$. This implies

$$|a_n x^n| \geq |a_n x^N| \left|\frac{a_N}{a_{N+1}}\right| \left|\frac{a_{n+1}}{a_{n+2}}\right| \cdots \left|\frac{a_{n-1}}{a_n}\right| \geq |a_N x^N|.$$

Therefore $a_n x^n \not\to 0$ and the series $\sum a_n x^n$ diverges. This shows that $\overline{\lim}_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right|$ is no smaller than the radius of convergence.

Suppose $|x| < \underline{\lim}_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right|$. Then there is a fixed number $r > |x|$ and a natural number N such that $|x| < r < \left|\frac{a_n}{a_{n+1}}\right|$ for $n \geq N$. This implies

$$|a_n x^n| \leq |a_n r^N| \left|\frac{a_N}{a_{N+1}}\right| \left|\frac{a_{n+1}}{a_{n+2}}\right| \cdots \left|\frac{a_{n-1}}{a_n}\right| \left(\frac{|x|}{r}\right)^n \leq |a_N r^N| \left(\frac{|x|}{r}\right)^n.$$

By the convergence of $\sum \left(\frac{|x|}{r}\right)^n$, the series $\sum a_n x^n$ converges. This shows that $\underline{\lim}_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right|$ is no bigger than the radius of convergence.

Exercise 5.86

In terms of the formula for the radius of convergence, by $\lim \sqrt[n]{n} = 1$ and similar limits, we have

$$\overline{\lim} \sqrt[n-1]{|na_n|} = \overline{\lim} \sqrt[n]{|a_n|} = \overline{\lim} \sqrt[n+1]{\left|\frac{a_n}{n}\right|}.$$

For direct argument, we note that $|x| <$ radius implies $\sum |a_n x^n|$ converges and $|x| >$ radius implies $\sum |a_n x^n|$ diverges.

Suppose r_0, r_1, r_{-1} are a radii of convergence of $\sum a_n x^n$, $\sum n a_n x^{n-1}$, $\sum \frac{a_n}{n+1} x^{n+1}$. If $\sum |n a_n x^{n-1}|$ converges, then by $|a_n x^n| \leq |x|\ |n a_n x^{n-1}|$ and the comparison test, $\sum |a_n x^n|$ converges. This means that $|x| < r_1$ implies $|x| \leq r_0$. Therefore $r_1 \geq r_0$.

If $\sum |a_n x^n|$ converges, then by $|\frac{a_n}{n+1} x^{n+1}| \leq |x|\ |a_n x^n|$ and the comparison test, $\sum |\frac{a_n}{n+1} x^{n+1}|$ converges. This implies $r_0 \geq r_{-1}$.

If $\sum|\frac{a_n}{n+1}x^{n+1}|$ converges. Then for any y satisfying $|y| < |x|$, we have $|na_ny^{n-1}| = \frac{n(n+1)}{|xy|}\left|\frac{y}{x}\right|^n|\frac{a_n}{n+1}x^{n+1}| \le |\frac{a_n}{n+1}x^{n+1}|$ for sufficiently big n. By the comparison test, $\sum|na_ny^{n-1}|$ converges. This means that $|y| < |x| < r_{-1}$ implies $|x| \le r_1$. Therefore $r_{-1} \ge r_1$.

Exercise 5.87

The function $f(x) = \sum_{n\ge 0} a_nx^n$ is defined for $|x| < R$. We have $f'(x) = \sum_{n\ge 1} na_nx^{n-1}$, again for $|x| < R$. Applying the proposition to the power series $\sum na_nx^{n-1}$, we further get $f''(x) = \sum_{n\ge 2} n(n-1)a_nx^{n-2}$ for $|x| < R$. Keep going, we get $f^{(k)}(x) = \sum_{n\ge k} n(n-1)\cdots(n-k+1)a_nx^{n-k}$ for $|x| < R$. Evaluating the series at $x = 0$, we get $f^{(k)}(0) = k!a_k$.

Exercise 5.88

The Taylor series of $\arctan x$ is

$$\arctan x = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2n+1}.$$

The radius of convergence is 1 and the series also converges at $x = 1$. By Proposition 5.4.6, the Taylor series uniformly converges on $[0,1]$, and we get

$$\frac{\pi}{4} = \lim_{x\to 1^-}\arctan x = \sum_{n=0}^{\infty}(-1)^n\lim_{x\to 1^-}\frac{x^{2n+1}}{2n+1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \cdots.$$

Moreover,

$$\begin{aligned}
&1 + \frac{1}{2} - \frac{1}{3} - \frac{1}{4} + \frac{1}{5} + \frac{1}{6} - \frac{1}{7} - \frac{1}{8} + \cdots \\
=&\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \cdots\right) + \left(\frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \frac{1}{10} - \frac{1}{11} + \cdots\right) \\
=&\frac{\pi}{4} + \frac{1}{2}\log 2.
\end{aligned}$$

Exercise 5.89

The Taylor series of $\arcsin x$ is

$$\arcsin x = x - \sum_{n=1}^{\infty}\frac{(2n)!}{4^n(4n^2-1)(n!)^2}x^{2n+1}.$$

The radius of convergence is 1.

Denote the coefficient in the Taylor series by $a_n = \dfrac{(2n)!}{4^n(2n-1)(n!)^2}$. Then we have

$$\frac{a_n}{a_{n-1}} = \frac{2n(2n-1)(2n-3)(2n-1)}{4(2n-1)(2n+1)n^2} = \frac{(2n-3)(2n-1)}{2n(2n+1)} \le \frac{2n-3}{2n} = 1 - \frac{\frac{3}{2}}{n}.$$

By Exercise 5.44, we find the series $\sum a_n$ converges. Therefore the Taylor series of $\arcsin x$ converges at ± 1. Since $\arcsin x$ is continuous at $x = \pm 1$, by Proposition **??**, the Taylor series uniformly converges on $[-1,1]$ and equals $\arcsin x$.

Exercise 5.90

Since $\sum a_n$ and $\sum b_n$ converge, the power series $\sum a_n x^n$, $\sum b_n x^n$ absolutely converge for $|x| < 1$. By Proposition 5.2.7, we have $(\sum a_n x^n)(\sum b_n x^n) = \sum c_n x^n$ for $|x| < 1$. Since the three power series converge at $x = 1$, by Proposition 5.4.6, they uniformly converge on $[0, 1]$. Therefore the limit $\lim_{x\to 1^-}$ and the sum can commute, and we get

$$\begin{aligned}
\left(\sum a_n\right)\left(\sum b_n\right) &= \left(\sum \lim_{x\to 1^-} a_n x^n\right)\left(\sum \lim_{x\to 1^-} b_n x^n\right) \\
&= \left(\lim_{x\to 1^-} \sum a_n x^n\right)\left(\lim_{x\to 1^-} \sum b_n x^n\right) && \text{(uniformly converge)} \\
&= \lim_{x\to 1^-} \left(\sum a_n x^n\right)\left(\sum b_n x^n\right) = \lim_{x\to 1^-} \sum c_n x^n && \text{(absolutely converge)} \\
&= \sum \lim_{x\to 1^-} c_n x^n = \sum c_n. && \text{(uniformly converge)}
\end{aligned}$$

Exercise 5.91

For convex $f(x)$, let $L(x)$ be the straight line connecting $f(k)$ to $f(k+1)$. We have $f(x) \le L(x)$ on $[k, k+1]$, and

$$\int_k^{k+1} f(x)dx \le \int_k^{k+1} L(x)dx = \frac{1}{2}(f(k) + f(k+1)).$$

Therefore

$$\int_1^n f(x)dx = \sum_{k=1}^{n-1} \int_k^{k+1} f(x)dx \le \sum_{k=1}^{n-1} \frac{1}{2}(f(k) + f(k+1)) = \sum_{k=1}^{n} f(k) - \frac{1}{2}f(1) - \frac{1}{2}f(n).$$

This is the same as $d_n \ge \frac{1}{2}(f(1) + f(n))$.

Exercise 5.92

For $n > m$, we have

$$\begin{aligned} d_n - d_m &= f(m+1) + f(m+2) + \cdots + f(n) - \int_m^n f(x)dx \\ &= \frac{f(m) + f(m+1)}{2} + \frac{f(m+1) + f(m+2)}{2} + \cdots + \frac{f(n-1) + f(n)}{2} \\ &\quad - \int_m^n f(x)dx + \frac{1}{2}(f(n) - f(m)). \end{aligned}$$

Since f is convex and differentiable, we know f' is increasing. Then by Exercise 4.102, we have

$$\left| \frac{1}{2}(f(k) + f(k+1)) - \int_k^{k+1} f(x)dx \right| \le \frac{\omega_{[k,k+1]}(f')}{8} = \frac{1}{8}(f'(k+1) - f'(k)).$$

Adding together, we get

$$\left| d_n - d_m - \frac{1}{2}(f(n) - f(m)) \right| \le \sum_{k=m}^{n-1} \frac{1}{8}(f'(k+1) - f'(k)) = \frac{1}{8}(f'(n) - f'(m)).$$

Exercise 5.93

For convex and differentiable $f(x)$, we have

$$f(x) - f(x-1) = \frac{f(x) - f(x-1)}{x - (x-1)} \le f'(x) \le f(x+1) - f(x) = \frac{f(x+1) - f(x)}{(x+1) - x}.$$

Then $\lim_{x\to+\infty} f(x) = 0$ implies $\lim_{x\to+\infty} f'(x) = 0$.

Taking $n \to \infty$ in the inequality obtained in Exercise 5.92, we get

$$\left| d_m - \gamma - \frac{1}{2}f(m) \right| \le -\frac{1}{8}f'(m).$$

Exercise 5.94

For twice differentiable $f(x)$, by Exercise 4.105, we have

$$\int_k^{k+1} f(x)dx = \frac{1}{2}(f(k)+f(k+1)) + \frac{1}{8}(f'(k)-f'(k+1)) + \frac{f''(c_k)}{24}, \quad c_k \in (k,k+1).$$

Taking $\sum_{k=m}^{n-1}$, we get

$$\begin{aligned}\int_m^n f(x)dx = & f(m+1)+f(m+2)+\cdots+f(n) \\ & -\frac{1}{2}(f(n)-f(m)) - \frac{1}{8}(f'(n)-f'(m)) \\ & + \frac{1}{24}\sum_{k=m}^{n-1} f''(c_k).\end{aligned}$$

By $\inf_{[k,k+1]} f'' \le f''(c_k) \le \sup_{[k,k+1]} f''$, we get the desired inequality.

Exercise 5.95

Take $f(x) = \dfrac{1}{x}$ in Exercise 5.94. We have

$$d_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n} - \log n, \quad \lim_{n\to\infty} d_n = \gamma, \quad \lim_{n\to\infty} f(n) = \lim_{n\to\infty} f'(n) = 0.$$

Taking $n \to \infty$ in the inequality obtained in Exercise 5.94, we get

$$\frac{1}{24}\sum_{k=m}^{\infty} \inf_{[k,k+1]} f'' \le d_m - \frac{1}{2}f(m) - \frac{1}{8}f'(m) \le \frac{1}{24}\sum_{k=m}^{\infty} \sup_{[k,k+1]} f''.$$

Since $f'' = \dfrac{2}{x^3}$ is decreasing, we have

$$\sum_{k=m}^{n-1} \sup_{[k,k+1]} f'' = \sum_{k=m}^{+\infty} f''(k) \le \int_{m-1}^{+\infty} f''dx = f'(+\infty) - f'(m-1) = \frac{1}{(m-1)^2},$$
$$\sum_{k=m}^{n-1} \inf_{[k,k+1]} f'' = \sum_{k=m}^{+\infty} f''(k+1) \ge \int_{m+1}^{+\infty} f''dx = f'(+\infty) - f'(m+1) = \frac{1}{(m+1)^2}.$$

This gives the desired estimation.

Exercise 5.96

Take $f(x) = \dfrac{1}{\sqrt{x}}$ in Exercise 5.94. We have ($0 < \gamma < 1$ and is not the Euler-Mascheroni constant)

$$d_n = 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n-1}} - 2\sqrt{n}, \quad \lim_{n\to\infty} d_n = \gamma, \quad \lim_{n\to\infty} f(n) = \lim_{n\to\infty} f'(n) = 0.$$

Moreover, since $f'' = \frac{3}{4}x^{-\frac{5}{2}}$ is decreasing, we have

$$\sum_{k=m}^{n-1} \sup_{[k,k+1]} f'' = \sum_{k=m}^{+\infty} f''(k) \le \int_{m-1}^{+\infty} f'' dx = f'(+\infty) - f'(m-1) = \frac{1}{2(m-1)^{\frac{3}{2}}},$$
$$\sum_{k=m}^{n-1} \inf_{[k,k+1]} f'' = \sum_{k=m}^{+\infty} f''(k+1) \ge \int_{m+1}^{+\infty} f'' dx = f'(+\infty) - f'(m+1) = \frac{1}{2(m+1)^{\frac{3}{2}}}.$$

This leads to the estimation (after replacing m by n)

$$\frac{1}{48(n+1)^{\frac{3}{2}}} \le 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}} - 2\sqrt{n} - \gamma - \frac{1}{2\sqrt{n}} + \frac{1}{16n^{\frac{3}{2}}} \le \frac{1}{48(n+1)^{\frac{3}{2}}},$$

and further implies

$$1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}} = 2\sqrt{n} + \gamma + \frac{1}{2\sqrt{n}} - \frac{1}{24n^{\frac{3}{2}}} + o\left(\frac{1}{n^{\frac{3}{2}}}\right).$$

Exercise 5.97

By the integral comparison test, $\sum \frac{1}{n(\log n)^p}$ converges if and only if $\int_2^{+\infty} \frac{dx}{x(\log x)^p}$ converge. By $\int_2^{+\infty} \frac{dx}{x(\log x)^p} = \int_{\log 2}^{+\infty} \frac{dy}{y^p}$, the improper integral converges if and only if $p > 1$. Therefore the series converges if and only if $p > 1$.

Exercise 5.98

By the comparison test and Exercise 5.97, if $\left|\frac{x_{n+1}}{x_n}\right| \le \frac{(n-1)(\log(n-1))^p}{n(\log n)^p}$ for some $p > 1$ and sufficiently big n, then $\sum |x_n|$ converges. Note that

$$\begin{aligned}\frac{(n-1)(\log(n-1))^p}{n(\log n)^p} &= \left(1 - \frac{1}{n}\right)\left(\frac{\log(n-1)}{\log n}\right)^p = \left(1 - \frac{1}{n}\right)\left(\frac{\log n + \log\left(1 - \frac{1}{n}\right)}{\log n}\right)^p \\ &= \left(1 - \frac{1}{n}\right)\left(1 + \frac{1}{\log n}\left(-\frac{1}{n} + o\left(\frac{1}{n}\right)\right)\right)^p \\ &= \left(1 - \frac{1}{n}\right)\left(1 - \frac{p}{n\log n} + o\left(\frac{1}{n\log n}\right)\right) \\ &= 1 - \frac{1}{n} - \frac{p}{n\log n} + o\left(\frac{1}{n\log n}\right).\end{aligned}$$

Then like the argument for the Raabe test (Exercise 5.29), the condition $\left|\frac{x_{n+1}}{x_n}\right| \le \frac{(n-1)(\log(n-1))^p}{n(\log n)^p}$ for some $p > 1$ and sufficiently big n is equivalent to $\left|\frac{x_{n+1}}{x_n}\right| \le 1 - \frac{1}{n} - \frac{q}{n\log n}$ for some $q > 1$ and sufficiently big n.

Exercise 5.99

By comparing with the divergent series $\sum \frac{1}{n\log n}$, if

$$\left|\frac{x_{n+1}}{x_n}\right| \le \frac{(n-1)\log(n-1)}{n\log n} = 1 - \frac{1}{n} - \frac{1}{n\log n} + o\left(\frac{1}{n\log n}\right)$$

for sufficiently big n, then $\sum |x_n|$ diverges. The condition will be satisfied when $\left|\frac{x_{n+1}}{x_n}\right| \le 1 - \frac{1}{n} - \frac{p}{n\log n}$ for some $p < 1$ and sufficiently big n.

Exercise 5.100

The inequality $\left|\frac{x_{n+1}}{x_n}\right| \le 1 - \frac{1}{n} - \frac{q}{n\log n}$ is the same as $\left(1 - \frac{1}{n} - \left|\frac{x_{n+1}}{x_n}\right|\right)\log n \ge q$. So $\underline{\lim}\left(1 - \frac{1}{n} - \left|\frac{x_{n+1}}{x_n}\right|\right)\log n > 1$ implies convergence and $\overline{\lim}\left(1 - \frac{1}{n} - \left|\frac{x_{n+1}}{x_n}\right|\right)\log n < 1$ implies divergence.

Exercise 5.101

In case $\left|\frac{x_{n+1}}{x_n}\right| = 1 - \frac{1}{n} - \frac{1}{n\log n} + o\left(\frac{1}{n\log n}\right)$, the convergence of $\sum |x_n|$ will depend on more details about $\left(\frac{1}{n\log n}\right)$.

For example, let us consider $y_n = \frac{1}{n\log n(\log\log n)^p}$. By the integral comparison test, the series converges if and only if $p > 1$. Now

$$\frac{y_n}{y_{n-1}} = \frac{n-1}{n}\frac{\log(n-1)}{\log n}\left(\frac{\log\log(n-1)}{\log\log n}\right)^p.$$

We have

$$\begin{aligned}
\frac{n-1}{n} &= 1 - \frac{1}{n}, \\
\frac{\log(n-1)}{\log n} &= \frac{\log n + \log\left(1 - \frac{1}{n}\right)}{\log n} \\
&= 1 - \frac{1}{\log n}\left(\frac{1}{n} + O\left(\frac{1}{n^2}\right)\right) = 1 - \frac{1}{n\log n} + o\left(\frac{1}{n\log n\log\log n}\right).
\end{aligned}$$

Moreover, by

$$\begin{aligned}
\log\log(n-1) &= \log\log n + \log\frac{\log(n-1)}{\log n} \\
&= \log\log n + \log\left(1 - \frac{1}{n\log n} + o\left(\frac{1}{n\log n}\right)\right) \\
&= \log\log n - \frac{1}{n\log n} + o\left(\frac{1}{n\log n}\right),
\end{aligned}$$

we have

$$\left(\frac{\log\log(n-1)}{\log\log n}\right)^p = \left(1-\frac{1}{n\log n\log\log n}+o\left(\frac{1}{n\log n\log\log n}\right)\right)^p$$
$$= 1-\frac{p}{n\log n\log\log n}+o\left(\frac{1}{n\log n\log\log n}\right).$$

Multiplying the three estimations together, we get

$$\frac{y_n}{y_{n-1}} \le 1-\frac{1}{n}-\frac{1}{n\log n}-\frac{p}{n\log n\log\log n}+o\left(\frac{1}{n\log n\log\log n}\right).$$

By the integral comparison test, $\sum y_n$ converges if and only if $p>1$. Similar to the Bertrand test, if

$$\left|\frac{x_n}{x_{n-1}}\right| \le 1-\frac{1}{n}-\frac{1}{n\log n}-\frac{p}{n\log n\log\log n}$$

for some constant $p>1$ and sufficiently big n, then $\sum x_n$ converges. Moreover, if

$$\left|\frac{x_n}{x_{n-1}}\right| \ge 1-\frac{1}{n}-\frac{1}{n\log n}-\frac{p}{n\log n\log\log n}$$

for some constant $p<1$ and sufficiently big n, then $\sum |x_n|$ diverges.

Exercise 5.102

If $\left|\frac{x_n}{x_{n+1}}\right| \ge 1+\frac{1}{n}+\frac{p}{n\log n}$ for some $p>1$ and sufficiently big n, then $\sum x_n$ absolutely converges.

If $\left|\frac{x_n}{x_{n+1}}\right| \le 1+\frac{1}{n}+\frac{p}{n\log n}$ for some $p<1$ and sufficiently big n, then $\sum |x_n|$ diverges.

Exercise 5.103

Suppose there are $c_n>0$, $\delta>0$ and N, such that $c_n-c_{n+1}\left|\frac{x_{n+1}}{x_n}\right| \ge \delta$ for $n>N$. Then $|c_nx_n|-|c_{n+1}x_{n+1}| \ge \delta|x_n|$. In particular, $|c_nx_n|$ is a decreasing sequence and must converge. Therefore the series $\sum(|c_nx_n|-|c_{n+1}x_{n+1}|)$ converges. Then by $|c_nx_n|-|c_{n+1}x_{n+1}| \ge \delta|x_n|$, $\delta>0$, and the comparison test, we see that $\sum x_n$ absolutely converges.

Exercise 5.104

The inequality $c_n-c_{n+1}\frac{x_{n+1}}{x_n} \le 0$ means $\frac{x_{n+1}}{x_n} \ge \frac{c_{n+1}^{-1}}{c_n^{-1}}$. This implies $x_n \ge Ac_n^{-1}$ for $A=x_1c_1$. Thus $\sum c_n^{-1}=+\infty$ implies $\sum x_n=+\infty$.

Exercise 5.105

We attempt to get $c_n-c_{n+1}\left|\frac{x_{n+1}}{x_n}\right| = 1$. This means $c_n|x_n|-c_{n+1}|x_{n+1}|=|x_n|$, which is the same as $c_1|x_1|-|c_n|x_n=|x_1|+|x_2|+\cdots+|x_{n-1}|$. Since we wish to keep c_n positive, we will choose $c_1|x_1|=\sum_{n=1}^{\infty}|x_n|$. Note that this choice of c_1 makes use of the assumption that $\sum x_n$ absolutely converges. Then we get $c_n=\dfrac{|x_n|+|x_{n+1}|+|x_{n+2}|+\cdots}{|x_n|}$.

Exercise 5.106

For $c_n = 1$, the Kummer test becomes the ratio test. For $c_n = n - 1$, the test becomes the Raabe test. For $c_n = n \log n$, the test becomes the Bertrand test.

Exercise 5.107

If $p \neq 0$, then $\lim x_n = \lim \dfrac{p_n}{p_{n-1}} = \dfrac{\lim p_n}{\lim p_{n-1}} = \dfrac{p}{p} = 1$.

Exercise 5.108

The convergence requires $p \neq 0$, which implies all $x_n \neq 0$. Therefore as far as the convergence is concerned, dropping finitely many terms changes all the partial sums by multiplying the same nonzero constant. This implies that the convergence is not changed.

Suppose all $x_n > 0$. Then the partial product p_n satisfies $\log p_n = \log x_1 + \log x_2 + \cdots + \log x_n$. Therefore $\log p_n$ converges if and only if $\sum \log x_n$ converges. Moreover, since log is an invertible continuous map between $(0, +\infty)$ and $(-\infty, +\infty)$, the convergence of $\log p_n$ is the same as the convergence of p_n to a positive (and therefore nonzero) number.

Exercise 5.109 (1)

The partial product $\dfrac{2}{1}\dfrac{3}{2}\cdots\dfrac{n+1}{n} = n + 1$. The infinite product diverges to $+\infty$.

Exercise 5.109 (2)

The partial product $\dfrac{3}{2}\dfrac{2}{3}\dfrac{5}{4}\dfrac{4}{5}\cdots\dfrac{n+(-1)^n}{n} = 1$ for odd n and $= \dfrac{n+1}{n}$ for even n. The infinite product converges to 1.

Exercise 5.109 (3)

The partial product $\prod_{k=2}^{n} \dfrac{n-1}{n+1} = \dfrac{\prod_1^{n-1} k}{\prod_3^{n+1} k} = \dfrac{2}{n(n+1)}$. The infinite product diverges to 0.

Exercise 5.109 (4)

By $(n+1)^2 - (n+1) + 1 = n^2 + n + 1$, we get the partial product $\prod_{k=2}^{n} \dfrac{k^2 - k + 1}{k^2 + k + 1} = \frac{2^2-2+1}{n^2+n+1}$. The infinite product diverges to 0.

Exercise 5.109 (5)

By $\dfrac{n^3 - 1}{n^3 + 1} = \dfrac{n-1}{n+1} \cdot \dfrac{n^2 - n + 1}{n^2 + n + 1}$ and Exercises 5.109 (3,4), the infinite product diverges to 0.

Exercise 5.109 (6)

The partial product is $2^{1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}}$. The infinite product diverges because $\sum \dfrac{1}{n}$ diverges.

Exercise 5.109 (7)

The partial product is $2^{1+\frac{1}{3}+\frac{1}{3^2}+\cdots+\frac{1}{3^n}}$. Since $\sum_{n=0}^{\infty} \dfrac{1}{3^n} = \frac{1}{1-\frac{1}{3}} = \frac{3}{2}$, the infinite product converges to $2^{\frac{3}{2}}$.

Exercise 5.109 (8)

The partial product is $2^{1-1+\frac{1}{2!}-\frac{1}{3!}+\cdots+\frac{(-1)^n}{n!}}$. Since $\sum_{n=0}^{\infty}\dfrac{(-1)^n}{n!}=e^{-1}$, the infinite product converges to $2^{e^{-1}-1}$.

EXERCISE 5.109 (9)

By $\sin x\cos x=\dfrac{1}{2}\sin 2x$, we have

$$\left(\prod_{k=1}^{n}\cos\frac{x}{2^k}\right)\sin\frac{x}{2^n}=\frac{1}{2}\left(\prod_{k=1}^{n-1}\cos\frac{x}{2^k}\right)\sin\frac{x}{2^{n-1}}=\frac{1}{2^2}\left(\prod_{k=1}^{n-2}\cos\frac{x}{2^k}\right)\sin\frac{x}{2^{n-2}}=\cdots=\frac{1}{2^n}\sin x.$$

The partial product is $\dfrac{\sin x}{2^n\sin\dfrac{x}{2^n}}$. The infinite product converges to $\dfrac{\sin x}{x}$ (and 1 if $x=0$).

EXERCISE 5.110

If $x_n\neq 0$, then x_n has non-zero limit if and only if $\prod\dfrac{x_{n+1}}{x_n}$ converges.

If $\prod x_n$ and $\prod y_n$ converges, then $\prod(x_ny_n)=(\prod x_n)(\prod y_n)$ and $\prod x_n^p=(\prod x_n)^p$.

EXERCISE 5.111

By Exercises 5.108, we may use the Cauchy criterion for the convergence of $\sum\log x_n$. This means that, for any $\epsilon>0$, there is N, such that

$$n\geq m>N\implies |\log(x_mx_{m+1}\cdots x_n)|=|\log x_m+\log x_{m+1}+\cdots+\log x_n|<\epsilon.$$

The right side is the same as

$$e^{-\epsilon}<x_mx_{m+1}\cdots x_n<e^{\epsilon}.$$

Since a number λ is of the form e^{ϵ} if and only if $\lambda>1$ and is of the form $e^{-\epsilon}$ if and only if $\lambda<1$. The Cauchy criterion can be rephrased as follows: For any $\delta>0$, there is N, such that

$$n\geq m>N\implies 1-\delta<x_mx_{m+1}\cdots x_n<1+\delta.$$

Another equivalent way is the following: For any $\lambda>1$, there is N, such that

$$n\geq m>N\implies \lambda^{-1}<x_mx_{m+1}\cdots x_n<\lambda.$$

EXERCISE 5.112

If $\sum x_n$ converges, then $x_n\to 0$. This implies

$$\log(1+x_n)=x_n+r_n,\quad \lim\frac{r_n}{x_n^2}=\frac{1}{2}.$$

By the comparison test and $x_n^2>0$, the convergence of $\sum x_n^2$ is equivalent to the convergence of $\sum r_n$. Since $\sum x_n$ already converges, we conclude that $\sum\log(1+x_n)$ converges if and only if $\sum x_n^2$ converges.

If $\sum x_n^2$ converges, then $x_n^2 \to 0$. This implies $x_n \to 0$ and

$$\log(1+x_n) = x_n + r_n, \quad \lim \frac{r_n}{x_n^2} = \frac{1}{2}.$$

By the comparison test and the convergence of $\sum x_n^2$, we know $\sum r_n$ converges. Therefore $\sum \log(1+x_n)$ converges if and only if $\sum x_n$ converges.

EXERCISE 5.113

For $x_n = \dfrac{(-1)^n}{\sqrt{n}}$, the series $\sum x_n$ converges, but the series $\sum x_n^2$ diverges. By Exercise 5.112, we see that the infinite product $\prod(1+x_n)$ diverges.

EXERCISE 5.114

The sequence $\log x_n$ is decreasing and has limit 0. By the Leibniz test (Exercise 5.37), the series $\sum(-1)^n \log x_n$ converges. Correspondingly, the infinite product $\prod x_n^{(-1)^n}$ converges.

Let $x_n = 1 + z_n$. The requirement on x_n means that z_n decreases and $\lim_{n\to\infty} z_n = 0$. Then $y_n = z_n$ for even n and

$$y_n = \frac{1}{1+z_n} - 1 = -z_n + r_n, \quad \lim_{n\to\infty} \frac{r_n}{z_n^2} = 1$$

for odd n. By the Leibniz test, $\sum(-1)^n z_n$ converges. Therefore $\sum y_n = \sum(-1)^n z_n + \sum r_{2k+1}$ converges if and only if $\sum r_{2k+1}$ converges, which is equivalent to $\sum z_{2k+1}^2$ converges.

The discussion leads to the construction $z_n = \dfrac{1}{\sqrt{n}}$, which is decreasing and has limit 0. Moreover, $\sum z_{2k+1}^2$ diverges. Therefore for the corresponding

$$y_n = \begin{cases} \dfrac{1}{\sqrt{n}}, & \text{if } n \text{ even}, \\ \dfrac{1}{1+\dfrac{1}{\sqrt{n}}} - 1 = -\dfrac{1}{\sqrt{n}+1}, & \text{if } n \text{ odd}, \end{cases}$$

we know $\prod(1+y_n)$ converges, but $\sum y_n$ diverges.

EXERCISE 5.115

If $\sum |x_n|$ converges, then $\lim x_n$, and $\lim \dfrac{\log(1+|x_n|)}{x_n} = 1$. By the comparison test, $\sum \log(1+|x_n|)$ converges, which is the same as the convergence of $\prod(1+|x_n|)$.

Conversely, if $\prod(1+|x_n|)$ converges, then $\sum \log(1+|x_n|)$ converges, which implies $\lim \log(1+|x_n|)$ and further implies $\lim x_n 0$. Then the similar comparison test shows that $\sum |x_n|$ converges.

EXERCISE 5.116

If $\prod(1+|x_n|)$ converges, then $\lim(1+|x_n|) = 1$, and $\lim |x_n| = \lim x_n = 0$. This implies $\lim \dfrac{|\log(1+x_n)|}{\log(1+|x_n|)} = 1$. By comparison test, $\sum \log(1+x_n)$ absolutely converges. This implies that $\prod(1+x_n)$ converges.

Exercise 5.117

If $\prod(1+x_n)$ converges, then $\lim x_n = 0$, and $\lim_{n\to\infty} \dfrac{\log(1-x_n)}{\log(1+x_n)} = -1$. Since $0 < x_n < 1$, we have $\log(1+x_n) > 0$ and $\log(1-x_n) < 0$. Thus by the comparison test, $\sum \log(1+x_n)$ converges if and only if $\sum \log(1-x_n)$ converges. In other words, $\prod(1+x_n)$ converges if and only if $\prod(1-x_n)$ converges.

Exercise 5.118

Suppose $\prod(1+x_n)$ converges to a positive number but $\prod(1+|x_n|)$ diverges. Then by rearranging the terms in $\prod(1+x_n)$, the infinite product can have any positive number as the limit, and can also diverge.

Exercise 5.119

The inequality $\left|\dfrac{x_{n+1}}{x_n}\right| \le 1 - y_n$ implies $|x_{n+1}| \le |x_1| \prod_{i=1}^n (1-y_i)$. By Exercises 5.115, $\sum y_n = +\infty$ implies $\prod(1+y_n) = +\infty$. By Exercises 5.117, this further implies $\lim \prod_{i=1}^n (1-y_n) = \prod(1-y_n) = 0$. Therefore $\lim x_n = 0$.

Exercise 5.120

The inequality $\left|\dfrac{x_{n+1}}{x_n}\right| \ge 1 + y_n$ implies $|x_{n+1}| \ge |x_1| \prod_{i=1}^n (1+y_i)$. By Exercises 5.115, $\sum y_n = +\infty$ implies $\lim \prod_{i=1}^n (1+y_n) = \prod(1+y_n) = +\infty$. Therefore $\lim |x_n| = +\infty$.

Exercise 5.121

By $0 < 1 - y_n \le \dfrac{x_{n+1}}{x_n}$, x_n does not change sign. So we may assume all $x_n > 0$. The inequality $1 - y_n \le \dfrac{x_{n+1}}{x_n} \le 1 + z_n$ then implies that for $n \ge m$, we have

$$\sum_{i=m}^n \log(1-y_i) \le \sum_{i=m}^n (\log x_{i+1} - \log x_i) = \log x_{n+1} - \log x_m \le \sum_{i=m}^n \log(1+z_i).$$

By Exercises 5.115 and 5.117, the convergence of $\sum y_n$ and $\sum z_n$ imply the convergence of $\sum \log(1-y_n)$ and $\sum \log(1+z_n)$. By the Cauchy criterion for the series $\sum \log(1-y_n)$ and $\sum \log(1+z_n)$, and the estimation above, we see that the sequence $\log x_n$ also satisfies the Cauchy criterion. This proves that x_n converges to a nonzero limit.

Exercise 5.122

For $x_n = \dfrac{(n+a)^{n+\frac{1}{2}}}{(\pm e)^n n!}$, the calculation in Exercise 3.143 tells us

$$\left|\frac{x_n}{x_{n-1}}\right| = 1 + \frac{\lambda}{n^2} + o\left(\frac{1}{n^2}\right), \quad \lambda = \frac{a^2}{2} - \frac{a}{2} + \frac{1}{12}.$$

Choose some $\mu < \lambda < \nu$. Then

$$1 + \frac{\mu}{n^2} < \left|\frac{x_n}{x_{n-1}}\right| < 1 + \frac{\nu}{n^2}$$

for big n. Then by the convergence of $\sum \frac{1}{n^2}$ and Exercise 5.121, $|x_n|$ converges to a nonzero limit. In case $\pm e$ is chosen as e, the sequence converges. In case $\pm e$ is chosen as $-e$, the sequence diverges.

Exercise 5.123

Since geometric series converge absolutely, we may apply Proposition 5.2.7 to get

$$\prod_{i=1}^{k}\left(1-\frac{1}{p_i^x}\right)^{-1} = \left(\sum_{j=0}^{\infty}\frac{1}{p_1^{jx}}\right)\left(\sum_{j=0}^{\infty}\frac{1}{p_2^{jx}}\right)\cdots\left(\sum_{j=0}^{\infty}\frac{1}{p_k^{jx}}\right) = \sum_{j_1,j_2,\dots,j_k=0}^{\infty}\frac{1}{p_1^{j_1x}p_2^{j_2x}\cdots p_k^{j_kx}}$$
$$= \sum_{j_1,j_2,\dots,j_k=0}^{\infty}\frac{1}{(p_1^{j_1}p_2^{j_2}\cdots p_k^{j_k})^x} = \sum_{n\in S_k}\frac{1}{n^x}.$$

Exercise 5.124

Since $S_k \subset S_{k+1}$, the partial product $\prod_{i=1}^{k}\left(1-\frac{1}{p_i^x}\right)^{-1}$ is increasing in k. Therefore the infinite product $\prod_{i=1}^{\infty}\left(1-\frac{1}{p_i^x}\right)^{-1}$ converges if and only if the sequence $\sum_{n\in S_k}\frac{1}{n^x}$ is bounded. We certainly have

$$\sum_{n\in S_k}\frac{1}{n^x} \le \sum_{n=1}^{\infty}\frac{1}{n^x}.$$

On the other hand, since the prime factors of natural numbers $\le k$ are all $\le k$, we have

$$\sum_{n\in S_k}\frac{1}{n^x} \ge \sum_{n=1}^{k}\frac{1}{n^x}.$$

Therefore $\sum_{n\in S_k}\frac{1}{n^x}$ is bounded if and only if $\sum_{n=1}^{k}\frac{1}{n^x}$ is bounded, which is the same as $\sum_{n=1}^{\infty}\frac{1}{n^x}$ convergent. We conclude that $\prod_{i=1}^{\infty}\left(1-\frac{1}{p_i^x}\right)$ converges if and only if $\sum_{n=1}^{\infty}\frac{1}{n^x}$ converges, which means $x > 1$.

On the other hand, by Exercise 5.123, $\lim_{k\to\infty}\sum_{n\in S_k}\frac{1}{n^x}$ converges if and only if $\prod_{n=1}^{\infty}\left(1-\frac{1}{p_n^x}\right)$ converges. Moreover, by Exercises 5.115 and 5.117, $\prod_{n=1}^{\infty}\left(1-\frac{1}{p_n^x}\right)$ converges if and only if $\sum\frac{1}{p_n^x}$ converges. Therefore $\sum\frac{1}{p_n^x}$ converges if and only if $x > 1$.

Exercise 5.125

Let $x_n = \frac{1}{p_n^x}$. Then $0 < x_n < 1$, $\log(1-x_n) < 0$, and $\lim\frac{\log(1-x_n)}{x_n} = -1$. By the comparison test, $\sum x_n$ converges if and only if $\sum\log(1-x_n)$ converges. Therefore $\sum\frac{1}{p_n^x}$ converges if and only if $\prod_{i=1}^{\infty}\left(1-\frac{1}{p_i^x}\right)$ converges, which means $x > 1$.

Exercise 5.126

The argument in Exercise 5.31 works also for expressions of numbers in other base. If we change the base from 10 to 1000000 and change the digit 9 to 123456 (a "digit" in base 1000000), then we see that the sum of $\dfrac{1}{n}$ for those n with no digit 123456 in base 1000000 expression converges.

By Exercise 5.125, the series $\sum \dfrac{1}{p_n}$ diverges. Therefore in base 1000000, there must be some (actually infinitely many) prime numbers that has 123456 appearing as a 1000000-based digit. If it appears as the first 1000000-based digit, then the prime number is presented as $\cdots X_7 123456$. If it appears as the second 1000000-based digit, then the prime number is presented as $\cdots X_{13} 123456 X_6 X_5 X_4 X_3 X_2 X_1$. And so on. These 1000000-based expressions are also the 10-based expressions. Therefore we see that the string 123456 appears in (infinitely many) the decimal expression of prime numbers.

Exercise 5.127

We have $\sum \dfrac{a_n}{n^x} = \sum \dfrac{1}{n^{x-r}} \dfrac{a_n}{n^r}$. For each $x \geq r$, the sequence $\dfrac{1}{n^{x-r}}$ is decreasing in n. Moreover, $\dfrac{1}{n^{x-r}}$ is uniformly bounded by 1 for $n \geq 1$ and $x \geq r$. Then by the Abel test for uniform convergence, the convergence of $\sum \dfrac{a_n}{n^r}$ implies the uniform convergence of $\sum \dfrac{a_n}{n^x}$ on $[r, +\infty)$. Furthermore, the uniform convergence implies that the limit $\lim_{x \to r^+}$ commutes with the infinite sum, and we get

$$\lim_{x \to r^+} \sum \frac{a_n}{n^x} = \sum \lim_{x \to r^+} \frac{a_n}{n^x} = \sum \frac{a_n}{n^r}.$$

Exercise 5.128

Exercise 5.127 tells us that if $\sum \dfrac{a_n}{n^x}$ converges for $x = r$, then the series converges for any $x \geq r$. Therefore if R is the infimum of those r such that $\sum \dfrac{a_n}{n^r}$. Then $\sum \dfrac{a_n}{n^x}$ diverges for $x < R$ and converges for $x > R$.

Suppose $x < \overline{\lim}_{n \to \infty} \dfrac{\log |a_n|}{\log n}$. Then there are infinitely many a_n satisfying $\dfrac{\log |a_n|}{\log n} = \log_n |a_n| > x$. In other words, we have $\dfrac{|a_n|}{n^x} > 1$ for infinitely many n. Therefore $\dfrac{a_n}{n^x}$ does not converge to 0, and $\sum \dfrac{a_n}{n^x}$ diverges. This proves that $R \geq \overline{\lim}_{n \to \infty} \dfrac{\log |a_n|}{\log n}$.

Exercise 5.129

By taking derivative of $\sum \dfrac{a_n}{n^x}$ term by term, we get series $\sum \dfrac{a_n}{n^x \log n}$. By Exercises 5.127 and 5.128. The series $\sum \dfrac{a_n}{n^x}$ uniformly converges on $[r, +\infty)$ for any $r > R$. Since $\dfrac{1}{\log n}$ is decreasing and has limit 0, by the Abel test, the series $\sum \dfrac{a_n}{n^x \log n}$ also uniformly converges on $[r, +\infty)$ for any $r > R$. This implies that $\left(\sum \dfrac{a_n}{n^x}\right)' = \sum \dfrac{a_n}{n^x \log n}$ for $x > r$. Since r can be any number $> R$, we conclude that we can take derivative term by term on $(R, +\infty)$.

The argument can be easily extended to high order derivatives and gives us $\dfrac{d^k}{dx^k}\sum \dfrac{a_n}{n^x} = \sum \dfrac{a_n}{n^x(\log n)^k}$ for $x > R$.

The term by term integration follows from the uniform convergence of the series $\sum \dfrac{a_n}{n^x}$ on $[r, +\infty)$ for any $r > R$.

EXERCISE 5.130

Suppose $r > \overline{\lim}_{n\to\infty} \dfrac{\log |a_n|}{\log n} + 1$. Then there are only finitely many a_n satisfying $r < \dfrac{\log |a_n|}{\log n} + 1 = \log_n |a_n| + 1$. In other words, we have $\left|\dfrac{a_n}{n^r}\right| \le \dfrac{1}{n}$ for sufficiently big n. Now for any $x > r$, we have $\left|\dfrac{a_n}{n^x}\right| \le \dfrac{1}{n^{1+x-r}}$ for sufficiently big n. The convergence of $\sum \dfrac{1}{n^{1+x-r}}$ implies the series $\sum \dfrac{a_n}{n^x}$ absolutely converges on $(r, +\infty)$. Since r can be any number bigger than $\overline{\lim}_{n\to\infty} \dfrac{\log |a_n|}{\log n} + 1$, we see that $R' \le \overline{\lim}_{n\to\infty} \dfrac{\log |a_n|}{\log n} + 1$.

Suppose $x < \underline{\lim}_{n\to\infty} \dfrac{\log |a_n|}{\log n} + 1$. Then we have $x < \dfrac{\log |a_n|}{\log n} + 1 = \log_n |a_n| + 1$ for sufficiently big n. This implies that $\dfrac{|a_n|}{n^x} \ge \dfrac{1}{n}$ for sufficiently big n. The divergence of $\sum \dfrac{1}{n}$ then implies that $\sum \dfrac{|a_n|}{n^x}$ diverges. This proves $R' \ge \underline{\lim}_{n\to\infty} \dfrac{\log |a_n|}{\log n} + 1$.

EXERCISE 5.132

If $x \notin \cup_n (r_n - c\sqrt{a_n}, r_n + c\sqrt{a_n})$, then $|x - r_n| \ge c\sqrt{a_n}$. Therefore $\dfrac{a_n}{|x - r_n|} \le \dfrac{\sqrt{a_n}}{c}$. By the convergence of $\sum \sqrt{a_n}$, we see that $\sum \dfrac{a_n}{|x - r_n|}$ converges.

EXERCISE 5.133

Suppose $[0, 1] \subset \cup_n (r_n - c\sqrt{a_n}, r_n + c\sqrt{a_n})$. Then by Heine-Borel theorem, $[0, 1]$ is contained in finitely many open intervals

$$[0, 1] \not\subset (r_{n_1} - c\sqrt{a_{n_1}}, r_{n_1} + c\sqrt{a_{n_1}}) \cup \cdots \cup (r_{n_k} - c\sqrt{a_{n_k}}, r_{n_k} + c\sqrt{a_{n_k}}).$$

Then we can show that the length of $[0, 1]$ is no more than the total length of the open intervals

$$1 \le \sum_{i=1}^{k} 2c\sqrt{a_{n_i}}.$$

Therefore if $\sum \sqrt{a_n} < \dfrac{1}{2c}$, then the inequality above fails, and we must conclude $[0, 1] \not\subset \cup_n (r_n - c\sqrt{a_n}, r_n + c\sqrt{a_n})$.

EXERCISE 5.134

Suppose $|a_n| < M$. Then $|u_n(x)| = |b|^n|x - a_n| \le |b|^n(R + M)$ on $[-R, R]$. Since $|b| < 1$, the series for $f(x)$ converges uniformly on $[-R, R]$ for any R. Since $u_n(x)$ is continuous, $f(x)$ is also continuous.

Suppose a is different for any a_n. Then $u_n'(a) = b^n$ when $a > a_n$ and $u_n'(a) = -b^n$ when $a < a_n$. On the other hand, for any fixed N, we have $\delta = \min\{|a-a_1|, |a-a_2|, \dots, |a-a_N|\} > 0$. Then

$$|h| < \delta, n \le N \implies \frac{u_n(a + h) - u_n(a)}{h} = u_n'(a).$$

and

$$n > N \implies \left|\frac{u_n(a + h) - u_n(a)}{h}\right| \le b^n.$$

Thus we conclude that $|h| < \delta$ implies

$$\left|\frac{f(a + h) - f(a)}{h} - \sum_{n=1}^{\infty} u_n'(a)\right| \le \sum_{n>N} \left|\frac{u_n(a + h) - u_n(a)}{h}\right| + \sum_{n>N} |u_n'(a)| \le \sum_{n>N} |b|^n + \sum_{n>N} |b|^n.$$

The convergence of $\sum |b|^n$ implies that

$$\lim_{h\to 0} \frac{f(a + h) - f(a)}{h} = \sum_{n=1}^{\infty} u_n'(a).$$

In particular, $f(x)$ is differentiable at $x = a$.

Now suppose $a = a_k$ for some k. Let K be the collection of k such that $a = a_k$. Then $|b| < 1$ implies $B = \sum_{k\in K} b^k \ne 0$, and

$$f(x) = B|x - a| + \sum_{n\notin K} b^n|x - a_n|.$$

By similar argument as above, we see that $\sum_{n\notin K} b^n|x-a_n|$ is differentiable at a. Since $B|x-a|$ is not differentiable at a, we conclude that $f(x)$ is not differentiable at a.

Exercise 5.145

For $x > 0$, we have $\lim_{t\to 0^+} \frac{t^{x-1}e^{-t}}{t^{x-1}} = 1$ and $\lim_{t\to +\infty} \frac{t^{x-1}e^{-t}}{2^{-t}} = 1$. By the convergence of $\int_0^1 t^{x-1}dt$, $\int_0^{+\infty} 2^{-t}dt$ and the comparison test, the integral $\int_0^{+\infty} t^{x-1}e^{-t}dt$ converges.

For any $0 < a < 1 < b < +\infty$, we prove the continuity of $\Gamma(x)$ on (a, b). For any $\epsilon > 0$, there is $N > 1$ and $1 > r > 0$, such that for any $x \in (a, b)$, we have

$$\int_N^{+\infty} t^{x-1}e^{-t}dt < N^{b-1}\int_N^{+\infty} e^{-t}dt = N^{b-1}e^{-N} < \epsilon,$$
$$\int_0^r t^{x-1}e^{-t}dt < \int_0^r t^{a-1}e^{-t}dt < \epsilon.$$

Now there is $\delta > 0$, such that $|r^\delta - 1|r^{a-1}e^{-r} < \frac{\epsilon}{N}$ and $|N^\delta - 1|N^b e^{-1} < \frac{\epsilon}{N}$. Then for $a < x < y < b$ satisfying $y - x < \delta$, we have

$$\begin{aligned}\left|\int_r^N t^{x-1}e^{-t}dt - \int_r^N t^{y-1}e^{-t}dt\right| &\leq \int_r^N |t^{y-x} - 1|t^{x-1}e^{-t}dt \\ &\leq \int_r^1 |r^\delta - 1|r^{a-1}e^{-r}dt + \int_1^N |N^\delta - 1|N^{b-1}e^{-1}dt \\ &= \frac{\epsilon}{N}((1-r) + (N-1)) < \epsilon.\end{aligned}$$

Combining the estimations, we find that if $a < x < y < b$ and $y - x < \delta$, then

$$\begin{aligned}|\Gamma(x) - \Gamma(y)| \leq &\left|\int_N^{+\infty} t^{x-1}e^{-t}dt - \int_N^{+\infty} t^{y-1}e^{-t}dt\right| + \left|\int_0^r t^{x-1}e^{-t}dt - \int_0^r t^{y-1}e^{-t}dt\right| \\ &+ \left|\int_r^N t^{x-1}e^{-t}dt - \int_r^N t^{y-1}e^{-t}dt\right| \leq 5\epsilon.\end{aligned}$$

Exercise 5.146

For $x > 1$, $\Gamma(x) > \displaystyle\int_2^{+\infty} t^{x-1}e^{-t}dt > 2^{x-1}\int_2^{+\infty} e^{-t}dt = 2^{x-1}e^{-2}$. Since $\lim_{x\to+\infty} 2^{x-1}e^{-2} = +\infty$, we get $\lim_{x\to+\infty}\Gamma(x) = +\infty$.

For $x > 0$, $\Gamma(x) > \displaystyle\int_0^1 t^{x-1}dt = \frac{1}{x}$. Since $\lim_{x\to0^+}\frac{1}{x} = +\infty$, we get $\lim_{x\to0^+}\Gamma(x) = +\infty$.

Exercise 5.147

$\Gamma(x) = \displaystyle\int_0^{+\infty} u^{2(x-1)}e^{-u^2}2udu = 2\int_0^{+\infty} u^{2x-2}e^{-u^2}du.$

$\Gamma(x) = \displaystyle\int_0^{+\infty} (at)^{x-1}e^{-at}adt = \int_0^{+\infty} a^x t^{x-1}e^{-at}dt = a^x\int_0^{+\infty} t^{x-1}e^{-at}dt.$

Exercise 5.148

$$\begin{aligned}\Gamma(x+1) &= \int_0^{+\infty} t^x e^{-t}dt = -\int_0^{+\infty} t^x de^{-t} \\ &= -\lim_{t\to+\infty} t^x e^{-t} + t^0 e^{-0} + \int_0^{+\infty} e^{-t}dt^x = \int_0^{+\infty} e^{-t}xt^{x-1}dt = x\Gamma(x).\end{aligned}$$

Then $\Gamma(n) = (n-1)\Gamma(n-1) = \cdots = (n-1)!\Gamma(1) = (n-1)!\displaystyle\int_0^{+\infty} e^{-t}dt = (n-1)!$.

Exercise 5.149

The potential improperness is at 0^+ and 1^-. By $\lim_{t\to0^+}\frac{t^{x-1}(1-t)^{y-1}}{t^{x-1}} = 1$, $\lim_{t\to1^-}\frac{t^{x-1}(1-t)^{y-1}}{(1-t)^{y-1}} = 1$ and the comparison test, the convergence of $\displaystyle\int_0^1 t^{x-1}(1-t)^{y-1}dt$ is the same as the convergence of $\displaystyle\int_0^1 t^{x-1}dt$ and $\displaystyle\int_0^1 (1-t)^{y-1}dt$. This means exactly $x > 0$ and $y > 0$.

Exercise 5.150

$$\begin{aligned}
B(x,y) &= \int_{+\infty}^{0} \frac{u^{y-1}}{(1+u)^{x-1}(1+u)^{y-1}} \frac{-du}{(1+u)^2} = \int_0^{+\infty} \frac{u^{y-1}}{(1+u)^{x+y}} du \\
&= \int_0^1 \frac{u^{y-1}}{(1+u)^{x+y}} du + \int_1^{+\infty} \frac{u^{y-1}}{(1+u)^{x+y}} du = \int_0^1 \frac{u^{y-1}}{(1+u)^{x+y}} du + \int_1^0 \frac{v^{-y+1}}{(1+v^{-1})^{x+y}} (-v^{-2}) dv \\
&= \int_0^1 \frac{u^{y-1}}{(1+u)^{x+y}} du + \int_0^1 \frac{v^{x-1}}{(v+1)^{x+y}} dv = \int_0^1 \frac{t^{x-1}+t^{y-1}}{(1+t)^{x+y}} dt.
\end{aligned}$$

Exercise 5.151

$$B(x,y) = \int_{\frac{\pi}{2}}^{0} (\cos^2 t)^{x-1}(1-\cos^2 t)^{y-1}(-2\cos t \sin t) dt = 2\int_0^{\frac{\pi}{2}} \cos^{2x-1} t \sin^{2y-1} t dt.$$

Exercise 5.152

$B(x,y) = B(y,x)$ by Exercise 5.150.

$B(x,y) = \int_0^1 [t^x + t^{x-1}(1-t)](1-t)^{y-1} dt = B(x+1,y) + \int_0^1 t^{x-1}(1-t)^y dt = B(x+1,y) + \frac{1}{x}\int_0^1 (1-t)^y dt^x = B(x+1,y) - \frac{1}{x}\int_0^1 t^x d(1-t)^y = B(x+1,y) + \frac{1}{x}\int t^x y(1-t)^{y-1} dt = B(x+1,y) + \frac{y}{x}B(x+1,y) = \frac{x+y}{x}B(x+1,y).$