BAIRE'S THEOREM AND SETS OF DISCONTINUITY

1. Sets of discontinuity. For $f : R \to R$, we define

$$D_f = \{x \in R; f \text{ is not continuous at } x\}.$$

Example 1. (Dirichlet's function) $f(x) = 1, x \in \mathbb{Q}; f(x) = 0, x \in \mathbb{I} := R \setminus \mathbb{Q}$. Here $D_f = R$.

Example 2. (Thomae's function) $f(0) = 1; f(r) = 1/q, r \in \mathbb{Q}$ if $r = p/q$ in lowest terms, with $p \neq 0; f(x) = 0, x \in \mathbb{I}$. Here $D_f = \mathbb{Q}$.

Example 3. If $f : R \to R$ is monotone, one-sided limits exist (and are finite) at each $a \in R$, all discontinuities are 'jump type' and and D_f is countable.(*Proof:* Define an injective map $D_f \to \mathbb{Q}$ by picking a rational number in each 'gap' in the image of f (each point of discontinuity defines such a gap, and the gaps are disjoint.)

Conversely, given a countable set $D = \{x_n, n \geq 1\} \subset R$, define:

$$f(x) = \sum_{\{n; x_n < x\}} \frac{1}{2^n}.$$

It is not hard to show that f is increasing and $D_f = D$.

To understand D_f for a general $f : R \to R$, define the *oscillation* $\omega_f(x)$ of f at $x \in R$ via:

$$\omega_f(x, \delta) = \sup_{y,z \in I_\delta(x)} |f(y) - f(z)|; \quad I_\delta(x) = (x - \delta, x + \delta).$$

$$\omega_f(x) = \lim_{\delta \to 0_+} \omega_f(x, \delta) = \inf_{\delta > 0} \omega_f(x, \delta)$$

(since $\omega_f(x, \delta)$ is increasing in δ). Clearly $D_f = \{x \in R; \omega_f(x) > 0\}$, or:

$$D_f = \bigcup_{n \geq 1} D_{1/n}, \text{ where } D_\epsilon = \{x \in R; \omega_f(x) \geq \epsilon\} \text{ for } \epsilon > 0.$$

Exercise 1: D_ϵ is closed in R, for each $\epsilon > 0$.

Hint: The complement of D_ϵ is:

$$D_\epsilon^c = \{x \in R; (\exists \delta > 0) \omega_f(x, \delta) < \epsilon\} = \{x \in R; (\exists \delta > 0)(\forall y, z \in I_\delta(x)) |f(y) - f(z)| < \epsilon\},$$

and this set is open.

Definitions. A subset $A \subset R$ is an F_σ if it is a countable union of closed sets; A is a G_δ if it is a countable intersection of open sets.

Examples. A closed interval $A = [a, b] = \cup_{n\geq 1}(a - \frac{1}{n}, b + \frac{1}{n})$ is both an F_σ and a G_δ.

The complement of an F_σ is a G_δ, and vice-versa. The rationals $\mathbb{Q}$, being countable, are an F_σ, hence the irrationals $\mathbb{I}$ are a G_δ.

It will turn out (as a consequence of Baire's theorem) that $\mathbb{I}$ is not an F_σ, hence cannot be D_f for any $f : R \to R$, since we just saw that:

Proposition. For any $f : R \to R$, D_f is an F_σ.

Remark: Conversely, given any F_σ set $D \subset R$, there exists an $f : R \to R$ so that $D_f = D$. [W.H. Young 1903].

2. Baire's Theorem. We need some terminology.

Definitions. A subset $A \subset R$ is *dense* if any open subset of R intersects A. A is *nowhere dense* if any open interval $I \subset R$ contains an open interval $J \subset I$ disjoint from A ($J \cap A = \emptyset$). (*Careful:* the complement of a dense set is not necessarily 'nowhere dense', as shown by $\mathbb{Q}$ and $\mathbb{I}$, both dense in R.)

Exercise 2. $A \subset R$ is nowhere dense if and only if its closure has empty interior: $\text{int}(\bar{A}) = \emptyset$. Thus for a closed set $A \subset R$, 'nowhere dense' is equivalent to 'empty interior'. (In general, 'nowhere dense' implies empty interior; but $\mathbb{Q}$ and $\mathbb{I}$ have empty interior, and are not nowhere dense.)

Exercise 3. The complement of an open and dense set $D \subset R$ is a closed, nowhere dense set (and conversely).

Remark: The statements in exercises 2 and 3 are also true in a general complete metric space X; you may prove this more general case if you prefer.

Definitions. A set $A \subset R$ is *meager* if it is a countable union of nowhere dense sets. A is *residual* if it is a countable intersection of *open* dense sets.

(Another common name for 'residual' is 'G_δ dense'. A residual set is both G_δ and (by Baire's theorem) dense. Conversely, if $G = \cap_{n\geq 1} G_n$ (with each G_n open) is dense, then no G_n can omit an interval, or G would; so each G_n is dense.)

Baire's Theorem: The real line R is not meager.

Remark: This implies no interval $I \subset R$–open, closed, or half-open–is meager, either.

Corollary. The irrationals $\mathbb{I} \subset R$ are not an F_σ.

Proof. By contradiction, suppose $\mathbb{I} = \cup_{n\geq 1} C_n$, with each $C_n \subset R$ closed. Then each C_n has empty interior (since $\mathbb{I}$ has empty interior), and hence is nowhere dense. Thus $\mathbb{I}$ would be meager, and since $\mathbb{Q}$ is meager (countable union of one-point sets) their union R would also be meager, contradiction.

By complementation, the rationals $\mathbb{Q}$ are not a G_δ, in particular not a residual subset of R (although they are dense in R). The irrationals, on the other hand, *are* residual in R: letting $(r_n)_{n\geq 1}$ be an enumeration of $\mathbb{Q}$, we have: $\mathbb{I} = \cap_{n\geq 1} G_n$, where each $G_n = R \setminus \{r_n\}$ is open and dense in R. (So the complement of any countable set is residual in R.)

Main Lemma. Residual sets are dense: if $A = \cap_{n\geq 1} G_n$ with each G_n open and dense in R, then A is dense in R.

Proof. Let $I \subset R$ be a non-empty open set. We need to show $I \cap A \neq \emptyset$. Since G_1 is open and dense dense, $I \cap G_1 \neq \emptyset$ is open; thus we may find a *compact* interval $I_1 \subset I \cap G_1$. Since G_2 is open and dense, $G_2 \cap \text{int}(I_1)$ is open and non-empty, hence contains a compact interval I_2.

Proceeding in this fashion we find a nested sequence of non-empty compact intervals I_n, with $I_{n+1} \subset \text{int}(I_n)$. Thus their intersection $\cap_{n\geq 1} I_n$ is non-empty, and taking a point x in this intersection we see that $x \in A$. Since also $x \in I_1 \subset I$, it follows that $x \in I \cap A$.

Corollary. If $F = \cup_{n\geq 1} C_n$, with each C_n closed and with empty interior, then F has empty interior (and is meager, by definition).

Proof. By complementation: the Main Lemma implies F^c is dense in R, so F has empty interior.

Remark: Sets F as in the corollary are often called 'F_σ meager': a countable union of closed sets of empty interior. The complement of a 'G_δ dense' set is 'F_σ meager'. (This can be confusing, since 'F_σ meager' is not the same as 'F_σ and meager'.)

Proof of Baire's Theorem. By contradiction, suppose $R = \cup_{n\geq 1} E_n$, with each E_n nowhere-dense in R ($\bar{E}_n$ has empty interior.) Then R is also the union of their closures: $R = \cup_{n\geq 1} \bar{E}_n$, that is, a countable union of closed sets with empty interior. By complementation, we find $\cap_{n\geq 1} (\bar{E}_n)^c = \emptyset$, and since each $(\bar{E}_n)^c$ is open and dense in R (see Exercise 3) this contradicts the Main Lemma.

Remark 1. It is common to use "Baire's Theorem" to refer to the Main Lemma.

Remark 2. The Main Lemma ("Baire's Theorem") is valid in more general settings, with similar proofs: (i) in locally compact topological spaces ([Dugundji p. 249]); (ii) in complete metric spaces ([Dugundji] p. 299).

Remark 3. 'Residual sets' are one way to make precise the informal idea of 'generic behavior' in Mathematics, one that has proved fruitful in many infinite-dimensional settings: spaces of real-valued functions, of vector fields, etc. Even better is when the behavior defines an open dense set: then it is not only stable under perturbations (open), but any object in the class (function, vector fields, etc.) may be approximated by one exhibiting the behavior in question (dense).

3. Pointwise limits of continuous functions.

Theorem. If $f : R \to R$ is a pointwise limit of continuous functions, then D_f is F_σ meager (that is, a countable union of closed sets with empty interior).

(In particular, by Baire's theorem, f is continuous on a dense subset of R.)

Proof. We know $D_f = \bigcup_{n\geq 1} D_{1/n}$ (see Section 1), so it suffices to show that the closed sets D_ϵ have empty interior, for any $\epsilon > 0$. By contradiction, suppose D_ϵ contains an open interval I. We'll find an open interval $J \subset I$ disjoint from D_ϵ!

Let $f_n \to f$ pointwise on R, with each $f_n : R \to R$ continuous. For each $N \geq 1$, consider the set:

$$C_N = \{x \in I; (\forall m, n \geq N)|f_m(x) - f_n(x)| \leq \epsilon/3\}.$$

Clearly $\cup_{N\geq 1} C_N = I$ (by pointwise convergence).

Exercise 4. Each C_N is closed in I.

By Baire's Theorem, some C_N must have non-empty interior (otherwise I is meager), so we have an open interval $J \subset C_N \subset I$ (for this N).

On the other hand, $J \cap D_\epsilon = \emptyset$. To see this, taking limits $m \to \infty$ (pointwise at each $x \in C_N$) in the inequality defining C_N, we find that $x \in C_N \Rightarrow |f(x) - f_N(x)| \leq \epsilon/3$.

Exercise 5. If $|f - g| \leq \epsilon/3$ in an open interval containing x, then $|\omega_f(x) - \omega_g(x)| \leq 2\epsilon/3$.

Hint: Assume $y, z \in I_\delta(x)$, with $f(y) \geq f(z)$. Then

$$|f(y)-f(z)| = f(y)-f(z) \leq g(y)-g(z)+2 \sup_{w\in I_\delta(x)} |f(w)-g(w)| \leq |g(y)-g(z)|+2\epsilon/3,$$

if δ is small enough. (If $f(z) \geq f(y)$, exchange y and z to get the same estimate.)

Since $|f - f_N| \leq \epsilon/3$ in the open interval J and $\omega_{f_N}(x) = 0 \forall x$ (by continuity), it follows from Exercise 5 that $\omega_f(x) \leq 2\epsilon/3$ for each $x \in J$; thus $J \cap D_\epsilon = \emptyset$.

Corollary. Let $f : R \to R$ be differentiable on R. Then $f' : R \to R$ is continuous on a dense set.

Proof: Consider the sequence of continuous functions $g_n : R \to R$:

$$g_n(x) = \frac{f(x + \frac{1}{n}) - f(x)}{\frac{1}{n}}.$$

It suffices to observe that $g_n \to f'$ pointwise on R.

Baire's hierarchy. (See [W. Dunham, *The Calculus Gallery*])

Definition. Let $f : R \to R$. f is *class 0* if it is continuous on R. f is *class 1* if it is a pointwise limit of continuous functions, but not continuous. f is *class 2* if it is a pointwise limit of class 1 functions, but not in class 0 or class 1. And so on...

Example. Dirichlet's function is neither in class 0 or class 1 (since it is discontinuous everywhere, and class 1 functions are continuous on a dense set.) But it is in class 2. Consider:

$$D(x) = \lim_{k\to\infty} [\lim_{j\to\infty} (\cos k!\pi x)^{2j}].$$

If $x = p/q$ is rational (in lowest terms), then $k!x$ is an integer as soon as $k \geq q$, so $|\cos k!\pi x| = 1$ for $k \geq q$, and $D(x) = 1$. But if x is irrational, $k!x$ is never an integer, so $|\cos k!\pi x| < 1$ and $\lim_j (\cos k!\pi x)^{2j} = 0$, thus $D(x) = 0$. This shows $D(x)$ is Dirichlet's function.

For each $k \geq 1$, the function $g_k(x) = \lim_j (\cos k!\pi x)^{2j}$ is in class 1: it is a pointwise limit of continuous functions, but not continuous! (Verify that.) And $D(x) = \lim_k g_k(x)$ (pointwise), showing $D(x)$ is in class 2.

Remark. It has been shown that all Baire classes are non-empty, and that functions belonging to no Baire class exist (H. Lebesgue 1905.)

4. Generic continuous functions are nowhere differentiable.

Theorem. Let $X = C[0, 1]$ (continuous, real-valued functions on the unit interval), a complete metric space endowed with the sup norm. Then:

$$D = \{f \in X; \exists x \in [0, 1] \text{ so that } f \text{ is differentiable at } x\}.$$

is contained in a countable union of closed sets with empty interior in X. In particular, its complement D^c (the set of continuous functions which are nowhere differentiable) *contains* a set which is residual in X (and therefore D^c is dense in X, by Baire's Theorem): any continuous function in $[0,1]$ is the uniform limit of continuous, nowhere differentiable functions!

Proof. (Outline; based on Abbott's *Understanding Analysis, p. 227.*)

For each $n \geq 1, m \geq 1$, define, with $m_f(x,y) = (f(x) - f(y))/(x-y)$:

$$A_{m,n} = \{f \in X; (\exists x \in [0,1])(\forall y) 0 < |x-y| \leq \frac{1}{m} \Rightarrow |m_f(x,y)| \leq n\}.$$

Step 1. $D \subset \bigcup_{m,n\geq 1} A_{m,n}$.

Step 2. Each $A_{m,n}$ is closed in X: if $f_k \to f$ uniformly in $[0,1]$ with $f_k \in A_{m,n}$, then $f \in A_{m,n}$.

Thus it suffices to show each $A_{m,n}$ has empty interior in X: if $f \in A_{m,n}$ and $\epsilon > 0$ is arbitrary, the open ball $B_\epsilon(f) = \{g \in X; ||g - f|| < \epsilon\}$ contains functions which are not in $A_{m,n}$. In fact we'll show that the complement of $A_{m,n}$ is (open and) dense in X.

Step 3. Given $f \in X$, approximate f uniformly in $[0,1]$ by a continuous, piecewise linear function $p \in X$, so that $||f - p|| < \epsilon/2$. Then if $h \in X$ with $||h|| \leq 1$, we have $g = p + \frac{\epsilon}{2}h \in X$ satisfies $||f - g|| < \epsilon$.

Step 4. We seek h piecewise linear so that g is not in $A_{m,n}$:

$$(A_{m,n})^c = \{g \in X; (\forall x \in [0,1])(\exists y \in [0,1]) 0 < |x-y| \leq 1/m \text{ and } |m_g(x,y)| > n\}.$$

Here $m_g(x,y)$ denotes the slope of the secant to the graph of g defined by the points $x, y \in [0,1]$, and clearly:

$$m_g(x,y) = m_p(x,y) + \frac{\epsilon}{2} m_h(x,y), \text{ so } |m_g(x,y)| \geq \frac{\epsilon}{2}|m_h(x,y)| - |m_p(x,y)|,$$

and we seek h with $||h|| \leq 1$ so that:

$$(\forall x \in [0,1])(\exists y \in [0,1])(0 < |x-y| \leq 1/m \text{ and } |m_h(x,y)| \geq \frac{2}{\epsilon}(|m_p(x,y)| + n).$$

Since p is piecewise linear, there exists a finite partition of $[0,1]$ (into, we may assume, sub-intervals of length less than $1/m$) with p of constant slope on each subinterval I_j. So we construct h with constant absolute slope $|m_h|$ in each I_j, so that $|m_h|$ satisfies this inequality in each I_j. (Then given $x \in [0,1]$, take any y in the same I_j as x.)

To ensure h is continuous in $[0,1]$ and remains bounded by 1 in absolute value, we start and end h at 0 (at each endpoint of an I_j), and alternate between positive and negative slope. (So the graphs of h and g will be 'jagged', but the graph of g stays uniformly close to that of f.)

Remark. An explicit example of a continuous, nowhere differentiable function was given by Weierstrass (1872), see [Dunham]:

$$f(x) = \sum_{k=0}^{\infty} b^k \cos(\pi a^k x), \text{ where } 0 < b < 1, a \geq 3, ab > 1 + \frac{3\pi}{2}.$$