Linear Algebra: Application to Engineering

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September 10, 2019

Contents

Chapter 1

Network Flow

1.1 Network Flow

A network consists of nodes and branches connecting two nodes or is connected to only one node. The basic assumption of network flows is that the total amount flowing into the network is the same as the total amount flowing out of the network. Moreover, at each node, the total amount of the node is the same as the total amount out of the node.

In the network flow in Figure 1.1.1, each branch has a direction and the amount of flow, some known and some unknown. The flow can be electron, digital data, water, or traffic, etc., and are measured by the corresponding units. When the amount is positive, we mean the flow is physically in the indicated direction. When the amount is negative, the flow is physically opposite to the indicated direction. When the direction of a branch is reversed, then the amount is changed by sign. So the two pictures represent the same network flow.

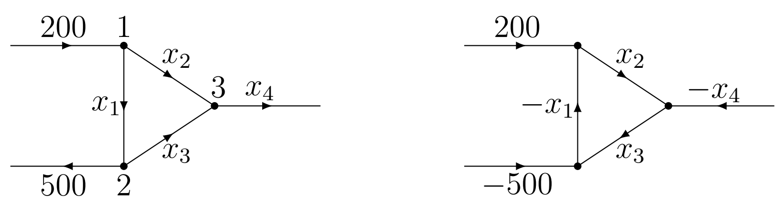


Figure 1.1.1: Network flow.

In Figure 1.1.1, the assumption on the network flow gives us a system of linear equations

$$x_4 + 500 = 200, \quad x_1 + x_2 = 200, \quad x_1 = x_3 + 500, \quad x_2 + x_3 = x_4.$$

The solution is

$$x_1 = 500 + x_3, \quad x_2 = -300 - x_3, \quad x_3 \text{ free.}$$

If we demand the flow from node 2 to node 3 to be $x_3 = -400$, i.e., the physical flow has amount 400 from node 3 to node 2, then this determines $x_1 = 100$ and $x_2 = 100$.

For the flow network in Figure 1.1.2, we have

$$x_1 + 200 = 100 + 300, \quad x_3 + 50 = x_4 + 100, \quad x_1 + x_2 = x_3,$$

$$x_2 + 50 + 300 = 400, \quad x_4 + 200 = 400.$$

Then we get unique solution

$$x_1 = 200, \quad x_2 = 50, \quad x_3 = 250, \quad x_4 = 200.$$

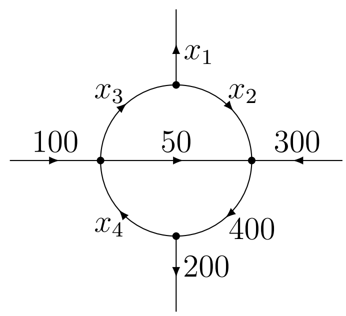


Figure 1.1.2: Network flow.

1.2 Closed Network

A closed network has no input and output. In this case, the network can be divided into several loops, and the assumption at the nodes means that we have loop flow X_i for each loop. Then the flow on a branch is the combination of all the loop flows passing through the branch.

The network on the left of Figure 1.2.1 has 4 nodes and 6 branches. We have $x_1, x_2, \dots, x_6$ for the flows along the 6 branches. They are determined by three *loop flows* X_1, X_2, X_3 by

$$x_1 = X_1,\ x_2 = X_2,\ x_3 = X_3,\ x_4 = -X_1 + X_2,\ x_5 = X_1 - X_3,\ x_6 = X_2 - X_3.$$

If we know certain three branch flows, then we may determine all branch flows. For example,

$$\begin{aligned} x_1 = 100, x_2 = 200, x_3 = 300 &\implies X_1 = 100, X_2 = 200, X_3 = 300 \\ &\implies X_4 = 100, x_5 = -200, x_6 = -100. \end{aligned}$$

In fact, given any value of x_1, x_4, x_6, we may also determine all the flows. On the other hand, the values of x_1, x_2, x_4 cannot be given arbitrarily, because they must

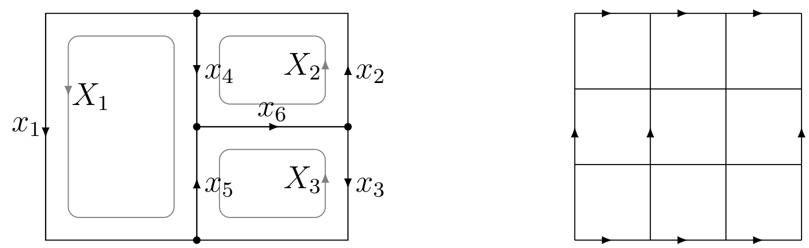


Figure 1.2.1: Closed network flow.

satisfy $x_1 + x_4 = x_2$. In other words, for certain values of x_1, x_2, x_4, the system has no solution. The problem with x_1, x_2, x_4 is that they are not related to X_3.

The right of Figure 1.2.1 is a closed network that can be divided into 9 loops. If we assign 9 branch flows as in the picture, such that each loop touches at least one branch, then the whole network flow is uniquely determined.

Finally, for the non-closed networks in Figures 1.1.1 and 1.1.2, we note that the given data sometimes uniquely determine the flow, and sometimes not. We can use loop flows of closed networks to analyse this.

Since the total input is the same as the total output, for the network in Figure 1.1.1, we may add an external node, indicated by $\circ$ and get Figure 1.2.2. Then we find one more value is needed to uniquely determine all the flows. The value could be x_1, x_2 or x_3, but not x_4.

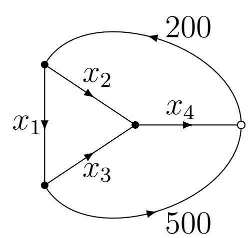
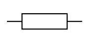
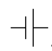


Figure 1.2.2: Closed network flow.

Chapter 2

Electric Circuit

2.1 Circuit

An electric circuit is a network in which electrons flow. Figure 2.1.1 is a DC (direct current) circuit with resistors  and batteries . The longer bar in a battery means positive side and the shorter bar means negative side. The battery provides certain voltage (usually in *volts*, denoted V). The resistor provides resistance R (usually in *ohm*, denoted Ω).

What we can physically measure in a circuit are the voltage differences and currents along each branch (or edge), both having directions. We would like to find out these indicators from the known resistors and batteries.

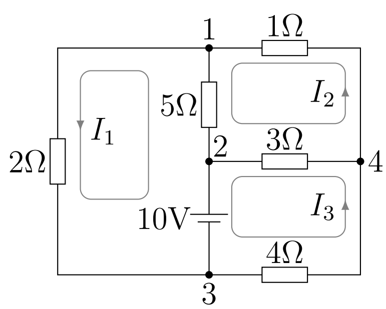


Figure 2.1.1: Circuit with resistors and batteries.

For the voltage difference, we imagine there is a voltage (potential) V_i at the i-th node. For a branch $\overline{ij}$ of the circuit, $V_{ij} = V_i - V_j$ is the *voltage drop* from i to j. If we switch the nodes, then the voltage drop from j to i is $V_{ji} = V_j - V_i = -V_{ij}$. For example, the branch $\overline{23}$ is occupied by a battery with 3 being the lower voltage end. Therefore we have $V_{23} = 10$. This is the same as $V_{32} = -10$.

For a branch $\overline{ij}$ of the circuit, we denote by I_{ij} the current flowing from i to j. The notation respects the choice of direction, in the sense that the flow I_{ji} in the opposite direction (from j to i) has value $I_{ji} = -I_{ij}$. Like the voltage difference, we may divide the circuit into several loops, and imagine there is a *loop current* I_k along the k-th loop in the indicated direction. Similar to I_{ij}, if we change the direction of the k-th loop, then we change I_k to $-I_k$. Then the current I_{ij} along a branch is the composite of all the loop currents flowing through the branch. For example, in Figure 2.1.1, we have

$$I_{12} = -I_1 + I_2.$$

Here I_1 flows through the branch $\overline{12}$. Since the direction of I_1 is from node 2 to node 1, opposite to node 1 to node 2, we have the negative sign. We also note that I_2 flows through the branch $\overline{12}$, and is from node 1 to node 2. Therefore we have the positive sign. Finally, I_3 does not appear because the loop does not intersect $\overline{12}$.

The voltage drops (differences) V_{ij} and the currents I_{ij} along branchs are the real physical quantities that we can measure. The voltages V_i and loops currents I_k are virtual quantities that help us to understand the calculate these real quantities.

If a branch $\overline{ij}$ is occupied by a resistor R_{ij}, then the voltage drop and the current are related by *Ohm's law*

$$V_{ij} = R_{ij} I_{ij}.$$

The resistor has no direction, which means $R_{ij} = R_{ji}$.

Knowing all the resistors and batteries in a circuit, we may calculate V_{ij} and I_{ij} in two ways, the KVL method that uses Kirchhoff's Voltage Law, and the KCL method that uses Kirchhoff's Current Law.

2.2 Kirchhoff's Voltage Law

Kirchhoff's voltage law (or *loop rule*, or *mesh rule*) says the following:

- The sum of potential (voltage) drops around any closed loop, counting the directions, is zero.

When applied to the first loop in Figure 2.1.1, the law means[1]

$$V_{13} - V_{23} - V_{12} = V_{13} + V_{32} + V_{21} = 0.$$

The application to the second and third loops give

$$\begin{aligned} V_{12} + V_{24} - V_{14} &= V_{12} + V_{24} + V_{41} = 0, \\ V_{23} + V_{34} - V_{24} &= V_{23} + V_{34} + V_{42} = 0. \end{aligned}$$

Each voltage drop can be calculated by Ohm's law

$$\begin{aligned} V_{12} &= R_{12} I_{12} = 5(-I_1 + I_2), & V_{13} &= R_{13} I_{13} = 2I_1, \\ V_{14} &= R_{14} I_{14} = 1(-I_2) = -I_2, & V_{23} &= 10, \\ V_{24} &= R_{24} I_{24} = 3(I_2 - I_3), & V_{34} &= R_{34} I_{34} = 4I_3. \end{aligned} \tag{2.2.1}$$

[1] The equality is a consequence of $V_{ij} = V_i - V_j$. Conversely, if such equality holds for each loop, then we can assign V_i to each node i so that $V_{ij} = V_i - V_j$ is always satisfied. In other words, the law is equivalent to the existence of the imaginary voltages V_i.

Therefore applying Kirchhoff's voltage law to the three loops gives a system of three equations

$$\begin{aligned} (2+5)I_1 - \quad 5I_2 \quad &= 10, \\ -5I_1 + (1+3+5)I_2 - \quad 3I_3 &= 0, \\ - \quad 3I_2 + (3+4)I_3 &= -10. \end{aligned}$$

The equation is of the form $I_1\vec{r}_1 + I_2\vec{r}_2 + I_3\vec{r}_3 = \vec{v}$, with *resistance vectors*

$$\begin{aligned} \vec{r}_1 &= (2+5, -5, 0) = (7, -5, 0), \\ \vec{r}_2 &= (-5, 1+3+5, -3) = (-5, 9, -3), \\ \vec{r}_3 &= (0, -3, 3+4) = (0, -3, 7), \end{aligned}$$

and *voltage vector*

$$\vec{v} = (10, 0, -10).$$

Note that the first coordinate of the resistance vector $\vec{r}_1$ counts all the resistors along the loop. The second and third coordinates count those resistors appearing in the interaction with the second and third loops. The same applies to the second and third resistance vectors.

By the same idea, for the circuit in Figure 2.2.1, we have

$$\begin{aligned} \vec{r}_1 &= (5+6+9, -6, 0) = (20, -6, 0), \\ \vec{r}_2 &= (-6, 2+3+6+7, -3) = (-6, 18, -3), \\ \vec{r}_3 &= (0, -3, 3+4+8) = (0, -3, 15), \\ \vec{v} &= (30, 0, -50). \end{aligned}$$

Then we get system of linear equations

$$\begin{aligned} 20I_1 - \quad 6I_2 \quad &= 30, \\ -6I_1 + 18I_2 - \quad 3I_3 &= 0, \\ - \quad 3I_2 + 15I_3 &= -50. \end{aligned}$$

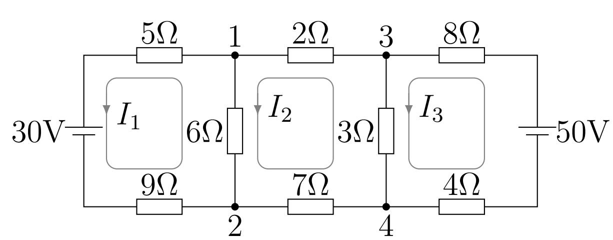


Figure 2.2.1: Another circuit with resistors and batteries.

2.3 Kirchhoff's Current Law

Kirchhoff's current law (or *point rule*, or *junction rule*, or *nodal rule*) says the following:

- The total currents going into a node must be the same as the total currents going out of the node.

At the node 1 Figure 2.1.1, the law says $I_{12} + I_{13} + I_{14} = 0$. By Ohm's law, this means

$$\frac{V_{12}}{R_{12}} + \frac{V_{13}}{R_{13}} + \frac{V_{14}}{R_{14}} = 0.$$

By $V_{ij} = V_i - V_j$, this becomes

$$\left(\frac{1}{R_{12}} + \frac{1}{R_{13}} + \frac{1}{R_{14}}\right) V_1 - \frac{1}{R_{12}} V_2 - \frac{1}{R_{13}} V_3 - \frac{1}{R_{14}} V_4 = 0.$$

We have similar equation derived from Kirchhoff's current law $I_{21} + I_{23} + I_{24} = 0$ at node 2

$$-\frac{1}{R_{12}} V_1 + \left(\frac{1}{R_{21}} + \frac{1}{R_{24}}\right) V_2 - \frac{1}{R_{24}} V_4 + I_{23} = 0.$$

The problem is that nodes 2 and 3 are connected by a battery, and we cannot use Ohm's law to express I_{23} in terms of the voltage drop. Instead of using Kirchhoff's current law at node 2, therefore we simply use the voltage drop caused by the battery

$$V_2 - V_3 = V_{23} = 10.$$

The other observation is that we may designate one node to be the "ground". Usually it is more convenient to designate the negative end of one battery as the ground. For example, we may declare node 3 to be the ground. This means setting $V_3 = 0$.

The reason we can set the round is that what we really care is the voltage drop $V_{ij} = V_i = V_j$. For any constant C, we may change V_i to $V_i + C$ without affecting V_{ij}. If we choose $C = -V_3$, then we get $V_3 = 0$.

With $V_3 = 0$, Kirchhoff's current law at node 1 becomes

$$\left(\frac{1}{1} + \frac{1}{2} + \frac{1}{5}\right) V_1 - \frac{1}{5} V_2 - \frac{1}{1} V_4 = 0.$$

Moreover, instead of Kirchhoff's current law at node 2, we use

$$V_2 = 10.$$

Then we add Kirchhoff's current law at node 4 to get a system

$$\begin{aligned} (\tfrac{1}{1} + \tfrac{1}{2} + \tfrac{1}{5}) V_1 - \tfrac{1}{5} V_2 - \qquad\qquad\qquad \tfrac{1}{1} V_4 &= 0, \\ -\tfrac{1}{1} V_1 - \tfrac{1}{3} V_2 + (\tfrac{1}{1} + \tfrac{1}{3} + \tfrac{1}{4}) V_4 &= 0, \\ V_2 \qquad\qquad\qquad &= 10. \end{aligned}$$

The system has unique solution.

Next we apply Kirchhoff's current law to the circuit in Figure 2.2.1. We set the ground and add nodes as in Figure 2.3.1. Then we have $V_5 = 30$ and $V_6 - V_7 = 50$. Applying Kirchhoff's current law to nodes 1, 2, 3, 4 and using $V_5 = 30$, $V_6 = V_7 + 50$, we get

$$\begin{aligned}
(\tfrac{1}{5} + \tfrac{1}{6} + \tfrac{1}{2})V_1 - \tfrac{1}{6}V_2 - \tfrac{1}{2}V_3 &= \tfrac{1}{5}30, \\
-\tfrac{1}{6}V_1 + (\tfrac{1}{6} + \tfrac{1}{7} + \tfrac{1}{9})V_2 - \tfrac{1}{7}V_4 &= 0, \\
-\tfrac{1}{2}V_1 + (\tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{8})V_3 - \tfrac{1}{3}V_4 - \tfrac{1}{8}V_7 &= \tfrac{1}{8}50, \\
-\tfrac{1}{7}V_2 - \tfrac{1}{3}V_3 + (\tfrac{1}{3} + \tfrac{1}{4} + \tfrac{1}{7})V_4 - \tfrac{1}{4}V_7 &= 0.
\end{aligned}$$

We need one more equation to determine five variables V_1, V_2, V_3, V_4, V_7. This can be achieved by combining Kirchhoff's current law $I_{63} + I_{67} = 0$ at node 6 and the law $I_{74} + I_{76} = 0$ at 7 to get $I_{63} + I_{74} = 0$ (using $I_{67} + I_{76} = 0$). Then by $I_{63} = \frac{1}{8}(V_6 - V_3) = \frac{1}{8}(V_7 + 50 - V_3)$ and $I_{74} = \frac{1}{4}(V_7 - V_4)$, we get the fifth equation

$$-\tfrac{1}{8}V_3 - \tfrac{1}{4}V_4 + (\tfrac{1}{4} + \tfrac{1}{8})V_7 = -\tfrac{1}{8}50.$$

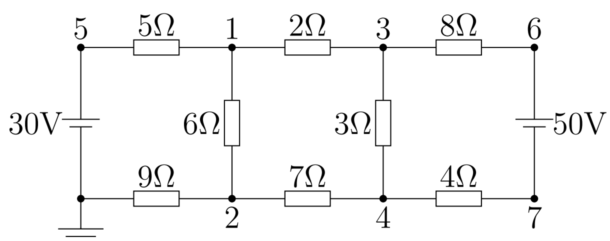


Figure 2.3.1: Applying Kirchhoff's current law to another circuit.

We may also find the fifth equation from the left side of the circuit. Denote by 0 the ground node. This means adding $I_{02} + I_{05} = 0$ and $I_{50} + I_{51} = 0$ together to get $I_{02} + I_{51} = 0$. By $I_{02} = \frac{1}{9}(V_0 - V_2) = -\frac{1}{9}V_2$ and $I_{51} = \frac{1}{5}(V_5 - V_1) = \frac{1}{5}(30 - V_1)$, this is

$$-\tfrac{1}{5}V_1 - \tfrac{1}{9}V_2 = -\tfrac{1}{5}30.$$

It turns out that adding either fifth equation is equivalent.

2.4 Exercise

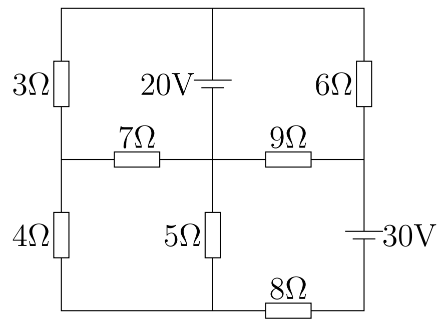


Figure 2.4.1: Exercise 1.

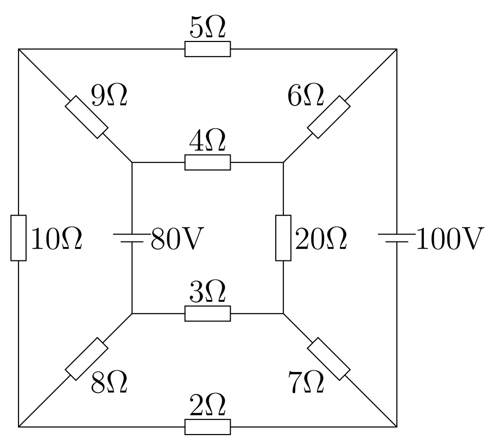


Figure 2.4.2: Exercise 2.