Math2131 Answer to Homework 3

Exercise 1.3.17

(2) Identify P_3 by $\mathbb{R}^4$, we have row operation

$$\begin{pmatrix} 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & -1 & 0 & -1 & 1 & 1 \\ 2 & -2 & 2 & 0 & 0 & 2 \\ 3 & -3 & 3 & 0 & 3 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & 0 & -1 & 1 & 1 \\ 0 & 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 3 & -3 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix}.$$

So these six vectors can span P_3, the minimal span set can be

$$\{t + t^2 + t^3, 1 + 2t^2 + 3t^2, 1 + t + 3t^3, 1 + t + 2t^2\}.$$

Remark. The solution is NOT unique.

Exercise 1.3.19

$''1 \Rightarrow 2''$: Let these vectors are $\alpha = \{\vec{v_1}, \vec{v_2}, \ldots, \vec{v_n}\}$, and suppose they are linearly dependent. WLOG, suppose can be expressed by $\vec{v_1}, \vec{v_2}, \ldots, \vec{v}_{n-1}$. Since α span V, we can conclude that $\vec{v_1}, \vec{v_2}, \ldots, \vec{v}_{n-1}$ can also span V. It contradicts to Proposition 1.3.7.

$''2 \Rightarrow 3''$: Suppose α can't span V, we will have a vector $\vec{u}$, where $\vec{u}$ can't expressed as a linear combination of α. So $n+1$ vectors $\vec{v_1}, \vec{v_2}, \ldots, \vec{v_n}, \vec{u}$ are linearly independent. It contradicts to Proposition 1.3.7. So α can span V. α is a basis for V.

$''3 \Rightarrow 1''$: Just use definition.

Exercise 1.3.20

(2) Since $\cos^2 t + \sin^2 t = 1$, they are not linearly independent. And $\cos 2t = 2\cos^2 t - 1$, so $f(t)$ is in the span of the given function. Suppose $g(t) = t$ is in the span of given function. We have

$$g(t) = x_1 \cos t + x_2 \sin t + x_3.$$

Evaluate $t = 0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}$. This system of linear equations doesn't have solution. It causes contradiction. So $g(t) = t$ is NOT in the span of given function.

Exercise 2.1.1

(2) Let $f : (x_1, x_2, x_3) \longmapsto (x_1, x_2 x_3)$. We have $f(2(0,2,2)) = (0,16)$, but $2f((0,2,2)) = (0,8)$. They are not equal. So f is NOT a linear transformation.

(3) Let $f : (x_1, x_2, x_3) \longmapsto (x_3, x_2, x_1)$. We have $f((x_1, x_2, x_3) + (y_1, y_2, y_3)) = (x_3 + y_3, x_2 + y_2, x_1 + y_1)$.

It equals $f(x_1, x_2, x_3) + f(y_1, y_2, y_3)$. And $f(a(x_1, x_2, x_3))$=(ax_1, ax_2, ax_3)=$a \cdot f((x_1, x_2, x_3))$. So this f is a linear transformation.

Exercise 2.1.2

(1) Let $L : f(t) \mapsto f^2(t)$. Choose $f(t) = 1$, $g(t) = t$. $L(f+g)(t) = (t+1)^2$, $Lf(t) + Lg(t) = t^2 + 1$. They are not equal. So L is NOT a linear transformation.

(2) Let $L : f(t) \mapsto f(t^2)$. Then $L(af)(t) = (af)(t^2) = af(t^2) = aL(f)(t)$ and $L(f+g)(t) = (f+g)(t^2) = f(t^2) + g(t^2) = L(f)(t) + L(g)(t)$. So L is a linear transformation.

(8) Let $L : f \mapsto f(0)f(1)$. Choose $f(t) = 1$, $g(t) = e^t$. $L(f+g) = 2(e+1)$. $Lf + Lg = e + 1$. They are not equal. So L is NOT a linear transformation.

Remark. Compare (1) and (2) carefully since I made mistakes in TA class.

Exercise 2.1.3

Suppose $\alpha = \{\vec{v_1}, \vec{v_2}, \ldots, \vec{v_n}\}$ is a basis for V, and v_i^* is the i-th coordinate map. i.e. If $\vec{u} = \sum_{i=1}^n x_i \vec{v_i}$, then $v_i^*(\vec{u}) = x_i$. Then we claim that $\{v_i^*\}$ is a basis for V^*.

If $\sum_{i=1}^n a_i v_i^* = 0^* \in V^*$, i.e. $(\sum_{i=1}^n a_i v_i^*)(\vec{u}) = 0 \in \mathbb{R}$ for all $\vec{u}$. And $\sum_{i=1}^n a_i v_i^*(\vec{u}) = a_i x_i = 0$ for all x_i. So $a_i = 0$. It shows $\{v_i^*\}$ is linearly independent.

Suppose $L \in V^*$, and $L(\vec{v_i}) = b_i \in \mathbb{R}$. Then we have $L = \sum_{i=1}^n b_i v_i^*$. It shows that any $L \in V^*$ can be expressed as a linear combination of $\{v_i^*\}$. So $\{v_i^*\}$ can span V^*.

Above all, $\{v_i^*\}$ is a basis for V^*.

Remark. If $f, g \in V^*$ and $\alpha = \{\vec{v_1}, \vec{v_2}, \ldots, \vec{v_n}\}$ is a basis for V. Then $f = g$ is equivalent to $f(\vec{v_i}) = g(\vec{v_i})$ for all index i. Use this proposition to prove $L = \sum_{i=1}^n b_i v_i^*$.

Exercise 2.1.5

Let $E : L \mapsto L(\vec{v})$. For any $L, K \in \mathrm{Hom}(V, W)$, $a \in \mathbb{R}$, we have

$$E(L + K) = (L + K)(\vec{v}) = L(\vec{v}) + K(\vec{v}) = E(L) + E(K),$$

and

$$E(aL) = (aL)(\vec{v}) = a \cdot L(\vec{v}) = aE(L).$$

So E is a linear transformation.

Exercise 2.1.7

For $L \in \mathrm{Hom}(V, W)$, any $K_1, K_2 \in \mathrm{Hom}(U, V)$, $a \in \mathbb{R}$, $\vec{u} \in U$, we have

$$L((K_1 + K_2)(\vec{u})) = L(K_1(\vec{u}) + K_2(\vec{u})) = L(K_1(\vec{u})) + L(K_2(\vec{u})).$$

It shows that $L \circ (K_1 + K_2) = L \circ K_1 + L \circ K_2$. We also have

$$L((aK_1)(\vec{u})) = L(aK_1(\vec{u})) = aL(K_1(\vec{u})).$$

It shows that $L \circ (aK_1) = a(L \circ K_1)$.

These mean that the map

$$\begin{array}{rccc} L_* & : & \mathrm{Hom}(U, V) & \to & \mathrm{Hom}(U, W) \\ & & K & \mapsto & L \circ K \end{array}$$

is a linear transformation.

Exercise 2.1.10

(1) Suppose $\vec{v_1}, \vec{v_2}, \ldots, \vec{v_n}$ are linearly dependent, WLOG, $\vec{v_1} = x_2\vec{v_2} + \ldots + x_n\vec{v_n}$. Let L operate on both side, we have $L(\vec{v_1}) = x_2 L(\vec{v_2}) + \ldots + x_n L(\vec{v_n})$. So $L(\vec{v_1}), L(\vec{v_2}), \ldots, L(\vec{v_n})$ are linearly dependent.

(2) Suppose $L(\vec{v_1}), L(\vec{v_2}), \ldots, L(\vec{v_n})$ are linearly independent. If $\vec{v_1}, \vec{v_2}, \ldots, \vec{v_n}$ are linearly dependent, use (1), we have $L(\vec{v_1}), L(\vec{v_2}), \ldots, L(\vec{v_n})$ are linearly dependent. It causes contradiction.