Math2131 Answer to Homework 1

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Ex. 1.1

Using axioms 1,2,7 and 8, we have

$$\begin{aligned}(a+b)(\vec{x}+\vec{y}) &= a(\vec{x}+\vec{y})+b(\vec{x}+\vec{y}) = (a\vec{x}+a\vec{y})+(b\vec{x}+b\vec{y}) = a\vec{x}+[a\vec{y}+(b\vec{x}+b\vec{y})] \\ &= a\vec{x}+[(b\vec{x}+b\vec{y})+a\vec{y}] = a\vec{x}+b\vec{y}+b\vec{x}+a\vec{y}.\end{aligned}$$

Ex. 1.3

We have

$$\begin{aligned}(y_1,y_2)+(x_1,x_2) &= (y_1+x_1,0) = (x_1+y_1,0) = (x_1,x_2)+(y_1,y_2), \\ \big((x_1,x_2)+(y_1,y_2)\big)+(z_1,z_2) &= (x_1+y_1,0)+(z_1,z_2) = (x_1+y_1+z_1,0) = (x_1,x_2)+\big((y_1,y_2)+(z_1,z_2)\big) \\ (ab)(x_1,x_2) &= (abx_1,0) = a(bx_1,0) = a\big(b(x_1,x_2)\big), \\ (a+b)(x_1,x_2) &= \big((a+b)x_1,0\big) = (ax_1,0)+(bx_1,0) = a(x_1,x_2)+b(x_1,x_2), \\ a\big((x_1,x_2)+(y_1,y_2)\big) = a(x_1+y_1,0) &= \big(a(x_1+y_1),0\big) = (ax_1,0)+(ay_1,0) = a(x_1,x_2)+a(y_1,y_2)\end{aligned}$$

which means axioms 1,2,6,7 and 8 hold. When $x_2\neq 0$, we have $(x_1,x_2)+(y_1,y_2)=(x_1+y_1,0)\neq(x_1,x_2)$ for any y_1 and y_2, thus axiom 3 does not hold and axiom 4 automatically fail. Also, when $x_2\neq 0$, we have $1(x_1,x_2)=(x_1,0)\neq(x_1,x_2)$, thus axiom 5 does not hold.

Ex. 1.9

Suppose $\vec{v}_1$ and $\vec{v}_2$ are two negative vectors such that

$$\begin{aligned}\vec{u}+\vec{v}_1 &= \vec{0} = \vec{v}_1+\vec{u}, \\ \vec{u}+\vec{v}_2 &= \vec{0} = \vec{v}_2+\vec{u}.\end{aligned}$$

Then we have

$$\begin{aligned}(\vec{v}_1+\vec{u})+\vec{v}_2 &= \vec{0}+\vec{v}_2 = \vec{v}_2, \\ \vec{v}_1+(\vec{u}+\vec{v}_2) &= \vec{v}_1+\vec{0} = \vec{v}_1, \\ (\vec{v}_1+\vec{u})+\vec{v}_2 &= \vec{v}_1+(\vec{u}+\vec{v}_2),\end{aligned}$$

which implies $\vec{v}_1=\vec{v}_2$.

Ex. 1.10
Exercis 1.9 justifies the notation $-\vec{u}$. We have

$$\begin{aligned} & (\vec{u}+\vec{v}_1)+(-\vec{u}) = (\vec{u}+\vec{v}_2)+(-\vec{u}) \\ \Rightarrow \quad & (\vec{u}-\vec{u})+\vec{v}_1 = (\vec{u}-\vec{u})+\vec{v}_2 \\ \Rightarrow \quad & \vec{0}+\vec{v}_1 = \vec{0}+\vec{v}_2 \\ \Rightarrow \quad & \vec{v}_1 = \vec{v}_2. \end{aligned}$$

Ex. 1.12
From the condition we have

$$\vec{w} = a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_n\vec{v}_n$$

for some real numbers a_i. We note a linear combination of $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n, \vec{w}$ be $b_1\vec{v}_1 + b_2\vec{v}_2 + \cdots + b_n\vec{v}_n + c\vec{w}$, where b_i and c are real numbers, and we have

$$\begin{aligned} b_1\vec{v}_1 + b_2\vec{v}_2 + \cdots + b_n\vec{v}_n + c\vec{w} &= b_1\vec{v}_1 + b_2\vec{v}_2 + \cdots + b_n\vec{v}_n + c(a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_n\vec{v}_n) \\ &= (b_1 + ca_1)\vec{v}_1 + (b_2 + ca_2)\vec{v}_2 + \cdots + (b_n + ca_n)\vec{v}_n, \end{aligned}$$

which is a linear combination of $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$.

Ex. 1.15
It is easily known from the operations of addition and scalar multiplication of matrix that for any matrix in $M_{3\times 2}$ we have

$$\begin{aligned} \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \\ x_{31} & x_{32} \end{pmatrix} &= x_{11}\begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} + x_{21}\begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{pmatrix} + x_{31}\begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{pmatrix} \\ &\quad + x_{12}\begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} + x_{22}\begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} + x_{32}\begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}, \end{aligned}$$

which indicates that these matrices span the vector space.

Ex. 1.17
From the condition we have

$$\vec{w} = a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_n\vec{v}_n$$

for some real numbers a_i, and for any $\vec{\alpha} \in V$, we have

$$\vec{\alpha} = b_1\vec{v}_1 + b_2\vec{v}_2 + \cdots + b_n\vec{v}_n + b_{n+1}\vec{w}$$

for some real numbers b_i. Then we have

$$\begin{aligned} \vec{\alpha} &= b_1\vec{v}_1 + b_2\vec{v}_2 + \cdots + b_n\vec{v}_n + b_{n+1}\vec{w} \\ &= b_1\vec{v}_1 + b_2\vec{v}_2 + \cdots + b_n\vec{v}_n + b_{n+1}(a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_n\vec{v}_n) \\ &= (b_1 + b_{n+1}a_1)\vec{v}_1 + (b_2 + b_{n+1}a_2)\vec{v}_2 + \cdots + (b_n + b_{n+1}a_n)\vec{v}_n, \end{aligned}$$

which is a linear combination of $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$. Thus we prove $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ span V.

Ex. 1.19

It suffices to show that all of (2), (3) and (4) are equivalent to (1). Apparently, by the commutativity of these vectors, (1) and (2) are equivalent. Also, we have

$$\begin{aligned} a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_n\vec{v}_n &= a_1\vec{v}_1 + \cdots + \frac{a_i}{c}(c\vec{v}_i) + a_n\vec{v}_n \\ &= a_1\vec{v}_1 + \cdots + a_i(\vec{v}_i + c\vec{v}_j) + \cdots + (a_j - ca_i)\vec{v}_j + \cdots + a_n\vec{v}_n, \end{aligned}$$

which shows (3) and (4) are equivalent to (1).

Ex. 1.22(2)

We form a matrix with the vectors as the columns and carry out row operations

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \\ 3 & 4 & 1 & 2 \\ 4 & 1 & 2 & 3 \end{pmatrix} \xrightarrow{\substack{R_2-2R_1 \\ R_3-3R_1 \\ R_4-4R_1}} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -2 & -7 \\ 0 & -2 & -8 & -10 \\ 0 & -7 & -10 & -13 \end{pmatrix} \xrightarrow{\substack{R_3-2R_2 \\ R_4-7R_2 \\ -1R_2}} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 7 \\ 0 & 0 & -4 & 4 \\ 0 & 0 & 4 & 46 \end{pmatrix}$$

$$\xrightarrow{\substack{R_4+R_3 \\ -\frac{1}{4}R_3 \\ \frac{1}{50}R_4}} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 7 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

For any vector (b_1, b_2, b_3, b_4), applying the same row operations, we get

$$\begin{pmatrix} 1 & 2 & 3 & 4 & b_1 \\ 2 & 3 & 4 & 1 & b_2 \\ 3 & 4 & 1 & 2 & b_3 \\ 4 & 1 & 2 & 3 & b_4 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 3 & 4 & b_1' \\ 0 & 1 & 2 & 7 & b_2' \\ 0 & 0 & 1 & -1 & b_3' \\ 0 & 0 & 0 & 1 & b_4' \end{pmatrix},$$

where b_1', b_2', b_3', b_4' are linear expressions of b_1, b_2, b_3, b_4. By solving the last equation and substituting back we can solve for all x_1, x_2, x_3, x_4, thus the four vectors span the Euclidean space.

Ex. 1.22(6)

We form a matrix with the vectors as the columns and carry out row operations

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 5 & 6 \\ 5 & 6 & 7 & a \\ 7 & 8 & 9 & b \end{pmatrix} \xrightarrow{\substack{R_2-3R_1 \\ R_3-5R_1 \\ R_4-7R_1}} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & -2 & -4 & -6 \\ 0 & -4 & -8 & a-20 \\ 0 & -6 & -12 & b-28 \end{pmatrix} \xrightarrow{\substack{R_2\leftrightarrow R_4 \\ R_3-\frac{2}{3}R_2 \\ R_4-\frac{1}{3}R_2}} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & -6 & -12 & b-28 \\ 0 & 0 & 0 & \frac{3a-2b-4}{3} \\ 0 & 0 & 0 & \frac{10-b}{3} \end{pmatrix}$$

$$\xrightarrow{\substack{-\frac{1}{6}R_2 \\ \frac{3}{3a-2b-4}R_3 \\ R_4-\frac{10-b}{3}R_3}} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & \frac{28-b}{6} \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

For any vector (c_1, c_2, c_3, c_4), applying the same row operations, we get

$$\begin{pmatrix} 1 & 2 & 3 & 4 & c_1 \\ 3 & 4 & 5 & 6 & c_2 \\ 5 & 6 & 7 & a & c_3 \\ 7 & 8 & 9 & b & c_4 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 3 & 4 & c_1' \\ 0 & 1 & 2 & \frac{28-b}{6} & c_2' \\ 0 & 0 & 0 & 1 & c_3' \\ 0 & 0 & 0 & 0 & c_4' \end{pmatrix},$$

where c_1', c_2', c_3', c_4' are linear expressions of c_1, c_2, c_3, c_4. Since we can make suitable choices of c_1', c_2', c_3', c_4' such that $c_4' \neq 0$, we find that the system of linear equations may not have solutions for all the right side. Therefore we conclude that the four vectors do not span the Euclidean space.

Ex. 1.25

First we consider the $n \times n$ case. It can be easily found out that, after the pivot columns are chosen to be fixed, we are able to determine one unique row echelon form (the pivot of the first pivot column has to lie on the first row, and the pivot of the second pivot column has to lie on the second row, and so on). Therefore, the number of row echelon forms is equal to the number of ways to choose pivot columns, which is:

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n.$$

When $m \geq n$, we can choose at most n pivot columns, so the case is the same as mentioned above, the number of row echelon forms is 2^n; when $m < n$, we should notice that we can only choose at most m pivot columns, thus the number of row echelon forms is

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{m}.$$

Ex. 1.26(6)

From Exercise 1.26(6) we get

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 5 & 6 \\ 5 & 6 & 7 & a \\ 7 & 8 & 9 & b \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & \frac{28-b}{6} \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

From Theorem 1.2.2 these vectors do not span the Euclidean space, since there exists zero row.

Ex. 1.28(2)

From Proposition 1.2.3 we know that 4 vectors cannot span $\mathbb{R}^5$. To interpret this, we have

$$\begin{pmatrix} 10 & 0 & 8 & 7 \\ -2 & 8 & -9 & -9 \\ 3 & -2 & 3 & 3 \\ 7 & 5 & 6 & -5 \\ 2 & -4 & 5 & 6 \end{pmatrix} \to \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

where $*$ can be any number. Row operations afterwards will not change the fact that zero row appears, thus these vectors cannot span $\mathbb{R}^5$.

Ex. 1.28(4)

When we form the matrix, the result becomes clear:

$$\begin{pmatrix} 6 & -4 & 6 & 8 & -2 \\ -2 & 8 & -9 & -5 & 4 \\ 3 & -2 & 3 & 4 & -1 \\ 7 & 5 & 6 & 2 & 3 \\ 2 & -4 & 5 & -7 & -6 \end{pmatrix} \xrightarrow[R_1 \leftrightarrow R_5]{R_1 - 2R_3} \begin{pmatrix} 2 & -4 & 5 & -7 & -6 \\ -2 & 8 & -9 & -5 & 4 \\ 3 & -2 & 3 & 4 & -1 \\ 7 & 5 & 6 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Row operations afterwards will not change the fact that zero row appears, thus these vectors cannot span $\mathbb{R}^5$.

Ex. 1.31(3)

The problem means that any 2×2 matrix $\begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix}$ can be expressed as a linear combination:

$$\begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix} = x_1 \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} + x_2 \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix}$$

We translate the problem into system of linear equations:

$$\begin{aligned} b_1 &= x_1 + 5x_2 \\ b_2 &= 2x_1 + 6x_2 \\ b_3 &= 3x_1 + 7x_2 \\ b_4 &= 4x_1 + 8x_2. \end{aligned}$$

Since $(1,2,3,4),(5,6,7,8)$ do not span $\mathbb{R}^4$ (simply by Proposition 1.2.2), we know that the matrices do not span the vector space.

Ex. 1.31(5)

We use Theorem 1.4.8 to quickly solve the problem. From the theorem, we know that if these n vectors span $\mathbb{R}^n$, their must be linearly independent. However, we have

$$1(\vec{e}_1 - \vec{e}_2) + 1(\vec{e}_2 - \vec{e}_3) + \cdots + 1(\vec{e}_{n-1} - \vec{e}_n) + 1(\vec{e}_n - \vec{e}_1) = \vec{0},$$

which implies that these n vectors are not linearly dependent. Therefore, they cannot span $\mathbb{R}^n$.

Ex. 1.32(1)

Since the span problem in P_3 can be translated into span problem in $\mathbb{R}^4$ and there are only 3 vectors, by Proposition 1.2.2 they do not span the vector space.

Ex. 1.32(3)

By similar translation in Exercise 1.31(3), we consider the matrix

$$\begin{pmatrix} \pi & \sqrt{2} & 3 & \sin 2 \\ \sqrt{3} & \pi & 100 & \pi \\ 1 & -10 & -77 & \sqrt{2}\pi \\ 2\pi & 2\sqrt{2} & 6 & 2\sin 2 \end{pmatrix} \xrightarrow{R_4 - 2R_1} \begin{pmatrix} \pi & \sqrt{2} & 3 & \sin 2 \\ \sqrt{3} & \pi & 100 & \pi \\ 1 & -10 & -77 & \sqrt{2}\pi \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Further row operations will not change the fact that zero row appears, thus the vectors do not span the vector space.

The question we need to prove is equivalent to the following: $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n, \vec{w}$ is linearly dependent if and only if $\vec{w}$ is a linear combination of $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$.

Ex. 1.38

We first consider the sufficiency. Assume $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n, \vec{w}$ is linearly independent. If $\vec{w}$ is a linear combination of $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$, that is $\vec{w} = a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_n\vec{v}_n$ for some numbers $a_1, \ldots, a_n$. Thus we have

$$a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_n\vec{v}_n - \vec{w} = 0\vec{v}_1 + 0\vec{v}_2 + \cdots + 0\vec{v}_n + 0\vec{w} = \vec{0}.$$

Since $-1 \neq 0$, there is a contradiction to the assumption. Thus we prove the sufficiency.
We then consider the necessity. Assume $\vec{w}$ is not a linear combination of $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$. If $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n, \vec{w}$ is linearly dependent, which means in the equation

$$a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_n\vec{v}_n + a_{n+1}\vec{w} = b_1\vec{v}_1 + b_2\vec{v}_2 + \cdots + b_n\vec{v}_n + b_{n+1}\vec{w},$$

where a_i and b_i are real numbers, there exist one $a_j \neq b_j$. If $a_{n+1} = b_{n+1}$, we cancel the $\vec{w}$ terms out, and find a contradiction to the condition that $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ is linearly independent. Therefore, $a_{n+1} \neq b_{n+1}$, and we can rewrite the equation into

$$\vec{w} = \frac{a_1 - b_1}{b_{n+1} - a_{n+1}}\vec{v}_1 + \frac{a_2 - b_2}{b_{n+1} - a_{n+1}}\vec{v}_2 + \cdots + \frac{a_n - b_n}{b_{n+1} - a_{n+1}}\vec{v}_n,$$

which means $\vec{w}$ is a linear combination of $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ and there is a contradiction to the assumption. Thus we prove the necessity.

Ex. 1.42(3)

We take $t = 0, 1, -1, 2$, form a matrix and do row operations

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & e & e \\ 1 & -1 & e^{-1} & -e^{-1} \\ 1 & 2 & e^2 & 2e^2 \end{pmatrix} \xrightarrow[R_4 - R_1]{\substack{R_2 - R_1 \\ R_3 - R_1}} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & e-1 & e \\ 0 & -1 & e^{-1}-1 & -e^{-1} \\ 0 & 2 & e^2-1 & 2e^2 \end{pmatrix}$$

$$\xrightarrow[R_3 \leftrightarrow R_4]{\substack{R_3 + R_2 \\ R_4 - 2R_2}} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & e-1 & e \\ 0 & 0 & (e-1)^2 & 2e(e-1) \\ 0 & 0 & e^{-1}+e-2 & e-e^{-1} \end{pmatrix} \xrightarrow[R_4 - (e^{-1}+e-2)R_3]{(e-1)^{-2}R_3} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & e-1 & e \\ 0 & 0 & 1 & \frac{2e}{e-1} \\ 0 & 0 & 0 & \bullet \end{pmatrix},$$

where $\bullet$ is a "complicated" non-zero number. Therefore, all columns are pivot. By Theorem 1.3.5, we conclude that the functions are linearly independent. Moreover,

$$\begin{aligned} f(t) &= 1 \cdot e^t + 1 \cdot te^t \\ g(t) &= (2+t)e^t = 2 \cdot e^t + 1 \cdot te^t, \end{aligned}$$

thus $f(t), g(t)$ can be expressed as linear combinations of given functions.

Ex. 1.44(2)

We have

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \\ 3 & 4 & 1 & 2 \\ 4 & 1 & 2 & 3 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 7 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

which shows that all columns are pivot. By Theorem 1.3.5, we conclude that the vectors are linearly independent.

Ex. 1.48(1)

By Proposition 1.3.6, 6 vectors in $\mathbb{R}^3$ must be linearly dependent. We interpret it into

$$\begin{pmatrix} 1 & 2 & 3 & 1 & 3 & 2 \\ 2 & 3 & 1 & 3 & 2 & 1 \\ 3 & 1 & 2 & 2 & 1 & 3 \end{pmatrix} \to \begin{pmatrix} * & * & * & 0 & 0 & 0 \\ * & * & * & 0 & 0 & 0 \\ * & * & * & 0 & 0 & 0 \end{pmatrix},$$

where $*$ means any number. The existence of (many) zero columns implies that the vectors are linearly dependent.

Ex. 1.48(3)

The vectors are actually linearly independent. I think there's a typo. If the last vector is $(\pi, 3\pi, 2\pi, -4\pi)$, we easily find that

$$-\pi(1,3,2,4) + 0(10,-2,3,7) + 0(0,8,-2,5) + 1(\pi, 3\pi, 2\pi, -4\pi) = (0,0,0,0),$$

which implies that these vectors are linearly dependent. We interpret it into

$$\begin{pmatrix} 1 & 10 & 0 & \pi \\ 3 & -2 & 8 & 3\pi \\ 2 & 3 & -2 & 2\pi \\ -4 & 7 & 5 & -4\pi \end{pmatrix} \xrightarrow{C_4-\pi C_1} \begin{pmatrix} 1 & 10 & 0 & 0 \\ 3 & -2 & 8 & 0 \\ 2 & 3 & -2 & 0 \\ -4 & 7 & 5 & 0 \end{pmatrix},$$

The existence of the zero column implies that the vectors are linearly dependent.

Ex. 1.50(1)

Similar to Example 1.3.7, we have

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 0 \\ 2x_1 + 3x_2 + 4x_3 &= 0 \\ 3x_1 + 4x_2 + x_3 &= 0 \end{aligned}$$

to be solved, which is equivalent to

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 1 \end{pmatrix} \xrightarrow[R_3-3R_1]{R_2-2R_1} \begin{pmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \\ 0 & -2 & -8 \end{pmatrix} \xrightarrow{R_3-2R_2} \begin{pmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \\ 0 & 0 & -4 \end{pmatrix}.$$

We note that all columns are pivot. Therefore, the homogeneous system of linear equations has only trivial solution $x_1 = x_2 = x_3 = 0$, which implies that these vectors are linearly independent.

Ex. 1.55(11)

We get $x_1 = b_1 - a_1x_3 - a_2x_5, x_2 = b_2 - a_3x_3 - a_4x_5, x_4 = b_3 - a_5x_5$; x_3, x_5 arbitrary.

Ex. 1.56(6)

We get

$$\begin{pmatrix} 1 & -2 & 0 & -3 & 0 & 1 \\ 0 & 0 & 1 & -5 & -6 & 4 \end{pmatrix}$$

where the last column is the right side. The system is not homogeneous since the right are not all zeros.

Ex. 1.58(3)

We form a matrix and do row operations

$$\begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & a \end{pmatrix} \xrightarrow[\substack{R_4+R_2 \\ R_2 \leftrightarrow R_3}]{\substack{R_2-R_1 \\ R_3-R_1 \\ R_4+R_3}} \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & a+2 \end{pmatrix}.$$

If and only if $a \neq -2$ the collection is a basis.

Ex. 1.61(3)

The question has the same meaning as Exercise 1.50, since span and independent and basis is equivalent in the case. If α is a basis , then for β we have the matrix

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

which is already in row echelon form. All rows and columns are pivot, which implies that β is a basis.
If β is a basis, we let $\beta = \{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$, then $\alpha = \{\vec{u}_1, \vec{u}_2 - \vec{u}_1, \vec{u}_3 - \vec{u}_2\}$ and we have the matrix

$$\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$$

which is already in row echelon form. All rows and columns are pivot, which implies that α is a basis.

Ex. 1.68(1)

After trivial translation, the question is equivalent to the matrix

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & a \end{pmatrix} \xrightarrow[R_3+R_2]{R_2-R_1} \begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & a+1 \end{pmatrix}.$$

If and only if $a \neq -1$ the collection is a basis.

Ex. 1.69(4)

Let $\alpha = \{(\cos\theta, \sin\theta), (-\sin\theta, \cos\theta)\}$. From the technique used in Example 1.4.5, we do row operations to the following matrix

$$\begin{pmatrix} \cos\theta & -\sin\theta & 1 & 0 \\ \sin\theta & \cos\theta & 0 & 1 \end{pmatrix} \xrightarrow[R_2-\sin\theta R_1]{\sec\theta R_1} \begin{pmatrix} 1 & -\tan\theta & \sec\theta & 0 \\ 0 & \sec\theta & -\tan\theta & 1 \end{pmatrix} \xrightarrow[R_1+\tan\theta R_2]{\cos\theta R_2} \begin{pmatrix} 1 & 0 & \cos\theta & \sin\theta \\ 0 & 1 & -\sin\theta & \cos\theta \end{pmatrix}$$

We get

$$[\vec{e}_1]_\alpha = \begin{pmatrix} \cos\theta \\ -\sin\theta \end{pmatrix}, \qquad [\vec{e}_2]_\alpha = \begin{pmatrix} \sin\theta \\ \cos\theta \end{pmatrix}.$$

By Proposition 1.4.2, the $\alpha-$coordinate of a general vector in $\mathbb{R}^2$ is

$$[(x,y)]_\alpha = x[\vec{e}_1]_\alpha + y[\vec{e}_2]_\alpha = x\begin{pmatrix} \cos\theta \\ -\sin\theta \end{pmatrix} + y\begin{pmatrix} \sin\theta \\ \cos\theta \end{pmatrix} = \begin{pmatrix} x\cos\theta + y\sin\theta \\ -x\sin\theta + y\cos\theta \end{pmatrix}.$$

Ex. 1.69(7)

Let $\alpha = \{(0,1,2), (0,0,1), (1,2,3)\}$. From the technique used in Example 1.4.5, we do row operations to the following matrix

$$\begin{pmatrix} 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 2 & 0 & 1 & 0 \\ 2 & 1 & 3 & 0 & 0 & 1 \end{pmatrix} \xrightarrow[\substack{R_1\leftrightarrow R_2 \\ R_2\leftrightarrow R_3}]{R_3-2R_2} \begin{pmatrix} 1 & 0 & 2 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 & -2 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{pmatrix} \xrightarrow[R_2+R_3]{R_1-2R_3} \begin{pmatrix} 1 & 0 & 0 & -2 & 1 & 0 \\ 0 & 1 & 0 & 2 & -2 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{pmatrix}.$$

We get

$$[\vec{e}_1]_\alpha = \begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix}, \qquad [\vec{e}_2]_\alpha = \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}, \qquad [\vec{e}_2]_\alpha = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

By Proposition 1.4.2, the $\alpha-$coordinate of a general vector in $\mathbb{R}^2$ is

$$[(x,y,z)]_\alpha = x[\vec{e}_1]_\alpha + y[\vec{e}_2]_\alpha + z[\vec{e}_3]_\alpha = x\begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix} + y\begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} + z\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -2 & 1 & 0 \\ 2 & -2 & 1 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Ex. 1.71(1)

Apparently, we have

$$t^n - 1 = -\big[1(1-t) + 1(t-t^2) + \cdots + 1(t^{n_1} - t^n)\big],$$

which means $t^n - 1$ is a linear combination of the other functions and therefore the collection is not a basis of P_n.

Ex. 1.71(3)

After trivial translation, the question is equivalent to the $(n+1) \times (n+1)$ matrix

$$A = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}$$

which is already in row echelon form. All rows and columns are pivot, which implies that the collection is a basis of P_n, noted as α. Moreover, we have

$$\begin{aligned} \begin{pmatrix} A \; I \end{pmatrix} &= \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 & 0 & 0 & \cdots & 1 \end{pmatrix} \\ &\xrightarrow{\bigcup_{i=1}^{n}(R_1 - R_i)} \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 1 & -1 & -1 & \cdots & -1 \\ 0 & 1 & 0 & \cdots & 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 & 0 & 0 & \cdots & 1 \end{pmatrix}. \end{aligned}$$

Therefore for any polynomial $x_0 + x_1 t + x_2 t^2 + \cdots + x_n t^n$, we have

$$\big[x_0 + x_1 t + x_2 t^2 + \cdots + x_n t^n\big]_\alpha = \begin{pmatrix} 1 & -1 & -1 & \cdots & -1 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}.$$

Ex. 1.76(1)

After trivial translation, we get the matrix

$$\begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix} \xrightarrow[R_4+R_3]{\substack{R_2-R_1 \\ R_3+R_2}} \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & -1 & -1 & 1 & 1 & 0 \\ 0 & 0 & -1 & 2 & 1 & 1 \\ 0 & 0 & 0 & 2 & 2 & 2 \end{pmatrix}$$

in row echelon form, where all rows are pivot. Delete the last 2 vectors, by doing the same operations, we still have the matrix with all rows pivot. Thus we have a spanning set $\alpha = \{1+t, 1+t^2, 1+t^3, t+t^2\}$. By Proposition 1.2.3, we cannot delete more than 2 vectors, and therefore α is a minimal spanning set.