Math2131 Answer to Homework 10

Exercise 7.2

Similar to **Example 7.1.2**, we have $(1,0) = \frac{1}{5}\vec{v}_1 - \frac{2}{5}\vec{v}_2$, then $A^n \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{5}A^n\vec{v}_1 - \frac{2}{5}A^n\vec{v}_2 = \frac{1}{5}5^n\vec{v}_1 - \frac{2}{5}15^n\vec{v}_2 = 5^{n-1}\begin{pmatrix} 1+4\cdot 3^n \\ 2-2\cdot 3^n \end{pmatrix}$. Hence $A^n = A^n \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 5^{n-1}\begin{pmatrix} 1+4\cdot 3^n & 2-2\cdot 3^n \\ 2-2\cdot 3^n & 4+3^n \end{pmatrix}$.

Exercise 7.4

The 1-dimensional invariant subspace $\mathbb{R}\vec{v}$ of a rotation brings $\vec{v}$ to another vector $\vec{v}' = c\vec{v}$ for some $c \in \mathbb{R}$. Also, $|\vec{v}| = |\vec{v}'| = |c||\vec{v}|$ with repsect to the dot product, so $c = \pm 1$. Hence only such rotations are I or $-I$.

Exercise 7.5

$L(\text{Ran}L) \subseteq \text{Ran}L$, so $\text{Ran}L$ is L-invariant. $L(\text{Ker}L) \subseteq \text{Ker}L$, so $\text{Ker}L$ is also L-invariant.

Exercise 7.9

We have $L(H) \subseteq H$. Then $\overline{L}(\overline{H}) = \overline{L(H)} \subseteq \overline{H}$, so $\overline{H}$ is an invariant subspace of $\overline{L}$.

Exercise 7.10

Let H be an invariant subspace of L. Then $K^{-1}L(H)$ is an invariant subspace of $K^{-1}LK$ because for each $\vec{v} \in K^{-1}L(H)$, $\vec{v} = K^{-1}L(\vec{u})$ for some $\vec{u} \in H$, then $K^{-1}LK(\vec{v}) = K^{-1}LKK^{-1}L(\vec{u}) = K^{-1}L(L(\vec{u})) \in K^{-1}L(L(H)) \subseteq K^{-1}L(H)$.
Conversely, for each invariant subspace H' of $K^{-1}LK$, $L^{-1}K^{-1}LK(H')$ is an invariant subspace of L because for each $\vec{v} \in L^{-1}K^{-1}LK(H')$, $\vec{v} = L^{-1}K^{-1}LK(\vec{u})$ for some $\vec{u} \in H'$, then $L(\vec{v}) = L(L^{-1}K^{-1}LK(\vec{u})) = K^{-1}LK(\vec{u}) \in K(H') \subseteq H$.

Exercise 7.11

Let this vector space be a $\mathbb{F}$-space. For each $\vec{u} \in H$, $\vec{u} = u_0\vec{v} + u_1L(\vec{v}) + u_2(L^2\vec{v}) + u_{k-1}L^{k-1}(\vec{v})$ for some $u_i \in \mathbb{F}$. Then $L(\vec{u}) = u_0L(\vec{v}) + u_1L^2(\vec{v}) + u_2L^3(\vec{v}) + u_{k-1}L^k(\vec{v}) = -a_0u_{k-1}\vec{v} + (u_0 - a_1u_{k-1})L(\vec{v}) + (u_1 - a_2u_{k-1})L^2(\vec{v}) + \cdots + (u_{k-2} - a_{k-1}u_{k-1})L^{k-1}(\vec{v})$, so $[L|_H]_{\alpha\alpha}$ has the given matrix form.

Exercise 7.14

For each $\vec{v}_i \in H_i$, we have $L(\vec{v}) = \lambda_i\vec{v}$, so λ_i are eigenvalues of L. To show that they are the only eigenvalues, suppose $L(\vec{v}) = \lambda\vec{v}$ for some $\vec{v} \neq \vec{0}$. Then we have $\vec{v} = \sum_{i=1}^{k}\vec{v}_i$ for some $\vec{v}_i \in H_i$ not all zero. Then $\sum_{i=1}^{k}\lambda\vec{v}_i = L(\vec{v}) = L\left(\sum_{i=1}^{k}\vec{v}_i\right) = \sum_{i=1}^{k}L(\vec{v}_i) = \sum_{i=1}^{k}\lambda_i\vec{v}_i$, so $\sum_{i=1}^{k}(\lambda_i - \lambda)\vec{v}_i = \vec{0}$.
Then as V is a direct sum of H_i, $(\lambda_i - \lambda) = 0$ for all i. But as $\vec{v}_i$ are not all zero, $\lambda = \lambda_i$ for some i.
Let W be an invariant subspace of V then $W \cap H_i$ are subspaces of V and $W \cap H_i \subseteq H_i$. Write $W_i = W \cap H_i$, and we claim that W is the direct sum of W_i. This sum is direct as V is a direct sum of H_i and $W_i \in H_i$. Then it suffices to show that each $\vec{w}$ is a sum of $\vec{w}_i \in W_i$.

For each $\vec{w} \in W \subseteq V$, $\vec{w} = \sum_{i=1}^{k} \vec{v}_i$ for some $\vec{v}_i \in H_i$. By applying L to $\vec{w}$ recursively, we have $L^n(\vec{w}) = \sum_{i=1}^{k} \lambda_i^n \vec{v}_i \in W$ for all nonnegative integers n. Then we have

$$\begin{pmatrix} \vec{w} \\ L(\vec{w}) \\ L^2(\vec{w}) \\ \vdots \\ L^{k-1}(\vec{w}) \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 & \cdots & \lambda_k \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 & \cdots & \lambda_k^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \lambda_1^{k-1} & \lambda_2^{k-1} & \lambda_3^{k-1} & \cdots & \lambda_k^{k-1} \end{pmatrix} \begin{pmatrix} \vec{v}_1 \\ \vec{v}_2 \\ \vec{v}_3 \\ \vdots \\ \vec{v}_{k-1} \end{pmatrix}$$

The Vandermonde matrix is invertible for distinct λ_i. Let A be its inverse. Then

$$A \begin{pmatrix} \vec{w} \\ L(\vec{w}) \\ L^2(\vec{w}) \\ \vdots \\ L^{k-1}(\vec{w}) \end{pmatrix} = \begin{pmatrix} \vec{v}_1 \\ \vec{v}_2 \\ \vec{v}_3 \\ \vdots \\ \vec{v}_{k-1} \end{pmatrix}$$

We have $\vec{v}_i = \sum_{j=1}^{k} a_{ij} L^{j-1}(\vec{w}) \in W$, so each $\vec{v}_i \in W$. Then $\vec{v}_i \in W \cap H_i = W_i$. We are done.

Exercise 7.16

Let L be a linear operator and A be its corresponding matrix. If $L(\vec{v}) = 0\vec{v} = \vec{0}$ for some non-zero vector $\vec{v}$, then $A\vec{x} = \vec{0}$ has non-trivial solution, so $\det L = \det A = 0$. If $\det L = \det A = 0$, then there exist non-trivial solution $\vec{x}$. Then $L(\vec{x}) = \vec{0} = 0\vec{x}$, so 0 is a eigenvalue of L.

Exercise 7.18

Each $\vec{v} \in V$ can be expressed as $\vec{v} = \sum_{i=1}^{k} \vec{h}_i$ for some $\vec{h}_i \in H_i$. Then $LK(\vec{v}) = L\left(\sum_{i=1}^{k} K(\vec{h}_i)\right) = \sum_{i=1}^{k} \lambda_i K(\vec{h}_i) = \sum_{i=1}^{k} K(\lambda_i \vec{h}_i) = K\left(\sum_{i=1}^{k} \lambda_i \vec{h}_i\right) = KL(\vec{v})$ for all $\vec{v}$, so $LK = KL$.

Exercise 7.19

Let λ be an eigenvalue of L, and $\vec{v}$ be the corresponding eigenvector. Then $\vec{0} = O(\vec{v}) = L^2(\vec{v}) + 3L(\vec{v}) + 2I(\vec{v}) = (\lambda^2 + 3\lambda + 2)\vec{v}$, so $\lambda = -1$ or -2. Furthermore, at least one of -1 and -2 are eigenvalues of L.

Exercise 7.20

For any $f \in P_n$, f is a polynomial of degree at most n. Then $\frac{d^{n+1}}{dx^{n+1}} f = 0$, so the derivative operator is nilpotent.

Let λ be an eigenvalue of a nilpotent linear operator L with eigenvector $\vec{v}$. Then $L^n = O$ for some n so $\vec{0} = L^n(\vec{v}) = \lambda^n \vec{v}$ and thus $\lambda = 0$. Conversely, if $L^n = O$, then $\text{Rank} L < \dim V$. This shows that L is not-invertible, and $L(\vec{v}) = \vec{0}$ has a non-trivial solution. This shows that 0 is the only eigenvalue of nilpotent operators.

Exercise 7.23

Let A be an upper or lower triangular matrix, and a_{ii} be its diagonal entries. Then tI_A is also an upper or lower triangular matrix with diagonal entries $t-a_{ii}$. Then $\det(tI-A) = \prod_{i=1}^{n}(t-a_{ii})$. This polynomial has only a_{ii} as its roots. Hence the eigenvalues of A can only be its diagonal entries.

Exercise 7.25

For $f(x) = a_0+a_1+\cdots+a_nx^n$, $f'(x) = a_1+2a_2+\cdots+na_nx^{n-1}$, so the matrix for the derivative operator D is $\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & n \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$. Then $\det(tI-D) = \det \begin{pmatrix} t & -1 & 0 & \cdots & 0 \\ 0 & t & -2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -n \\ 0 & 0 & 0 & \cdots & t \end{pmatrix} = t^{n+1}$.

Exercise 7.27

$\det(tI-A) = \det\begin{pmatrix} t-a & -b \\ -c & t-d \end{pmatrix} = (t-a)(t-d)-bc = t^2-(a+d)t+(ad-bc) = t^2-(\mathrm{tr}A)t+\det A$.

Exercise 7.31

We proceed by induction. The statement is true for $n=2$. Suppose the statement is true for a $n-1\times n-1$ matrix. Then $\det(tI-L|_H) = \det\begin{pmatrix} t & 0 & 0 & \cdots & 0 & a_0 \\ -1 & t & 0 & \cdots & 0 & a_1 \\ 0 & -1 & t & \cdots & 0 & a_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & t & a_{n-2} \\ 0 & 0 & 0 & \cdots & -1 & t+a_{n-1} \end{pmatrix} =$ $t\det\begin{pmatrix} t & 0 & \cdots & 0 & a_1 \\ -1 & t & \cdots & 0 & a_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & t & a_{n-2} \\ 0 & 0 & \cdots & -1 & t+a_{n-1} \end{pmatrix} + a_0(-1)^{n-1}\det\begin{pmatrix} -1 & t & 0 & \cdots & 0 \\ 0 & -1 & t & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & t \\ 0 & 0 & 0 & \cdots & -1 \end{pmatrix} = t(t^{n-1}+a_{n-1}t^{n-2}+\cdots+a_1)+a_0 = t^n+a_{n-1}t^{n-1}+\cdots+a_1t+a_0$.

Exercise 7.32

(1) $\det(tI-A) = \det\begin{pmatrix} t & -1 \\ -1 & t \end{pmatrix} = t^2+1$ which has no solution in reals. Hence this matrix has no (real) eigenspace and is not (real) diagonisable.

(3) A is an upper triagonal matrix so the eigenvalues of A is 1, 4, 6. $A-I = \begin{pmatrix} 0 & 2 & 3 \\ 0 & 3 & 5 \\ 0 & 0 & 5 \end{pmatrix}$,

the eigenspace is $\mathbb{R}(1,0,0)$. $A-4I = \begin{pmatrix} -3 & 2 & 3 \\ 0 & 0 & 5 \\ 0 & 0 & 2 \end{pmatrix}$, the eigenspace is $\mathbb{R}(2,3,0)$. $A-6I = \begin{pmatrix} -5 & 2 & 3 \\ 0 & -2 & 5 \\ 0 & 0 & 0 \end{pmatrix}$, the eigenspace is $\mathbb{R}(4,25,-10)$. Writing $P = \begin{pmatrix} 1 & 2 & 4 \\ 0 & 3 & 25 \\ 0 & 0 & -10 \end{pmatrix}$, all row pivot so we have a basis of eigenvectors. Then $A = P \begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 6 \end{pmatrix} P^{-1}$.

(4) Applying transpose to result of **7.32 (3)**, we have $A^T = (PDP^{-1})^T = (P^{-1})^T DP^T$. Then for $P^T = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 4 & 25 & -10 \end{pmatrix}$, we obtain the basis of eigenvectors $\{(1,2,4),(0,3,25),(0,0,-10)\}$, and the eigenspaces $\mathbb{R}\vec{v}$ for the eigenvectors. Furthermore, $(P^T)^{-1}$ diagonalizes A^T with the same D as **7.32 (3)**.

Exercise 7.33

(2) Let $P = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & -1 & 2 \end{pmatrix}$. Then P is invertible with $P^{-1} = \begin{pmatrix} -\frac{1}{2} & \frac{3}{2} & -\frac{1}{2} \\ 1 & -1 & 0 \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{pmatrix}$. Then the matrix is $A = PDP^{-1} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} -\frac{1}{2} & \frac{3}{2} & -\frac{1}{2} \\ 1 & -1 & 0 \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} -1 & 3 & -1 \\ 2 & -2 & 0 \\ -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 3 & -3 \\ -3 & 7 & -3 \\ -6 & 6 & -2 \end{pmatrix}$.

(3) For the same P as **7.33 (2)**, we see that A acts on a basis of vectors the same as identity, so this matrix must be the identitiy matrix.

Exercise 7.35

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then $A^2 = \begin{pmatrix} a^2+bc & (a+d)b \\ (a+d)c & d^2+bc \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. If $a+d=0$, then we have $bc = 1-a^2$, so we write $A = \begin{pmatrix} a & b \\ \frac{1-a^2}{b} & -a \end{pmatrix}$ for arbitrary a and arbitrary $b \neq 0$. If $a+d \neq 0$, then $b=c=0$, so a, $d = \pm 1$ and $A = \begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix}$.

Exercise 7.36

The operator has a basis of eigenvectors, which all eigenvalues are 0. Then this operator is the zero operator. Then as the derivative operator is nilpotent and not constantly zero, it is not diagonalizable.

Exercise 7.39

If the diagonalisable matrix has the same characteristic polynomial, they must have the same eigenvalues. Then we can write $A = PDP^{-1}$ and $B = QDQ^{-1}$ for the same D. Hence

$QP^{-1}A(QP^{-1})^{-1} = QP^{-1}PDP^{-1}PQ^{-1} = QDQ^{-1} = B$, so A, B are similar.

EXERCISE 7.31

(3)Let $\vec{v}_i = (x_i, y_i)$. Then $\vec{v}_{i+1} = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix} \vec{v}_i$. Let $A = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}$. The characteristic polynomial of A is $t^2 - 4t - 1$, with eigenvalues $2 \pm \sqrt{5}$. The eigenvectors of A are $\vec{v}_1 = \left(\frac{-1+\sqrt{5}}{2}, 1 \right)$ and $\vec{v}_2 = \left(\frac{-1-\sqrt{5}}{2}, 1 \right)$. Then $(1,0) = \frac{1}{\sqrt{5}}(\vec{v}_1 - \vec{v}_2)$ and $(0,1) = \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2}\vec{v}_1 - \frac{1-\sqrt{5}}{2}\vec{v}_2 \right)$. Hence $(a,b) = \frac{1}{\sqrt{5}} \left(\frac{2a+b+\sqrt{5}}{2}\vec{v}_1 - \frac{2a+b-\sqrt{5}}{2}\vec{v}_2 \right)$, so

$$(x_n, y_n) = \frac{1}{\sqrt{5}} \left(\frac{2a+b+\sqrt{5}}{2}(2+\sqrt{5})^n \vec{v}_1 - \frac{2a+b-\sqrt{5}}{2}(2-\sqrt{5})^n \vec{v}_2 \right)$$

$$x_n = \frac{1}{\sqrt{5}} \left(\frac{2a+b+\sqrt{5}}{2}(2+\sqrt{5})^n \frac{-1+\sqrt{5}}{2} + \frac{2a+b-\sqrt{5}}{2}(2-\sqrt{5})^n \frac{1+\sqrt{5}}{2} \right)$$

$$y_n = \frac{1}{\sqrt{5}} \left(\frac{2a+b+\sqrt{5}}{2}(2+\sqrt{5})^n - \frac{2a+b-\sqrt{5}}{2}(2-\sqrt{5})^n \right)$$