Math2131 Answer to Homework 11

Exercise 7.44

(2) We can find an orthogonal basis of $H^\perp = \mathbb{R}(1,-1,-1,1)\oplus\mathbb{R}(1,-1,1,-1)$. Then the matrix of the linear operation with respect to this ordered orthogonal basis is $\begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$.

Hence with respect to the standard basis, the matrix is

$$
\begin{aligned}
A &= \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{pmatrix}^{-1} \\
&= \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \frac{1}{4} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{pmatrix} \\
&= \begin{pmatrix} \sqrt{2} & 0 & -1 & -1 \\ \sqrt{2} & 0 & 1 & 1 \\ 0 & -\sqrt{2} & 1 & -1 \\ 0 & -\sqrt{2} & -1 & 1 \end{pmatrix} \frac{1}{4} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{pmatrix} \\
&= \frac{1}{4} \begin{pmatrix} \sqrt{2}-2 & \sqrt{2}+2 & \sqrt{2} & \sqrt{2} \\ \sqrt{2}+2 & \sqrt{2}-2 & \sqrt{2} & \sqrt{2} \\ -\sqrt{2} & -\sqrt{2} & \sqrt{2}-2 & \sqrt{2}+2 \\ -\sqrt{2} & -\sqrt{2} & \sqrt{2}+2 & \sqrt{2}-2 \end{pmatrix}
\end{aligned}
$$

(5) The matrix with respect to the same orthogonal basis as above is $\begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

Hence with respect to the standard basis, the matrix is

$$
\begin{aligned}
A &= \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \frac{1}{4} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{pmatrix} \\
&= \begin{pmatrix} \sqrt{2} & 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 & 0 \\ 0 & -\sqrt{2} & 0 & 0 \\ 0 & -\sqrt{2} & 0 & 0 \end{pmatrix} \frac{1}{4} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{pmatrix} \\
&= \frac{1}{4} \begin{pmatrix} \sqrt{2} & \sqrt{2} & \sqrt{2} & \sqrt{2} \\ \sqrt{2} & \sqrt{2} & \sqrt{2} & \sqrt{2} \\ -\sqrt{2} & -\sqrt{2} & \sqrt{2} & \sqrt{2} \\ -\sqrt{2} & -\sqrt{2} & \sqrt{2} & \sqrt{2} \end{pmatrix}
\end{aligned}
$$

Exercise 7.45

(1) → (2): $L^*(L^*)^* = L * L = I$, so L^* is normal.

(2) → (3): $\langle L(\vec{v}), L(\vec{v})\rangle = \langle \vec{v}, L^*(L(\vec{v}))\rangle = \langle \vec{v}, L(L^*(\vec{v}))\rangle = \langle L^*(\vec{v}), L^*(\vec{v})\rangle$, so $||L(\vec{v})|| = ||L^*(\vec{v})||$.

(3) → (1): $\langle L(\vec{v}), L(\vec{v})\rangle = \langle \vec{v}, L^*(L(\vec{v}))\rangle$ and $\langle L^*(\vec{v}), L^*(\vec{v})\rangle = \langle \vec{v}, L(L^*(\vec{v}))\rangle$, so $\langle \vec{v}, L(L^*(\vec{v})) - L^*((\vec{v}))\rangle$ for all $\vec{v}$, hence $L(L^*(\vec{v})) - L^*((\vec{v})) = \vec{0}$, $LL^* = L^*L$.

(1) → (4): $L^* = L_1^* + L_2^* = L_1 - L_2$. Then $LL^* = (L_1+L_2)(L_1-L_2) = L_1^2 + L_2L_1 - L_1L_2 - L_2^2$ and $L^*L = (L_1-L_2)(L_1+L_2) = L_1^2 - L_2L_1 + L_1L_2 - L_2^2$. As $LL^* = L^*L$, $L_2L_1 - L_1L_2 = -L_2L_1 + L_1L_2$, so $L_1L_2 = L_2L_1$.

(4) → (1): $LL^* = (L_1 + L_2)(L_1 - L_2) = L_1^2 + L_2L_1 - L_1L_2 - L_2^2 = L_1^2 - L_2L_1 + L_1L_2 - L_2^2 = (L_1 - L_2)(L_1 + L_2) = L^*L$.

Exercise 7.48

As L is normal, it has there exist a orthogonal basis of eigenvectors. Then there exist a decompostion of L and V of the same form as in **Exercise 7.14**. As H is an invariant subspace, by **Exercise 7.14** there exist a decomposotion of H into a direct sum of $W_i \subseteq H_i$. For each i we construct $W_i^\perp$ such that $H_i = W_i \oplus W_i^\perp$. Then $H^\perp$ is the direct sum of $W_i^\perp$. As the $W_i^\perp$ are each contained in an eigenspace, $L(W_i^\perp) \subseteq W_i^\perp$, hence $H^\perp$ is L-invariant. Then by **Lemma 7.2.5**, $H^\perp$ is L^*-invariant and $H = (H^\perp)^\perp$ is L^*-invariant.

Exercise 7.53

(2) The characteristic polynomial is $(t-a)^2 - b^2 = (t-a-b)(t-a+b)$. Then the eigenvalues are $a+b$ and $a-b$. Then $A-(a+b)I = \begin{pmatrix} -b & b \\ b & -b \end{pmatrix}$ with kernel $\mathbb{R}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $A-(a+b)I = \begin{pmatrix} b & b \\ b & b \end{pmatrix}$ with kernel $\mathbb{R}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$. Thus $A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} a+b & 0 \\ 0 & a-b \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}^{-1}$.

(3) The characteristic polynomial is $\det\begin{pmatrix} t+1 & -2 & 0 \\ -2 & t & -2 \\ 0 & -2 & t-1 \end{pmatrix} = (t+1)(t^2-t-4) - 4(t-1) = t^3 - 9t$ with eigenvalues ± 3 and 0. Then A has kernel $\mathbb{R}\begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix}$, $A - 3I = \begin{pmatrix} -4 & 2 & 0 \\ 2 & -3 & 2 \\ 0 & 2 & -2 \end{pmatrix}$ has kernel $\mathbb{R}\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$ and $A + 3I = \begin{pmatrix} 2 & 2 & 0 \\ 2 & 3 & 2 \\ 0 & 2 & 4 \end{pmatrix}$ has kernel $\mathbb{R}\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$.

Thus $A = \begin{pmatrix} 2 & 1 & 2 \\ -1 & 2 & -2 \\ 2 & 2 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -3 \end{pmatrix}\begin{pmatrix} 2 & 1 & 2 \\ -1 & 2 & -2 \\ 2 & 2 & 1 \end{pmatrix}^{-1}$.

Exercise 7.54

Let this matrix be A. As A is real symmetric, it has an orthogonal basis of eigenvectors. Then its eigenvector with eigenvalue 3 must be orthogonal to both $(1, 1, 0)$ and $(1, -1, 0)$. We can take it to be $(0,0,1)$. Then

$$\begin{aligned} A &= \begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1} \\ &= \begin{pmatrix} 1 & 2 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} \frac{3}{2} & -\frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{3}{2} & 0 \\ 0 & 0 & 3 \end{pmatrix} \end{aligned}$$

Exercise 7.58

Let $A = a_{ij}$, $A_1 = b_{ij}$ and $A_2 = c_{ij}$ be the matrices of L, L_1 and L_2 respectively. For the existence, we assign $b_{ij} = \dfrac{\Re(a_{ij}) + \Re(a_{ji})}{2} + \dfrac{\Im(a_{ij}) - \Im(a_{ji})}{2}$, $c_{ij} = \dfrac{\Re(a_{ij}) - \Re(a_{ji})}{2} + \dfrac{\Im(a_{ij}) + \Im(a_{ji})}{2}$ for all i, j. Then A_1 and A_2 satisfies the properties: $A = A_1 + A_2$, A_1 is hermitian and A_2 is skew-Hermitian. For the uniqueness, suppose $L = L_1 + L_2 = L_1' + L_2'$. Then $O = (L_1 - L_1') + (L_2 - L_2')$, and $L_1 - L_1'$ is Hermitian while $L_2 - L_2'$ is skew-Hermitian. Applying adjoint on both sides, $O = (L_1 - L_1') - (L_2 - L_2')$, so $L_1 - L_1' = O$, $L_1 = L_1'$.

Exercise 7.61

Using the same decompostion of L as in the lecture notes, $L^3 = O \perp \lambda_1^3 \begin{pmatrix} O & I \\ -I & O \end{pmatrix} \perp \lambda_2^3 \begin{pmatrix} O & I \\ -I & O \end{pmatrix} \perp \cdots \perp \lambda_q^3 \begin{pmatrix} O & I \\ -I & O \end{pmatrix} = O$. Hence the λ_i must be all zero, and thus A itself is O. Similarly, $L^2 = O \perp \lambda_1^2 \begin{pmatrix} -I & O \\ O & -I \end{pmatrix} \perp \lambda_2^2 \begin{pmatrix} -I & O \\ O & -I \end{pmatrix} \perp \cdots \perp \lambda_q^2 \begin{pmatrix} -I & O \\ O & -I \end{pmatrix} = -I$, so the eigenspace H with eigenvalue zero must be the zero subspace (somewhat contradicting the definition, but in any case this H should not be in the direct sum), and the λ_i are ± 1. Hence L composed of rotation by 90° either clockwise or anticlockwise on pairwise orthogonal subspaces of dimension 2, and we can write $A = \begin{pmatrix} A_1 & O & O & \cdots & O \\ O & A_2 & O & \cdots & O \\ O & O & A_2 & \cdots & O \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ O & O & O & \cdots & A_q \end{pmatrix}$, where each of $A_i = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ or $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

Exercise 7.62

All orthogonal operators on $\mathbb{R}^2$ are the identity, flipping and rotation.

Exercise 7.64

Using the ordered orthogonal basis in **7.44**, this operator flips H and as it fixes $(1,-1,1,-1)$, it must fix $H^{\perp}$. Hence with respect to the standard basis, the matrix is

$$\begin{aligned} A &= \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \frac{1}{4} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{pmatrix} \\ &= \begin{pmatrix} -1 & -1 & 1 & 1 \\ -1 & -1 & -1 & -1 \\ -1 & 1 & -1 & 1 \\ -1 & 1 & 1 & -1 \end{pmatrix} \frac{1}{4} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{pmatrix} \\ &= \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \end{aligned}$$