Math2131 Answer to Homework 2

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Ex. 2.1(2)

We denote the map as L. The map is not a linear transformation. We give a counter example. We have

$$\begin{aligned} & L\big(2(0,2,2)\big) = L(0,4,4) = (0,16), \qquad 2L(0,2,2) = 2(0,4) = (0,8) \\ \Rightarrow \quad & L\big(2(0,2,2)\big) \neq 2L(0,2,2). \end{aligned}$$

Ex. 2.1(4)

We note the map as L. The map is a linear transformation. To justify this, we have

$$\begin{aligned} L(x_1+y_1, x_2+y_2, x_3+y_3) &= (x_1+y_1) + 2(x_2+y_2) + 3(x_3+y_3) \\ &= (x_1+2x_2+3x_3) + (y_1+2y_2+3y_3) \\ &= L(x_1,x_2,x_3) + L(y_1,y_2,y_3), \end{aligned}$$
$$L\big(c(x_1,x_2,x_3)\big) = L(cx_1,cx_2,cx_3) = cx_1+2cx_2+3cx_3 = c(x_1+2x_2+3x_3) = cL(x_1,x_2,x_3).$$

Ex. 2.2(2)

We note the map as L. The map is a linear transformation. To justify this, we note $g(t) = f_1(t) + f_2(t)$ and we have

$$L\big(f_1(t)+f_2(t)\big) = L\big(g(t)\big) = g(t^2) = f_1(t^2) + f_2(t^2) = L\big(f_1(t)\big) + L\big(f_2(t)\big),$$
$$L\big(cf_1(t)\big) = cf_1(t^2) = cL\big(f_1(t)\big).$$

Ex. 2.2(7)

We note the map as L. The map is a linear transformation. To justify this, we note $g(t) = f_1(t) + f_2(t)$ and we have

$$\begin{aligned} L\big(f_1(t)+f_2(t)\big) &= L\big(g(t)\big) \\ &= \big(g(0)+g(1), g(2)\big) \\ &= \big(f_1(0)+f_1(1), f_1(2)\big) + \big(f_2(0)+f_2(1), f_2(2)\big) \\ &= L\big(f_1(t)\big) + L\big(f_2(t)\big), \\ L\big(cf_1(t)\big) &= \big(cf_1(0)+cf_1(1), cf_1(2)\big) \\ &= c\big(f_1(0)+f_1(1), f_1(2)\big) \\ &= cL\big(f_1(t)\big). \end{aligned}$$

Ex. 2.2(8)

We note the map as L. The map is not a linear transformation. We give a counter example. We let $f(t) = 1$ and we have

$$\begin{aligned} & L\big(2f(t)\big) = \big(cf(0)\big)\big(cf(1)\big) = 4f(0)f(1) = 4, \\ & 2L(f(t)) = 2f(0)f(1) = 2, \\ \Rightarrow \quad & L\big(2f(t)\big) \neq 2L(f(t)). \end{aligned}$$

Ex. 2.3

Statement 1 and 2 are equivalent, so it suffices to prove 1. Suppose $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ are linearly dependent, then we have

$$\vec{v}_1 = a_2\vec{v}_2 + a_3\vec{v}_3 + \cdots + a_n\vec{v}_n$$

for some numbers a_i. Thus we easily get

$$L(\vec{v}_1) = L(a_2\vec{v}_2 + a_3\vec{v}_3 + \cdots + a_n\vec{v}_n) = a_2L(\vec{v}_2) + a_3L(\vec{v}_3) + \cdots + a_nL(\vec{v}_n)$$

which means $L(\vec{v}_1), L(\vec{v}_2), \ldots, L(\vec{v}_n)$ are linearly dependent.

Ex. 2.6

Using technique similar to that in Example 2.1.13, we note

$$\vec{v}_1 = (1, -1, 0),\ \vec{v}_2 = (1, 0, 1),\ \vec{v}_3 = (1, 1, 1), \quad \alpha = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$$

and form a matrix

$$\begin{pmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 & \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 0 & 0 & 1/3 & -2/3 & 1/3 \\ 0 & 1 & 0 & 1/3 & 1/3 & -2/3 \\ 0 & 0 & 1 & 1/3 & 1/3 & 1/3 \end{pmatrix}.$$

We find

$$[\vec{e}_1]_\alpha = \Big(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\Big), \quad [\vec{e}_2]_\alpha = \Big(-\frac{2}{3}, \frac{1}{3}, \frac{1}{3}\Big), \quad [\vec{e}_3]_\alpha = \Big(\frac{1}{3}, -\frac{2}{3}, \frac{1}{3}\Big).$$

This implies

$$\begin{aligned} L(\vec{e}_1) &= \frac{1}{3}L(\vec{v}_1) + \frac{1}{3}L(\vec{v}_2) + \frac{1}{3}L(\vec{v}_3) = \frac{1}{3}(1,2,3,4) + \frac{1}{3}(2,3,4,1) + \frac{1}{3}(3,4,1,2) = \Big(2, 3, \frac{8}{3}, \frac{7}{3}\Big), \\ L(\vec{e}_2) &= -\frac{2}{3}L(\vec{v}_1) + \frac{1}{3}L(\vec{v}_2) + \frac{1}{3}L(\vec{v}_3) = -\frac{2}{3}(1,2,3,4) + \frac{1}{3}(2,3,4,1) + \frac{1}{3}(3,4,1,2) = \Big(1, 1, -\frac{1}{3}, -\frac{5}{3}\Big), \\ L(\vec{e}_3) &= \frac{1}{3}L(\vec{v}_1) - \frac{2}{3}L(\vec{v}_2) + \frac{1}{3}L(\vec{v}_3) = \frac{1}{3}(1,2,3,4) - \frac{2}{3}(2,3,4,1) + \frac{1}{3}(3,4,1,2) = \Big(0, 0, -\frac{4}{3}, \frac{4}{3}\Big), \end{aligned}$$

and we conclude the matrix of L is

$$\begin{pmatrix} L(\vec{e}_1) & L(\vec{e}_2) & L(\vec{e}_3) \end{pmatrix} = \begin{pmatrix} 2 & 1 & 0 \\ 3 & 1 & 0 \\ \frac{8}{3} & -\frac{1}{3} & -\frac{4}{3} \\ \frac{7}{3} & -\frac{5}{3} & \frac{4}{3} \end{pmatrix}.$$

Ex. 2.10

For any $L, P \in \mathrm{Hom}(V, W)$, we have

$$(L+P)(\vec{v}) = L(\vec{v}) + P(\vec{v}), \qquad (cL)(\vec{v}) = cL(\vec{v}),$$

from the definition of addition and scalar multiplication of a linear transformation. Since it exactly is the conditions required for a evaluation map, we conclude that the map is a linear transformation.

Ex. 2.14

From Example 2.1.10 we knows that the matrix of F_ρ is $\begin{pmatrix} \cos 2\rho & \sin 2\rho \\ \sin 2\rho & -\cos 2\rho \end{pmatrix}$ and we have

$$F_\rho^2 = \begin{pmatrix} \cos 2\rho & \sin 2\rho \\ \sin 2\rho & -\cos 2\rho \end{pmatrix} \begin{pmatrix} \cos 2\rho & \sin 2\rho \\ \sin 2\rho & -\cos 2\rho \end{pmatrix} = \begin{pmatrix} \cos^2 2\rho + \sin^2 2\rho & \cos 2\rho \sin 2\rho - \sin 2\rho \cos 2\rho \\ \sin 2\rho \cos 2\rho - \cos 2\rho \sin 2\rho & \sin^2 2\rho + (-\cos 2\rho)^2 \end{pmatrix}$$

It is clear that

$$\begin{pmatrix} \cos^2 2\rho + \sin^2 2\rho & \cos 2\rho \sin 2\rho - \sin 2\rho \cos 2\rho \\ \sin 2\rho \cos 2\rho - \cos 2\rho \sin 2\rho & \sin^2 2\rho + (-\cos 2\rho)^2 \end{pmatrix} = I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \Leftrightarrow \quad \cos^2 2\rho + \sin^2 2\rho = 1$$

Ex. 2.16

It is clear that rotation of 30° followed by reflection with respect to y-axis is different from reflection with respect to y-axis followed by rotation of 30°. To justify it, we consider the matrices and have

$$\begin{pmatrix} \cos 30^\circ & -\sin 30^\circ \\ \sin 30^\circ & \cos 30^\circ \end{pmatrix} \begin{pmatrix} \cos \pi & \sin \pi \\ \sin \pi & -\cos \pi \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -\frac{\sqrt{3}}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix},$$

$$\begin{pmatrix} \cos \pi & \sin \pi \\ \sin \pi & -\cos \pi \end{pmatrix} \begin{pmatrix} \cos 30^\circ & -\sin 30^\circ \\ \sin 30^\circ & \cos 30^\circ \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix} = \begin{pmatrix} -\frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix},$$

and the two are not equal.

Ex. 2.23

Here are some examples.

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} + \begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix} = \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 & 2 \\ 4 & 3 & 4 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \end{pmatrix}.$$

Ex. 2.24(1)

The matrix which we note as A_θ is the matrix of R_θ. From Example 2.1.18 we have $R_{\theta_1} \circ R_{\theta_2} = R_{\theta_1 + \theta_2}$, from which we can derive that

$$A_\theta^n = A_{n\theta} = \begin{pmatrix} \cos n\theta & -\sin n\theta \\ \sin n\theta & \cos n\theta \end{pmatrix}.$$

Ex. 2.24(5)
We note

$$I = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad J = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad O = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

and we have

$$\begin{aligned} I^n &= I, \\ J^0 &= I, \\ J^2 &= \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \\ J^3 = J^2 J &= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \\ J^4 = J^3 J &= \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = O = J^5 = J^6 = \cdots = J^n \text{ (for any } n > 4). \end{aligned}$$

Therefore, we get

$$\begin{aligned} A^n = (aI + bJ)^n &= \sum_{i=0}^{n} \binom{n}{i} (aI)^{n-i} (bJ)^i = \sum_{i=0}^{3} \binom{n}{i} (aI)^{n-i} (bJ)^i + \sum_{i=4}^{n} O \\ &= a^n I + n a^{n-1} b J + n(n-1) a^{n-2} b^2 J^2 + n(n-1)(n-2) a^{n-3} b^3 J^3 + \sum_{i=4}^{n} O \\ &= \begin{pmatrix} a^n & n a^{n-1} & n(n-1) a^{n-2} b^2 & n(n-1)(n-2) a^{n-3} b^3 \\ 0 & a^n & n a^{n-1} & n(n-1) a^{n-2} b^2 \\ 0 & 0 & a^n & n a^{n-1} \\ 0 & 0 & 0 & a^n \end{pmatrix}. \end{aligned}$$

Ex. 2.25(3)
We carry out the row operation on the matrix

$$\begin{pmatrix} A & B \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & b \end{pmatrix} \xrightarrow[R_3 - 9R_1]{R_2 - 5R_1} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & -4 & -8 & -12 \\ 0 & -8 & -16 & b - 36 \end{pmatrix} \xrightarrow[R_1 - 2R_2]{\substack{R_3 - 2R_2 \\ -\frac{1}{4}R_2}} \begin{pmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & b - 12 \end{pmatrix}.$$

If $b = 12$, similar to Example 2.1.19, we get

$$X = \begin{pmatrix} -1 & -2 \\ 2 & 3 \end{pmatrix}.$$

If $b \neq 12$, from the last row we conclude that X does not exist.

Ex. 2.26(2)

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $B = \begin{pmatrix} x & y \\ z & w \end{pmatrix}$, then $A^T = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$ and $B^T = \begin{pmatrix} x & z \\ y & w \end{pmatrix}$ and we have

$$(AB)^T = \begin{pmatrix} ax+bz & ay+bw \\ cx+dz & cy+dw \end{pmatrix}^T = \begin{pmatrix} ax+bz & cx+dz \\ ay+bw & cy+dw \end{pmatrix} = B^T A^T.$$

To solve the matrix equation

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} X \begin{pmatrix} 4 & -3 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

we first let $X \begin{pmatrix} 4 & -3 \\ -2 & 1 \end{pmatrix} = Y$, then by row operations, we have

$$\begin{pmatrix} 1 & 2 & 1 & 0 \\ 3 & 4 & 0 & 1 \end{pmatrix} \xrightarrow[-\frac{1}{2}R_2]{R_2-3R_1} \begin{pmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & 3/2 & -1/2 \end{pmatrix} \xrightarrow{R_1-2R_2} \begin{pmatrix} 1 & 0 & -2 & 1 \\ 0 & 1 & 3/2 & -1/2 \end{pmatrix}$$

which means $Y = \begin{pmatrix} -2 & 1 \\ 3/2 & -1/2 \end{pmatrix}$. Thus

$$\begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix} X^T = Y^T = \begin{pmatrix} -2 & 3/2 \\ 1 & -1/2 \end{pmatrix}$$

and by row operations, we have

$$\begin{pmatrix} 4 & -2 & -2 & 3/2 \\ -3 & 1 & 1 & -1/2 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & -1/4 \\ 0 & 1 & 1 & -5/4 \end{pmatrix}$$

which means $X^T = \begin{pmatrix} 0 & -1/4 \\ 1 & -5/4 \end{pmatrix}$ and therefore $X = \begin{pmatrix} 0 & 1 \\ -1/4 & -5/4 \end{pmatrix}$.

Ex. 2.27

Let $X = \begin{pmatrix} x & y \\ z & w \end{pmatrix}$, then

$$AX = \begin{pmatrix} -x & -y \\ z & w \end{pmatrix} = XA = \begin{pmatrix} -x & y \\ -z & w \end{pmatrix} \quad \Rightarrow \quad y = -y,\ z = -z \quad \Rightarrow \quad y = z = 0$$

Therefore, we have $X = \begin{pmatrix} x & 0 \\ 0 & w \end{pmatrix}$ where x and w can be any number.

For the general case, we let

$$X = \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{pmatrix},$$

and we have

$$AX = \begin{pmatrix} a_1b_{11} & a_1b_{12} & \cdots & a_1b_{1n} \\ a_2b_{21} & a_2b_{22} & \cdots & a_2b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_nb_{n1} & a_nb_{n2} & \cdots & a_nb_{nn} \end{pmatrix}, \qquad XA = \begin{pmatrix} a_1b_{11} & a_2b_{12} & \cdots & a_nb_{1n} \\ a_1b_{21} & a_2b_{22} & \cdots & a_nb_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_1b_{n1} & a_2b_{n2} & \cdots & a_nb_{nn} \end{pmatrix}.$$

To make $AX = XA$, we have

$$a_ib_{ij} = a_jb_{ij} \quad \Leftrightarrow \quad b_{ij} = 0 \text{ if } a_i \neq a_j$$

for any $i, j = 1, 2, \ldots, n$. Then the general solution of X is

$$X = \begin{pmatrix} b_{ij} \end{pmatrix} \quad \text{where } b_{ij} = \begin{cases} 0 & \text{if } a_i \neq a_j \\ \text{any number} & \text{if } a_i = a_j \end{cases}.$$

Ex. 2.34

It is clear that the map is not onto. For example, for any x_1, x_2, x_3, we have

$$L(x_1.x_2, x_3) = x_1 \cos t + x_2 \sin t + x_3 e^t \neq f(t) \in C^\infty \quad \text{where } f(t) = t.$$

Also, it is clear that the map is one-to-one. By Example 1.3.3 we already derive that $\cos t, \sin t, e^t$ are linearly independent, which means

$$x_1 \cos t + x_2 \sin t + x_3 e^t = y_1 \cos t + y_2 \sin t + y_3 e^t \quad \Rightarrow \quad x_1 = y_1, \quad x_2 = y_2, \quad x_3 = y_3.$$

This proves that L is one-to-one.

Ex. 2.38

We note the linear transformation as $L : V \to W$. By the condition, there exist a set of vectors $\{\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_n\}$ that span W, and a set of vectors $\{\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n\}$ in V such that $L(\vec{a}_i) = \vec{b}_i$. Then, for any vector $\vec{x}$ in W, we have

$$\vec{x} = x_1\vec{b}_1 + x_2\vec{b}_2 + \cdots + x_n\vec{b}_n.$$

Therefore, we can always find a vector $\vec{y} = x_1\vec{a}_1 + x_2\vec{a}_2 + \cdots + x_n\vec{a}_n$ in V such that

$$L(\vec{y}) = x_1L(\vec{a}_1) + x_2L(\vec{a}_2) + \cdots + x_nL(\vec{a}_n) = x_1\vec{b}_1 + x_2\vec{b}_2 + \cdots + x_n\vec{b}_n = \vec{x}.$$

This proves that L is onto.

Ex. 2.40

If we let A be a matrix of a linear transformation $L : \mathbb{R}^n \to \mathbb{R}^m$, then a system of linear equations $A\vec{x} = \vec{b}$ has solution for all $\vec{b} \in \mathbb{R}^m$ is equivalent to L is onto. By Proposition 2.2.4, L is onto if and only if there is a linear transformation $K : \mathbb{R}^m \to \mathbb{R}^n$ such that $L \circ K = I_W$. We note that the corresponding matrix of I_W is I_m, thus the matrix of K is the B we want. Moreover, the uniqueness of the solution is equivalent to the one-to-one property of L. By Proposition 2.2.5, L is one-to-one if and only if there is a linear transformation $K : \mathbb{R}^m \to \mathbb{R}^n$ such that $K \circ L = I_V$. We note that the corresponding matrix of I_V is I_n, thus the matrix of K is the B we want.

Since $L \circ K$ and L are linear transformations, we have

$$L \circ K(\vec{w}_1 + \vec{w}_2) = L \circ K(\vec{w}_1) + L \circ K(\vec{w}_2), \qquad L \circ K(c\vec{w}_1) = c(L \circ K)(\vec{w}_1)$$

Ex. 2.42

We let $L : U \to V$ and $K : W \to U$, then $L \circ K : W \to V$. For any $\vec{w}$ in W, there is $\vec{v} \in V$ such that $K(\vec{w}) = \vec{v}$. Since $L \circ K$ and L are linear transformations and L is one-to-one, we have

$$\begin{aligned}
& (L \circ K)(c\vec{w}) = L\big(K(c\vec{w})\big) = c(L \circ K)(\vec{w}) = cL\big(K\vec{w}\big) = cL(\vec{v}), \\
& L(c\vec{v}) = cL(\vec{v}), \quad \text{and} \quad L(\vec{v}_x) = L(\vec{v}_y) \ \Rightarrow \ \vec{v}_x = \vec{v}_y, \\
\Rightarrow \quad & K(c\vec{w}) = c\vec{v} = cK(\vec{w}).
\end{aligned}$$

For any $\vec{w}_1, \vec{w}_2$ in W, there is $\vec{v}_1, \vec{v}_2 \in V$ such that $K(\vec{w_1}) = \vec{v_1}, K(\vec{w_2}) = \vec{v_2}$. Since $L \circ K$ and L are linear transformations and L is one-to-one, we have

$$\begin{aligned}
& (L \circ K)(\vec{w}_1 + \vec{w}_2) = L\big(K(\vec{w}_1 + \vec{w}_2)\big) = (L \circ K)(\vec{w}_1) + (L \circ K)(\vec{w}_2) = L\big(K(\vec{w}_1)\big) + L\big(K(\vec{w}_2)\big) \\
& \qquad = L(\vec{v}_1) + L(\vec{v}_2), \\
& L(\vec{v}_1 + \vec{v}_2) = L(\vec{v}_1) + L(\vec{v}_2), \quad \text{and} \quad L(\vec{v}_x) = L(\vec{v}_y) \ \Rightarrow \ \vec{v}_x = \vec{v}_y, \\
\Rightarrow \quad & K(\vec{w}_1 + \vec{w}_2) = \vec{v}_1 + \vec{v}_2 = K(\vec{w}_1) + K(\vec{w}_2).
\end{aligned}$$

Therefore, we conclude that K is a linear transformation.