Selected Solutions to Section 7.3

Abstract

The order of all exercises is based on: www.math.ust.hk/~mamyan/ma2131/lecture.pdf

Ex 7.74

L is diagonalisable $\Leftrightarrow$ under some basis, $L \leftrightarrow \operatorname{diag}[a_1, ..., a_2, ..., a_k, ...]$ $\Leftrightarrow$ eigenvectors can form a basis $\Leftrightarrow V = \bigoplus \operatorname{Ker}(L - a_i I) \Leftrightarrow m(t) = \prod (t - a_i)$.

Ex 7.76

$V = \bigoplus \operatorname{Ker} p_i^{m_i}$, since p_i is irreducible, $p_i(t) = t - a_i$. $\gcd(p_i, p_j) = 1$, so $\operatorname{Ker} p_i^r \cap \operatorname{Ker} p_j^s = \{\vec{0}\}$. We know $\operatorname{Ker} p^{m_i} \subset \operatorname{Ker} p^{m_i+1}$ and $\operatorname{Ker} p^{m_i+1} \cap \operatorname{Ker} p_j^s = \{\vec{0}\}$. By direct sum, $\operatorname{Ker} p^{m_i+1} = \operatorname{Ker} p^{m_i}$.

It is obvious that $\operatorname{Ker} p^{m_i-1} \subset \operatorname{Ker} p^{m_i}$. If $\operatorname{Ker} p^{m_i-1} = \operatorname{Ker} p^{m_i}$, then for any $v \in \operatorname{Ker} p^{m_i}$. We have $p^{m_i-1}(L)(v) = \vec{0}$. So for any $u \in V$, $p_1^{m_1} \cdots p_i^{m_i-1} \cdots p_k^{m_k}(L)(v) = \vec{0}$. So $p_1^{m_1} \cdots p_i^{m_i-1} \cdots p_k^{m_k} \in \operatorname{Ann} L$, which contradicts to the minimality of $m(t)$.

Ex 7.77

In Example 7.1.5, for L, we can find a_i, s.t. $L^k + a_{k-1}L^{k-1} + \cdots + a_0 I = O$. So $t^k + \cdots + a_0 \in \operatorname{Ann} L$.

Since k is the biggest integer such that $S = \{v, L(v), ..., L^{k-1}(v)\}$ is linearly independent for any v. So if $\deg m(t) < k$, it can't span whole space V. So $m(t) = t^k + \cdots + a_0$.

Ex 7.78 (1)

$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, $\det(tI - A) = t^2 - 5t - 2$. So it has two different eigenvalues, each eigenspace is $\dim 1$. So A is diagonalisable.

Ex 7.80

Let $D = \frac{d}{dt}$, and for any $f \in P_n$, we have $D^{n+1} f = 0$. $D^n(t^n) = 1 \neq 0$, so $m(t) = t^{n+1}$.

Ex 7.81

For any $A \in M_n$, let $A^T = T(A)$. We have $T^2(A) - A = 0$ so $m(t) | t^2 - 1$. $T(E_{12}) - E_{12} \neq 0$ and $T(E_{12}) + E_{12} \neq 0$. So $m(t) = t^2 - 1$.

Ex 7.84

Suppose $A = \begin{pmatrix} 0 & \bullet & \cdots & \bullet & \bullet \\ & 0 & \cdots & \bullet & \bullet \\ & & \ddots & \vdots & \vdots \\ & & & 0 & \bullet \\ & & & & 0 \end{pmatrix}$ and we have

$$A^2 = \begin{pmatrix} 0 & 0 & \bullet & \cdots & \bullet & \bullet \\ & 0 & 0 & \cdots & \bullet & \bullet \\ & & 0 & \ddots & \vdots & \vdots \\ & & & \ddots & 0 & \bullet \\ & & & & 0 & 0 \\ & & & & & 0 \end{pmatrix}, \ldots\ldots, A^{n-1} = \begin{pmatrix} 0 & 0 & \cdots & 0 & \bullet \\ & 0 & \cdots & 0 & 0 \\ & & \ddots & \vdots & \vdots \\ & & & 0 & 0 \\ & & & & 0 \end{pmatrix}, A^n = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ & 0 & \cdots & 0 & 0 \\ & & \ddots & \vdots & \vdots \\ & & & 0 & 0 \\ & & & & 0 \end{pmatrix}.$$

So A is nilpotent. Similarly, we know any lower triangular matrix is nilpotent.

Ex 7.87

A is diagonalisable $\Leftrightarrow$ Each Jordan block of A is 1×1, i.e. $J_{a_i}=[a_i]$.

Ex 7.88

We only need to consider each Jordan block and then joint all Jordan blocks.

$$PAP^{-1}=\begin{pmatrix} a & & & & \\ 1 & a & & & \\ & 1 & \ddots & & \\ & & \ddots & a & \\ & & & 1 & a \end{pmatrix} \text{ and } \begin{pmatrix} & & & 1 \\ & & 1 & \\ & \iddots & & \\ 1 & & & \end{pmatrix}\begin{pmatrix} a & & & & \\ 1 & a & & & \\ & 1 & \ddots & & \\ & & \ddots & a & \\ & & & 1 & a \end{pmatrix}\begin{pmatrix} & & & 1 \\ & & 1 & \\ & \iddots & & \\ 1 & & & \end{pmatrix}=\begin{pmatrix} a & 1 & & & \\ & a & \ddots & & \\ & & \ddots & 1 & \\ & & & a & 1 \\ & & & & a \end{pmatrix}.$$

$$\text{So } A\sim\begin{pmatrix} a & & & & \\ 1 & a & & & \\ & 1 & \ddots & & \\ & & \ddots & a & \\ & & & 1 & a \end{pmatrix}\sim\begin{pmatrix} a & 1 & & & \\ & a & \ddots & & \\ & & \ddots & 1 & \\ & & & a & 1 \\ & & & & a \end{pmatrix}\sim A^T.$$

Ex 7.89

$J_a=\begin{pmatrix} a & & & & \\ 1 & a & & & \\ & 1 & \ddots & & \\ & & \ddots & a & \\ & & & 1 & a \end{pmatrix}=J+aI$, so $J_a^k=(J+aI)^k=\sum_{i=0}^n C_i^n a^{n-i}J^i$. The details are shown in previous homework.