**Math 204 Final, Spring 2006**

(1) Suppose $f(x,y)$ and $f_y(x,y)$ are continuous near $(x_0,y_0)$. Prove that the equation $y = y_0 + \int_{x_0}^{x} f(t,y)dt$ determines $y$ as a function of $x$ near $x_0$.

(2) Let $n$ be a natural number. Show that $x+y+z+(x^2+y^2)z^n = 0$ uniquely determines $z$ as a function of $(x,y)$ at $(0,0,0)$. Then find the $(n+2)$-nd order Taylor expansion of $z = z(x,y)$ at $(0,0)$.

(3) Suppose $p_i > 0$, $\frac{1}{p_1} + \frac{1}{p_2} + \cdots + \frac{1}{p_n} = 1$. Suppose $x_i \geq 0$. Prove that

$$x_1 x_2 \cdots x_n \leq \frac{1}{p_1} x_1^{p_1} + \frac{1}{p_2} x_2^{p_2} + \cdots + \frac{1}{p_n} x_n^{p_n}.$$

(4) Compute integrals.

1. $\int_{0 \leq x,y \leq \pi} \cos x \cos y \max\{x,y\} dxdy$.
2. $\int_{S^2} \frac{dx \wedge dy}{z} + \frac{dy \wedge dz}{x} + \frac{dz \wedge dx}{y}$. The orientation of the sphere $S^2$ is compatible with the outward normal vector.

(5) If possible, find the potential function.

1. $\frac{y(xy+1)dx + (2y-x)dy}{y^2}$.
2. $(x^2 + ay, bx + y^2)$.
3. $(x+y)dx + xydy$.

(6) Suppose a function $f(x,y)$ defined on a rectangle $[a,b] \times [c,d]$ is increasing in $x$ for fixed $y$ and increasing in $y$ for fixed $x$. Prove that $f$ is Riemann integrable.

**Answer to Math 204 Final, Spring 2006**

(1) Consider the function $g(x,y) = y - y_0 - \int_{x_0}^{x} f(t,y)dt$. We have $g(x_0,y_0) = 0$. Moreover, $g_x = (x,y)$ is continuous and $g_y = 1 - \int_{x_0}^{x} f_y(t,y)dt$ is also continuous by the continuity of $f_y(x,y)$. Therefore $g(x,y)$ is differentiable near $(x_0,y_0)$. By $g_y(x_0,y_0) = 1 \neq 0$ and the implicit function theorem, we conclude that $g(x,y) = 0$ determines $y$ as a function of $x$ near $x_0$.

Note that the argument made use of $\frac{\partial}{\partial y} \int_a^b f(t,y)dt = \int_a^b f_y(t,y)dt$ when $f_y$ is continuous. This can be proved by

$$\frac{1}{h}\left(\int_a^b f(t,y+h)dt - \int_a^b f(t,y)dt\right) - \int_a^b f_y(t,y)dt = \int_a^b (f_y(t,y^*) - f_y(t,y))dt,$$

where $y^*$ is a number (depending on $t$) between $y$ and $y+h$. The continuity implies uniform continuity and the integral converges to 0 as $h \to 0$.
(2) By $(x+y+z+(x^2+y^2)z^n)_z = 1+n(x^2+y^2)z^{n-1} = 1 \neq 0$ at $(0,0,0)$ and the implicit function theorem, $z$ is uniquely determined as a function of $(x,y)$ near $(0,0,0)$.

By $(1+2xz^n)dx + (1+2yz^n)dy + (1+n(x^2+y^2)z^{n-1})dz = 0$, we get $z_x(0,0) = -1$, $z_y(0,0) = -1$. Thus $z(x,y) = -x-y+R_2$ with $R_2 = O(\|(x,y)\|^2)$. Substitute into the equality, we get

$$\begin{aligned}0 &= R_2 + (x^2+y^2)(-x-y+R_2)^n \\ &= R_2 + (x^2+y^2)[(-1)^n(x+y)^n + n(-1)^{n-1}(x+y)^{n-1}R_2 + \cdots] \\ &= R_2 + (-1)^n(x^2+y^2)(x+y)^n + O(\|(x,y\|^{n+3}).\end{aligned}$$

Therefore $R_2 = (-1)^{n-1}(x^2+y^2)(x+y)^n + R_{n+3}$, with $R_{n+3} = O(\|(x,y\|^{n+3})$. We conclude that the Taylor expansion

$$z = -x-y+(-1)^{n-1}(x^2+y^2)(x+y)^n + R_{n+3}.$$

(3) We will show that the minimum of $f = \dfrac{1}{p_1}x_1^{p_1} + \dfrac{1}{p_2}x_1^{p_2} + \cdots + \dfrac{1}{p_n}x_1^{p_n}$ under the constraint $g = x_1x_2\cdots x_n = c$ is $\geq c$. The possible local extremes satisfy $\nabla f = \lambda \nabla g$, which means

$$\frac{x_1^{p_1-1}}{x_2x_3\cdots x_n} = \frac{x_2^{p_2-1}}{x_1x_3\cdots x_n} = \cdots = \frac{x_n^{p_1-n}}{x_1x_2\cdots x_{n-1}} = \lambda,$$

or

$$x_1^{p_1} = x_2^{p_2} = \cdots = x_n^{p_n} = \lambda x_1x_2\cdots x_n = \lambda c.$$

Therefore $c = \lambda x_1x_2\cdots x_n = (\lambda c)^{\frac{1}{p_1}+\frac{1}{p_2}+\cdots+\frac{1}{p_n}} = \lambda c$, so that $\lambda = 1$, $x_i^{p_i} = c$, and $f = \left(\dfrac{1}{p_1} + \dfrac{1}{p_2} + \cdots + \dfrac{1}{p_n}\right)c = c$.

If one $x_i$ is big, then $f$ becomes big. Therefore there is $M$, such that $f(\vec{x}) > c$ when $\|\vec{x}\| \geq M$. Since $f = c$ at the candidate above, this means that $f$ cannot have absolute minimum outside the ball $\|\vec{x}\| < M$. On the other hand, the continuous function must reach its minimum on the compact subset $\|\vec{x}\| \leq M$. The minimum must appear inside the ball and will be an absolute minimum. Since there is only one candidate for the local extreme, the candidate is the minimum.
(4.1)

$$\begin{aligned}\int_{0\leq x,y\leq\pi} \cos x\cos y \max\{x,y\}dxdy &= \int_{0\leq x\leq y\leq\pi} y\cos x\cos y dxdy + \int_{0\leq y\leq x\leq\pi} x\cos x\cos y dxdy \\ &= \int_0^\pi \left(\int_0^y \cos x dx\right) y\cos y dy + \int_0^\pi \left(\int_0^x \cos y dy\right) x\cos x dx \\ &= \int_0^\pi y\cos y\sin y dy + \int_0^\pi x\cos x\sin x dx \\ &= \int_0^\pi x\sin 2x dx = \left(-\frac{x}{2}\cos 2x + \frac{1}{4}\sin 2x\right)_0^\pi = -\frac{\pi}{2}.\end{aligned}$$

(4.2)

$$\int_{S^2} \frac{dx\wedge dy}{z} + \frac{dy\wedge dz}{x} + \frac{dz\wedge dx}{y} = \int_{S^2} \left(\frac{1}{x}, \frac{1}{y}, \frac{1}{z}\right)\cdot \vec{n} dA$$
$$= \int_{S^2} \left(\frac{1}{x}, \frac{1}{y}, \frac{1}{z}\right)\cdot (x,y,z) dA$$
$$= 3\int_{S^2} 1 dA = 12\pi.$$

(5.1) $\frac{d}{dy}\left(\frac{y(xy+1)}{y^2}\right) = -\frac{1}{y^2} = \frac{d}{dx}\left(\frac{2y-x}{y^2}\right)$. The 1-form is defined on $y>0$ and $y<0$, both simply connected. Therefore it has potential function $\varphi$. By $\varphi_x = \frac{y(xy+1)}{y^2}$, we get $\varphi = \int \frac{y(xy+1)}{y^2} dx + C(y) = \frac{x^2}{2} + \frac{x}{y} + C(y)$. Then by $\varphi_y = -\frac{x}{y^2} + C'(y) = \frac{2y-x}{y^2}$, we get $C'(y) = \frac{2}{y}$, so that $C(y) = 2\log|y| + C$. The potential function is

$$\varphi = \frac{x^2}{2} + \frac{x}{y} + 2\log|y| + C.$$

(5.2) $(x^2+ay)_y = a$, $(bx+y^2)_x = b$. The potential exists if and only if $a=b$. In this case, $\varphi_x = x^2+ay$ implies $\varphi = \frac{x^3}{3} + axy + C(y)$. Then $\varphi_y = ax + C'(y) = ax + y^2$ implies $C'(y) = y^2$. Therefore $C'(y) = \frac{y^3}{3} + C$ and $\varphi = \frac{x^3}{3} + \frac{y^3}{3} + axy + C$.
(5.3) $(x+y)_x = 1 \neq y = (xy)_x$. The 1-form has no potential.

(6) Consider rectangular partition of $[a,b]\times[c,d]$ given by $P\colon a = x_0 < x_1 < \cdots < x_m = b$ and $Q\colon c = y_0 < y_1 < \cdots < y_n = d$. Then by the monotone assumption, we have

$\omega_{[x_{i-1},x_i]\times[y_{j-1},y_j]}(f) = f(x_i,y_j) - f(x_{i-1},y_{j-1})$, $|f(x,y)-f(x',y')| \le f(b,d)-f(a,c)$, and

$$\begin{aligned}
\sum_{I\in P\times Q}\omega_I(f)\mu(I) &= \sum_{i,j}(f(x_i,y_j)-f(x_{i-1},y_{j-1}))\Delta x_i\Delta y_j \\
&= \sum_{j=1}^n\left(\sum_{i=1}^n(f(x_i,y_j)-f(x_{i-1},y_j))\Delta x_i\right)\Delta y_j \\
&\quad+\sum_{i=1}^n\left(\sum_{j=1}^n(f(x_{i-1},y_j)-f(x_{i-1},y_{j-1}))\Delta y_j\right)\Delta x_i \\
&\le \sum_{j=1}^n\left(\sum_{i=1}^n(f(x_i,y_j)-f(x_{i-1},y_j))\right)\|P\|\Delta y_j \\
&\quad+\sum_{i=1}^n\left(\sum_{j=1}^n(f(x_{i-1},y_j)-f(x_{i-1},y_{j-1}))\right)\|Q\|\Delta x_i \\
&\le \sum_{j=1}^n(f(b,y_j)-f(a,y_j))\|P\|\Delta y_j+\sum_{i=1}^n(f(x_{i-1},d)-f(x_{i-1},c))\|Q\|\Delta x_i \\
&\le (f(b,d)-f(a,c))(d-c)\|P\|+(f(b,d)-f(a,c))(b-a)\|Q\|.
\end{aligned}$$

Then $\|P\times Q\|<\epsilon$ implies $\|P\|<\epsilon$ and $\|Q\|<\epsilon$, which further implies $\sum_{I\in P\times Q}\omega_I(f)\mu(I) < (f(b,d)-f(a,c))(d-c+b-a)\epsilon$. Thus the criterion for integrability is verified.

**Math 204 Final, Spring 2007**

(1) Suppose $f\le g\le h$ and $A$ is a subset with volume. Suppose $f$ and $h$ are Riemann integrable on $A$ with $\int_A fd\mu = \int_A hd\mu$. Prove that $g$ is Riemann integrable on $A$.

(2) Suppose $f(t)$ is Riemann integrable on $[a,b]$. Suppose $A\subset\mathbb{R}^2$ is a subset with area, such that $(x,y)\in A \implies a\le x+y\le b$. Prove that $f(x+y)$ is integrable on $A$.

(3) Suppose $C$ is any closed curve on the sphere $x^2+y^2+z^2=R^2$. Prove that $\int_C(y^2+z^2)dx+(z^2+x^2)dy+(x^2+y^2)dz=0$. In general, what is the condition for $f$, $g$, $h$ so that $\int_C fdx+gdy+hdz=0$ for any closed curve $C$ on any sphere centered at the origin?

(4) Let $C_R$ be the counterclockwise circle of radius $R$ centered at the origin. Prove that

$$\int_{C_R} e^y[(x\sin x+y\cos x)dx+(y\sin x-x\cos x)]dy = cR^2$$

for a constant $c$. Then find the constant $c$.

Hint: $\dfrac{I}{R^2} = \displaystyle\int_{C_R}\frac{e^y}{x^2+y^2}[(x\sin x+y\cos x)dx+(y\sin x-x\cos x)]dy$.

(5) If possible, find the potential function.

1. Differential form $\omega = \dfrac{(\alpha x+\beta y)dx+(\gamma x+\delta y)dy}{(ax^2+by^2)^p}$.

2. Vector field $F = \dfrac{\vec{a}}{\vec{b} \cdot \vec{x}}$, $\vec{b} \neq \vec{0}$.

(6) True or false (no reason needed).

1. If $A$ has volume, then $\partial A$ has volume.
2. If $\partial A$ has volume, then $A$ has volume.
3. If $A, B \neq \emptyset$ and $A \times B$ has volume, then $A$ and $B$ have volumes.
4. If $A$ and $B$ have volumes, then $A \times B$ has volume.
5. If $A \subset \mathbb{R}^{m+n}$ has volume, then the projections $\pi_1(A) \subset \mathbb{R}^m$ and $\pi_2(A) \subset \mathbb{R}^n$ have volumes.
6. If $A \subset \mathbb{R}^{m+n}$ has the property that the projections $\pi_1(A) \subset \mathbb{R}^m$ and $\pi_2(A) \subset \mathbb{R}^n$ have volumes, then $A$ has volume.
7. If $f$ is Riemann integrable on $A$, then $|f|$ is Riemann integrable on $A$.
8. If $|f|$ is Riemann integrable on $A$, then $f$ is Riemann integrable on $A$.
9. If a parametrized curve $C \subset \mathbb{R}^{m+n}$ is rectifiable, then the projections $\pi_1(C) \subset \mathbb{R}^m$ and $\pi_2(C) \subset \mathbb{R}^n$ are rectifiable.
10. If a parametrized curve $C \subset \mathbb{R}^{m+n}$ has the property that the projections $\pi_1(C) \subset \mathbb{R}^m$ and $\pi_2(C) \subset \mathbb{R}^n$ are rectifiable, then $C$ is rectifiable.
11. If $\varphi$ and $\psi$ are potential functions of a vector field $F$ on an open subset $U$, then $\varphi = \psi + C$ for some constant $C$.
12. If a vector field $F$ has potential functions on open subsets $U$ and $V$, then $F$ has potential function on $U \cup V$.
13. If a vector field $F$ has potential functions on open subsets $U$ and $V \subset U$, then $F$ has potential function on $V$.
14. If vector fields $F$ and $G$ have potential functions on open subsets $U$ and $V$, then the vector field $(F, G)$ has potential function on $U \times V$.
15. If a vector field $F(\vec{x}, \vec{y})$ has potential functions for fixed $\vec{x}$ and for fixed $\vec{y}$, then $F$ has a potential function.

**Answer to Math 204 Final, Spring 2007**

(1) Let $I = \int_A f d\mu = \int_A h d\mu$. For any $\epsilon > 0$, there is $\delta > 0$, such that for any partition $P$ of $A$ satisfying $\|P\| < \delta$, we have

$$\left| \sum_{I \in P} f(x_I^*)\mu(I) - I \right| < \epsilon, \quad \left| \sum_{I \in P} h(x_I^*)\mu(I) - I \right| < \epsilon.$$

This implies

$$I-\epsilon \le \sum_{I\in P} f(x_I^*)\mu(I) \le \sum_{I\in P} g(x_I^*)\mu(I) \le \sum_{I\in P} h(x_I^*)\mu(I) \le I+\epsilon.$$

Therefore $\left|\sum_{I\in P} g(x_I^*)\mu(I) - I\right| < \epsilon$. By the definition of Riemann integration, we see that $g$ is Riemann integrable on $A$, with $\displaystyle\int_A g d\mu = I$.

(2) Suppose $A\subset[-R,R]^2$. Denote $\sigma(x,y)=x+y$ and for any $a\le s<t\le b$, denote $B_{[s,t]}=\sigma^{-1}[s,t]\cap[-R,R]^2$. Then $B_{[s,t]}$ has area $\mu(B_{[s,t]})\le 2\sqrt{2}R(t-s)$, so that $\sigma^{-1}[s,t]\cap A = B_{[s,t]}\cap A$ also has area, and

$$\mu(\sigma^{-1}[s,t]\cap A)\le \mu(B_{[s,t]})\le 2\sqrt{2}R(t-s).$$

Denote $g(x,y)=f(x+y)$ for $(x,y)\in A$. By the integrability of $f$, for any $\epsilon>0$, there is a partition of $[a,b]$, such that $\sum \omega_{[t_{i-1},t_i]}(f)\Delta t_i<\epsilon$. Then there is the induced partition of $A$ given by $\sigma^{-1}[t_{i-1},t_i]\cap A = B_{[t_{i-1},t_i]}\cap A$. With respect to this partition, we have

$$\omega_{\sigma^{-1}[t_{i-1},t_i]\cap A}(g)\le \omega_{[t_{i-1},t_i]}(f),\quad \mu(\sigma^{-1}[t_{i-1},t_i]\cap A)\le 2\sqrt{2}R\Delta t_i.$$

Therefore

$$\sum \omega_{\sigma^{-1}[t_{i-1},t_i]\cap A}(g)\mu(\sigma^{-1}[t_{i-1},t_i]\cap A)\le 2\sqrt{2}R\sum \omega_{[t_{i-1},t_i]}(f)\Delta t_i \le 2\sqrt{2}R\epsilon.$$

Since $R$ is fixed once $A$ is given, the criterion for the integrability of $g$ on $A$ is verified.

An alternative explanation is the following: We can find another function $\rho(x,y)$, such that $(\sigma,\rho)$ is a continuously differentiable change of variable (take $\rho=x-y$ or $\rho=x$, for examples). Let $h(t,s)=f(t)$ be the single variable function $f$ considered as a two variable function. Then $f(x,y)=h(\sigma(x,y),\rho(x,y))$. By the change of variable theorem, it suffices to show that $h(t,s)=f(t)$ is integrable on subsets $A'\subset\mathbb{R}^2$ with area. This is quite easy to prove.

The alternative explanation leads to many other possible $\sigma(x,y)$ for which the integrability of $f(t)$ implies the integrability of $f(\sigma(x,y))$. Moreover, the idea can be easily extended to more variable case.

(3) If $C$ is simple, then $C$ divides the sphere into two parts. With the normal vector $\vec{n}=\dfrac{\vec{x}}{R}=\dfrac{(x,y,z)}{R}$, one part $S$ is compatible with the orientation of $C$. Then by Stokes Theorem,

$$\begin{aligned}
&\int_C (y^2+z^2)dx+(z^2+x^2)dy+(x^2+y^2)dz\\
=&\int_S (2z-2y)dy\wedge dz+(2x-2z)dz\wedge dx+(2x-2y)dx\wedge dy\\
=&2\int_S (z-y,x-z,x-y)\cdot\frac{(x,y,z)}{R}dA\\
=&\frac{2}{R}\int_S[(z-y)x+(x-z)y+(x-y)z]dA=\frac{2}{R}\int_S 0dA=0.
\end{aligned}$$

In general, the rectifiable $C$ can be approximated by a closed curve $L$ consisting of finitely many big arcs on the sphere. Although $L$ many not be simple, it can be divided into finitely many simple closed curves. Therefore $\int_L (y^2+z^2)dx+(z^2+x^2)dy+(x^2+y^2)dz=0$. By taking the limit, we get $\int_C (y^2+z^2)dx+(z^2+x^2)dy+(x^2+y^2)dz=0$.

In general, for a simple closed curve $C$ and the choice of $S$ as above, we have

$$\begin{aligned}\int_C fdx+gdy+hdz &= \int_S (h_y-g_z)dy\wedge dz+(f_z-h_x)dz\wedge dx+(g_x-f_y)dx\wedge dy\\ &= \frac{1}{R}\int_S [(h_y-g_z)x+(f_z-h_x)y+(g_x-f_y)z]dA.\end{aligned}$$

Similar argument as above shows that if $(h_y-g_z)x+(f_z-h_x)y+(g_x-f_y)z=0$, then $\int_C fdx+gdy+hdz=0$ for any closed curve $C$ on the sphere.

(4) We have $\frac{I}{R^2}=\int_{C_R}\frac{e^y}{x^2+y^2}[(x\sin x+y\cos x)dx+(y\sin x-x\cos x)]dy$. By

$$\begin{aligned}&\left(\frac{e^y(x\sin x+y\cos x)}{x^2+y^2}\right)_y=\left(\frac{e^y(y\sin x-x\cos x)}{x^2+y^2}\right)_x\\ &=\frac{e^y}{(x^2+y^2)^2}[(x(x^2+y^2)-2xy)\sin x+(y(x^2+y^2)+x^2-y^2\cos x)]\end{aligned}$$

and Green Theorem, we see that $\frac{I}{R^2}$ is independent of the choice of $R$. Then

$$\begin{aligned}\frac{I}{R^2} &= \lim_{R\to 0^+}\frac{1}{R^2}\int_0^{2\pi} e^{R\sin\theta}R^2[-[\cos\theta\sin(R\cos\theta)+\sin\theta\cos(R\cos\theta)]\sin\theta\\ &\qquad +[\sin\theta\sin(R\cos\theta)-\cos\theta\cos(R\cos\theta)]\cos\theta]d\theta\\ &= \int_0^{2\pi}[-[\cos\theta\cdot 0+\sin\theta\cdot 1]\sin\theta+[\sin\theta\cdot 0-\cos\theta\cdot 1]\cos\theta]d\theta=-2\pi.\end{aligned}$$

(5.1) We have

$$\begin{aligned}\frac{d}{dy}\left(\frac{\alpha x+\beta y}{(ax^2+by^2)^p}\right) &= \frac{a\beta x^2+b\beta(1-2p)y^2-2pb\alpha xy}{(ax^2+by^2)^{p+1}},\\ \frac{d}{dx}\left(\frac{\gamma x+\delta y}{(ax^2+by^2)^p}\right) &= \frac{a\gamma(1-2p)x^2+b\gamma y^2-2pa\delta xy}{(ax^2+by^2)^{p+1}}\end{aligned}$$

The condition for the existence of potential for the differential form is $\beta=\gamma(1-2p)$, $\beta(1-2p)=\gamma$, $pb\alpha=pa\delta$. The solution has several cases.

First $\beta=\gamma$, $p=0$. Then $\omega=\alpha xdx+\delta ydy+\beta(ydx+xdy)$ has potential $\varphi=\frac{\alpha x^2+\delta y^2}{2}+\beta xy+C$.

Second $\beta = \gamma = 0$, $p \neq 0$. We have $\dfrac{\alpha}{a} = \dfrac{\delta}{b}$. Denote the quotient by $\lambda$. Then $\omega = \lambda\dfrac{axdx + bydy}{(ax^2+by^2)^p} = \dfrac{\lambda}{2}\dfrac{d(ax^2+by^2)}{(ax^2+by^2)^p}$ has potential $\varphi = \dfrac{(1-p)\lambda}{2(ax^2+by^2)^{p-1}} + C$ when $p \neq 1$ and $\varphi = \dfrac{\lambda}{2}\log|ax^2+by^2| + C$ when $p = 1$.

Third $\beta = -\gamma$, $p = 1$. We have $\dfrac{\alpha}{a} = \dfrac{\delta}{b}$ again and denote the quotient by $\lambda$. Then $\omega = \dfrac{\lambda(axdx+bydy) + \beta(ydx - xdy)}{ax^2+by^2}$ has potential $\varphi = \dfrac{\lambda}{2}\log|ax^2+by^2| + \beta\psi + C$, where

$$\psi = \begin{cases} \dfrac{1}{\sqrt{ab}}\arctan\dfrac{\sqrt{a}x}{\sqrt{b}y} & \text{if } a>0, b>0 \\ \dfrac{-1}{\sqrt{ab}}\arctan\dfrac{\sqrt{-a}x}{\sqrt{-b}y} & \text{if } a<0, b<0 \\ \dfrac{1}{\sqrt{-ab}}\log\left|\dfrac{\sqrt{a}x - \sqrt{-b}y}{\sqrt{a}x + \sqrt{-b}y}\right| & \text{if } a>0, b<0 \\ \dfrac{-1}{\sqrt{-ab}}\log\left|\dfrac{\sqrt{-a}x - \sqrt{b}y}{\sqrt{-a}x + \sqrt{b}y}\right| & \text{if } a<0, b>0 \end{cases}$$

Note that $\arctan\dfrac{s}{t}$ really means the angle of the vector $(s,t)$, which is not defined on the whole plane, but is really defined on the plane with a line from the origin to the infinity deleted.

(5.2) The $i$-th coordinate of $F = \dfrac{\vec{a}}{\vec{b}\cdot\vec{x}}$ is $f_i = \dfrac{a_i}{\vec{b}\cdot\vec{x}} = \dfrac{a_i}{b_1x_1 + \cdots + b_nx_n}$. Then $\dfrac{\partial f_i}{\partial x_j} = -\dfrac{a_ib_j}{(\vec{b}\cdot\vec{x})^2}$. Therefore the condition for the potential to exist is $a_ib_j = a_jb_i$. In other words, $\vec{a} = \lambda\vec{b}$, and the potential of $F = \lambda\dfrac{\vec{b}}{\vec{b}\cdot\vec{x}}$ is $\varphi = \log|\vec{b}\cdot\vec{x}| + C$.

(6) T, F, F, T, F; F, T, F, T, T; F, F, T, T, F.

**Math 204 Final, Spring 2008**

(1) Suppose $u^3 + xu^2 + yu + z = 0$. Find the absolute maximum and minimum of $u(x,y,z)$ on $x^2+y^2+z^2 \leq 1$.

(2) Suppose $f(x,y)$ is second order differentiable at $(0,0)$ and satisfying $f(0,0) = 0$, $f_y(0,0) \neq 0$. Prove that the function $y(x)$ implicitly given by $f(x,y) = 0$ near $(0,0)$ is also second order differentiable.

(3) Which subsets have volume? Explain.

1. $A = \{(x,y,z)\colon |x| < 1, |y| \leq 2, |z| < 3, x^2 - y \leq z \leq x^2 + y, y \geq z\sin x^4\}$.
2. $B = \{(x,y,z)\colon x \in \mathbb{Q}, z \notin \mathbb{Q}, |x| < 1, |z| < 1, y^2 = x^4 + xz^3 + x^3z + z^4\}$.
3. $C = \{(x,y,z)\colon x \in \mathbb{Q}, z \notin \mathbb{Q}, |x| < 1, |z| < 1, y^2 \leq x^4 + xz^3 + x^3z + z^4\}$.

(4) Compute the integrals.

1. $\displaystyle\int_{x^4+y^4\le 1}(|x|^3+|y|^3)dxdy$.

2. $\displaystyle\int_S (x+z^2)dy\wedge dz$, where $S$ is the parabola $z=x^2+y^2\le 1$ with orientation given by downward pointing normal vector.

(5) Find continuously differentiable $f(x)$, such that $\displaystyle\int_C f(yz)xdx+f(zx)ydy+f(xy)zdz$ depends only on the beginning and end of $C$. Then compute the integral from $(0,0,0)$ to $(1,1,1)$.

**Answer to Math 204 Final, Spring 2008**

(1) By $d(u^3+xu^2+yu+z)=(3u^2+2xu+y)du+u^2dx+udy+dz$, we have $u_z\neq 0$ in $x^2+y^2+z^2<1$. Therefore the absolute extremes must lie on the boundary $x^2+y^2+z^2=1$. The problem becomes the extremes of $f(x,y,z,u)=u$ subject to the constraints $g(x,y,z,u)=u^3+xu^2+yu+z=0$ and $h(x,y,z,u)=x^2+y^2+z^2=1$. At local constrained extremes, either $\nabla h$ and $\nabla g$ are parallel (the hypersurface is not regular), which means

$$\nabla g=(u^2,u,1,3u^2+2xu+y)=\mu\nabla h=2\mu(x,y,z,0)$$

for some $\mu$, or we can use Lagrange multiplier method to get

$$\nabla f=(0,0,0,1)=\lambda\nabla g+\mu\nabla h=\lambda(u^2,u,1,3u^2+2xu+y)+2\mu(x,y,z,0),$$

for some $\lambda,\mu$. In both cases, the constraints $g=0$, $h=1$ are also satisfied. In the first case, we have

$$x=u^2z,\quad y=uz,\quad 3u^2+2xu+y=0,\quad u^3+xu^2+yu+z=0,\quad x^2+y^2+z^2=1.$$

It turns out that such system has no solution. In the second case, we have

$$x=uy,\quad y=uz,\quad u^3+xu^2+yu+z=0,\quad x^2+y^2+z^2=1.$$

This reduces to

$$u^6-u^4-u^2-1=0,\quad (x,y,z)=-\frac{(u^2,u,1)u^3}{u^4+u^2+1}.$$

Let $a$ be the biggest root of $v^3-v^2-v-1=0$ (in fact, the cubic equation has only one root). Then $\sqrt{a}$ and $-\sqrt{a}$ are the absolute maximum and minimum.

(2) The differentiability condition means that there is a quadratic function

$$q(x,y)=a_1x+a_2y+c_{11}x^2+c_{22}y^2+c_{12}xy,$$

such that for any $\epsilon>0$, there is $\delta>0$, such that

$$|x|<\delta,|y|<\delta\implies |f(x,y)-a_1x-a_2y-c_{11}x^2-c_{22}y^2-c_{12}xy|\le\epsilon(|x|+|y|)^2.$$

By implicit function theorem, $g(x)$ is differentiable with $g(0) = 0$ and $g'(0) = -\dfrac{a_1}{a_2}$. Write $g(x) = -\dfrac{a_1}{a_2}x + R(x)$, where $\lim_{x\to 0} \dfrac{R(x)}{x} = 0$, and substitute into the estimation, we get (note that $f(x, g(x)) = 0$)

$$\begin{aligned}&\left|a_1x + a_2\left(-\frac{a_1}{a_2}x + R(x)\right) + c_{11}x^2 + c_{22}\left(-\frac{a_1}{a_2}x + R(x)\right)^2 + c_{12}x\left(-\frac{a_1}{a_2}x + R(x)\right)\right|\\ =&\left|a_2R(x) + \left(c_{11} + c_{22}\frac{a_1^2}{a_2^2}\right)x^2 + \left(c_{12} - 2c_{22}\frac{a_1}{a_2}\right)xR(x) + c_{22}R(x)^2\right|\\ <&\epsilon\left(|x| + \left|-\frac{a_1}{a_2}x + R(x)\right|\right)^2\end{aligned}$$

By $\lim_{x\to 0} \dfrac{R(x)}{x} = 0$, for sufficiently small $\delta$, $|x| < \delta$ will imply $|R(x)| < \epsilon|x|$. Substituting into the estimation above, we get

$$\left|a_2R(x) + \left(c_{11} + c_{22}\frac{a_1^2}{a_2^2}\right)x^2\right| \le \epsilon\left(\left|c_{12} - 2c_{22}\frac{a_1}{a_2}\right| + |c_{22}| + \left(2 + \left|\frac{a_1}{a_2}\right|\right)^2\right)|x|^2.$$

This shows that

$$g(x) = -\frac{a_1}{a_2}x - \frac{1}{a_2}\left(c_{11} + c_{22}\frac{a_1^2}{a_2^2}\right)x^2 + R_2(x),\quad \lim_{x\to 0}\frac{R_2(x)}{x^2} = 0.$$

(3) $A$ is the intersection of

$$\begin{aligned}A_1 &= (-1, 1) \times [-2, 2] \times (-3, 3),\\ A_2 &= \{(x, y, z)\colon (x, y) \in (-1, 1) \times [-2, 2], x^2 - y \le z \le x^2 + y\},\\ A_3 &= \{(x, y, z)\colon (x, z) \in (-1, 1) \times (-3, 3), z\sin x^4 \le y \le 2\}.\end{aligned}$$

$A_1$ has volume because it is a rectangle. $A_2$ has volume because $x^2 - y$ and $x^2 + y$ are integrable functions on $(-1, 1) \times [-2, 2]$. $A_3$ has volume because $z \sin x^4$ is an integrable function on $(-1, 1) \times (-3, 3)$. Therefore the intersection $A$ of the three subsets with volume also has volume.

We have $B \subset B_1 = \{(x, y, z)\colon |x| < 1, |z| < 1, y^2 = x^4 + xz^3 + x^3z + z^4\}$. Note that $B_1$ is part of the boundary of $B_2 = \{(x, y, z)\colon (x, z) \in (-1, 1) \times (-1, 1), -f(x, z) \le y \le f(x, z)\}$, where $f(x, z) = \sqrt{\max\{0, x^4 + xz^3 + x^3z + z^4\}}$ is a continuous function. Since $f$ integrable, $B_2$ has volume. Then the boundary of $B_2$ has volume 0. As a subset of the boundary of $B_2$, $B$ also has volume (and the volume is 0).

If $x, z \ge \epsilon$, then $x^4 + xz^3 + x^3z + z^4 \ge 4\epsilon^4$. By taking $\epsilon = \dfrac{1}{2}$, then we find $(x, y, z) \in [\epsilon, 1)\times[-\epsilon, \epsilon]\times[\epsilon, 1)$ satisfies $y^2 \le x^4 + xz^3 + x^3z + z^4$. Therefore $C \cap [\epsilon, 1) \times [-\epsilon, \epsilon] \times [\epsilon, 1) = C_1 = ([\epsilon, 1) \cap \mathbb{Q}) \times [-\epsilon, \epsilon] \times ([\epsilon, 1) - \mathbb{Q})$. Similar to the argument that $[0, 1] \cap \mathbb{Q}$ has no volume, $C_1$ has no volume. Since $[\epsilon, 1) \times [-\epsilon, \epsilon] \times [\epsilon, 1)$ has volume, $C$ cannot have volume.

(4.1) By symmetry, we have

$$\int_{x^4+y^4\le 1}(|x|^3+|y|^3)dxdy = 8\int_{x^4+y^4\le 1, x\ge 0, y\ge 0} x^3dxdy = 8\int_0^1\left(\int_0^{(1-y^4)^{\frac{1}{4}}} x^3dx\right)dy$$
$$= 2\int_0^1(1-y^4)dy = \frac{8}{5}.$$

(4.2) The surface has parametrization $(x,y,z)=(x,y,x^2+y^2)$ on the disk $x^2+y^2\le 1$. The parametrization is not compactible with the downward orientation. Therefore

$$\int_S (x+z^2)dy\wedge dz = -\int_{x^2+y^2\le 1}(x+z^2)\det\frac{\partial(y,x^2+y^2)}{\partial(x,y)}dxdy$$
$$= \int_{x^2+y^2\le 1}(x+(x^2+y^2)^2)2xdxdy = 2\int_{x^2+y^2\le 1}x^2dxdy$$
$$= 2\int_0^1\int_0^{2\pi} r^3\cos^2\theta drd\theta = \frac{\pi}{2}.$$

Alternatively, we may use Gauss Theorem. Let $A$ be the region between the parabola $z=x^2+y^2$ and the plane $z=1$. Then the boundary $A$ has two parts: $S$ and the disk $D$ given by $x^2+y^2\le 1$ and $z=1$. The outward normal vector of the boundary is downward on $S$ and is $(0,0,1)$ on $D$. Then

$$\int_S (x+z^2)dy\wedge dz = \int_A (x+z^2)_x dxdydz - \int_D (x+z^2)dy\wedge dz = \int_A dxdydz$$

is the volume of $A$. The section of $A$ at $z$ is a disk of radius $\sqrt{z}$ and has area $\pi z$. Therefore the volume of $A$ is

$$\int_S (x+z^2)dy\wedge dz = \int_0^1 \pi z dz = \frac{\pi}{2}.$$

(5) The condition for the independence of the integral on the path is

$$(f(yz)x)_y = (f(zx)y)_x, \quad (f(zx)y)_z = (f(xy)z)_y, \quad (f(xy)z)_x = (f(yz)x)_z.$$

This is the same as

$$f'(yz)xz = f'(zx)yz, \quad f'(zx)xy = f'(xy)xz, \quad f'(xy)yz = f'(yz)xy.$$

This is further equivalent to $\dfrac{f'(t)}{t}$ being constant. Therefore the integral is independent of the path if and only if $f(t)=at^2+b$.

If $f(t)=at^2+b$, then

$$f(yz)xdx + f(zx)ydy + f(xy)zdz = axyz(yzdx+zxdy+xydz) + b(xdx+ydy+zdz)$$
$$= \frac{a}{2}d(x^2y^2z^2) + \frac{b}{2}d(x^2+y^2+z^2),$$

and

$$\int_{(0,0,0)}^{(1,1,1)} f(yz)xdx+f(zx)ydy+f(xy)zdz = \left(\frac{a}{2}(x^2y^2z^2) + \frac{b}{2}(x^2+y^2+z^2)\right)_{(0,0,0)}^{(1,1,1)} = \frac{a+3b}{2}.$$

**Math 204 Final, Spring 2010**

1. (20 points) Suppose $f(x)$ is Riemann integrable on $[0,1]$. Prove that the function $g(x,y) = f(x^2+y^2)$ is Riemann integrable on $x^2+y^2 \leq 1$. Moreover,

$$\int_{x^2+y^2\leq 1} g(x,y)\,\mathrm{d}\mu = \pi \int_0^1 f(\tau)\,\mathrm{d}\tau.$$

Solution: We can partition the unit disk by a family of eccentric rings $\{R_i\}$, characterizing by their radii:

$$P : 0 = r_0 < r_1 < .... < r_n = 1.$$

In this partition, the Riemann sum of the oscillation is

$$\sum \omega_g(R_i)\mu(I_i)$$

that is the same as

$$\sum \omega_f([r_{i-1}, r_i]) \cdot \pi(r_i - r_{i-1}).$$

By the condition, the later goes to zero as $\max_i(r_i - r_{i-1}) \to 0$. Thus, the function $g$ is Riemann integrable on the unit disk. By Fubini theorem, we have

$$\int_{x^2+y^2\leq 1} g(x,y)\,\mathrm{d}\mu = \int_0^{2\pi}\int_0^1 f(r^2)r\mathrm{d}r\mathrm{d}\theta = \pi\int_0^1 f(\tau)\,\mathrm{d}\tau.$$

2. (20 points) Use the Lagrange multipliers to find the shortest distance from the straight line $3x+4y=10$ to the unit circle $x^2+y^2=1$.

Solution:

Let $(x,y)$ be on the unit circle $x^2+y^2=1$ and $(X,Y)$ be the point on the straight line $3X+4Y=10$ such that the distance between them is the shortest. This is equivalent to minimize the function $f(x,y,X,Y) = (X-x)^2+(Y-y)^2$. By the Lagrange Multipliers, we have

$$(-2(X-x), -2(Y-y), 2(X-x), 2(Y-y)) = \lambda(2x, 2y, 0, 0) + \mu(0, 0, 3, 4).$$

Together with the two equations, we have the following system

$$\begin{cases} -2(X-x) = \lambda \cdot 2x \\ -2(Y-y) = \lambda \cdot 2y \\ 2(X-x) = 3\mu \\ 2(Y-y) = 4\mu \\ x^2+y^2 = 1 \\ 3X+4Y = 10. \end{cases}$$

Since $|x| \leq 1$ and $|y| \leq 1$, we know that the straight line has no intersection with the circle. Hence, $X \neq x$ and $Y \neq y$. These imply $\lambda \neq 0$ and $\mu \neq 0$. The first four equations give

$$x = \left(-\frac{3}{2}\right) \cdot \frac{\mu}{\lambda}, \qquad y = (-2) \cdot \frac{\mu}{\lambda},$$

The unit circle gives

$$\left(\frac{\mu}{\lambda}\right)^2 = \frac{4}{25}.$$

Thus,

$$X = (1-\lambda)x = (1-\lambda)\left(-\frac{3}{2}\right) \cdot \frac{\mu}{\lambda}, \qquad Y = (1-\lambda)y = (1-\lambda)(-2) \cdot \frac{\mu}{\lambda}.$$

Substituting these into the straight line equation gives

$$\lambda = 1 + \frac{4}{5}\frac{\lambda}{\mu}.$$

Thus,

$$\mu = \frac{\mu}{\lambda} \cdot \lambda = \frac{\mu}{\lambda}\left(1 + \frac{4}{5}\frac{\lambda}{\mu}\right).$$

In summary, we have

$$\begin{aligned} x &= \left(-\frac{3}{2}\right) \cdot \frac{\mu}{\lambda}, \qquad & y &= (-2) \cdot \frac{\mu}{\lambda}, \\ X &= \frac{6}{5}, \qquad & Y &= \frac{8}{5} \\ \lambda &= 1 + \frac{4}{5}\frac{\lambda}{\mu}, \qquad & \mu &= \frac{\mu}{\lambda} + \frac{4}{5}. \end{aligned}$$

where

$$\left(\frac{\mu}{\lambda}\right)^2 = \frac{4}{25}.$$

These gives two sets of points:

$$(x, y) = (3/5, 4/5), \qquad (X, Y) = (6/5, 8/5)$$

with the distance $D = \sqrt{(3/5)^2 + (4/5)^2} = 1$; and

$$(x, y) = (-3/5, -4/5), \qquad (X, Y) = (6/5, 8/5)$$

with the distance $D = \sqrt{(9/5)^2 + (12/5)^2} = 3$. Therefore, the shortest distance between the straight line is 1.

3. (20 points) Suppose a change of variable $x = x(u, v), y = y(u, v)$ maps a Jordan measurable region $D \subset \mathbb{R}^2$ to $\Omega \subset \mathbb{R}^2$. If the transform satisfies

$$\frac{\partial x}{\partial u} = \frac{\partial y}{\partial v}, \qquad \frac{\partial x}{\partial v} = -\frac{\partial y}{\partial u},$$

prove:

$$\int_D \left[\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2\right] \mathrm{d}\mu_{(x,y)} = \int_\Omega \left[\left(\frac{\partial f}{\partial u}\right)^2 + \left(\frac{\partial f}{\partial v}\right)^2\right] \mathrm{d}\mu_{(u,v)}$$

Solution: Since

$$\begin{aligned}
\left(\frac{\partial f}{\partial u}\right)^2 + \left(\frac{\partial f}{\partial v}\right)^2 &= \left(\frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u}\right)^2 + \left(\frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v}\right)^2 \\
&= \left(\frac{\partial f}{\partial x}\right)^2\left(\frac{\partial x}{\partial u}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2\left(\frac{\partial y}{\partial u}\right)^2 + 2\frac{\partial f}{\partial x}\frac{\partial x}{\partial u}\frac{\partial f}{\partial y}\frac{\partial y}{\partial u} \\
&+ \left(\frac{\partial f}{\partial x}\right)^2\left(\frac{\partial x}{\partial v}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2\left(\frac{\partial y}{\partial v}\right)^2 + 2\frac{\partial f}{\partial x}\frac{\partial x}{\partial v}\frac{\partial f}{\partial y}\frac{\partial y}{\partial v} \\
&= \left[\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2\right]\left[\left(\frac{\partial x}{\partial u}\right)^2 + \left(\frac{\partial y}{\partial v}\right)^2\right] \\
&= \left[\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2\right]\left|\left[\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}\right]\right| \\
&= \left[\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2\right]\left|\det\frac{\partial(x,y)}{\partial(u,v)}\right|,
\end{aligned}$$

by the Theorem of Change of Variables, we have

$$\begin{aligned}
&\int_D \left[\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2\right] \mathrm{d}\mu_{(x,y)} \\
= &\int_\Omega \left[\left(\frac{\partial f}{\partial u}\right)^2 + \left(\frac{\partial f}{\partial v}\right)^2\right]\left|\det\frac{\partial(x,y)}{\partial(u,v)}\right|^{-1}\cdot\left|\det\frac{\partial(x,y)}{\partial(u,v)}\right| \mathrm{d}\mu_{(u,v)} \\
= &\int_\Omega \left[\left(\frac{\partial f}{\partial u}\right)^2 + \left(\frac{\partial f}{\partial v}\right)^2\right] \mathrm{d}\mu_{(u,v)}
\end{aligned}$$

4. (20 points) Show that the curve integral

$$I = \int_C [f(x) + ay + bz]\,\mathrm{d}x + [ax + g(y) + cz]\,\mathrm{d}y + [bx + cy + h(z)]\,\mathrm{d}z$$

is independent of path. Evaluate the integral from $(0,0,0)$ to $(X,Y,Z)$. Verify your answer by using a potential function $\varphi$.

Solution: It can be easily seen that

$$\begin{aligned}
&\mathrm{d}\{[f(x) + ay + bz]\,\mathrm{d}x + [ax + g(y) + cz]\,\mathrm{d}y + [bx + cy + h(z)]\,\mathrm{d}z\} \\
= &(a\,\mathrm{d}y \wedge \mathrm{d}x + b\,\mathrm{d}z \wedge \mathrm{d}x) + (a\,\mathrm{d}x \wedge \mathrm{d}y + c\,\mathrm{d}z \wedge \mathrm{d}y) + (b\,\mathrm{d}x \wedge \mathrm{d}z + c\,\mathrm{d}y \wedge \mathrm{d}z) \\
= &0.
\end{aligned}$$

By Stokes Theorem, we know that the curve integral is independent of path.

The integral can be evaluate by integrating along successive straight lones connecting $(0,0,0),(X,0,0),(X,Y,0),(X,Y,Z)$. Thus, the curve integral equals

$$\begin{aligned}
& \int_{(0,0,0)}^{(X,0,0)} + \int_{(X,0,0)}^{(X,Y,0)} + \int_{(X,Y,0)}^{(X,Y,Z)} [f(x)+ay+bz]\,\mathrm{d}x + [ax+g(y)+cz]\,\mathrm{d}y + [bx+cy+h(z)]\,\mathrm{d}z \\
= & \int_0^X [f(x)]\,\mathrm{d}x + \int_0^Y [aX+g(y)]\,\mathrm{d}y + \int_0^Z [bX+cY+h(z)]\,\mathrm{d}z \\
= & aXY+bXZ+cYZ+\int_0^X f(x)\,\mathrm{d}x + \int_0^Y g(y)\,\mathrm{d}y + \int_0^Z h(z)\,\mathrm{d}z.
\end{aligned}$$

A potential function $\varphi$ satisfies

$$\nabla\varphi = (f(x)+ay+bz, ax+g(y)+cz, bx+cy+h(z)).$$

Thus, from

$$\varphi_x = f(x)+ay+bz,$$

we have

$$\varphi = axy+bxz+\int_0^x f(x)\,\mathrm{d}x + \psi(y,z).$$

From $\varphi_y = ax+g(y)+cz$, we have

$$\psi_y = cz+g(y),$$

which gives

$$\psi = cyz + \int_0^y g(y)\,\mathrm{d}y + \chi(z).$$

At last, from $\varphi_z = bx+cy+h(z)$, we have

$$\chi_z = h(z),$$

which gives

$$\chi = \int_0^z h(z)\,\mathrm{d}z + C.$$

Therefore, the potential function is

$$\varphi = C+axy+bxz+cyz+\int_0^x f(x)\,\mathrm{d}x + \int_0^y g(y)\,\mathrm{d}y + \int_0^z h(z)\,\mathrm{d}z.$$

It is obvious that

$$\varphi(X,Y,Z)-\varphi(0,0,0) = I.$$

5. (20 points) Show that the area $\Omega$ enclosed by the curve

$$Ax^2+2Bxy+Cy^2=1, \qquad A>0, \quad AC-B^2>0$$

is given by the curve integral

$$I = \frac{1}{2} \oint_{x^2+y^2=R^2} \frac{x\,\mathrm{d}y - y\,\mathrm{d}x}{Ax^2 + 2Bxy + Cy^2}.$$

What is the area of $\Omega$?

Solution: Since, for any $(xy) \neq (0,0)$,

$$Ax^2 + 2Bxy + Cy^2 = A\left(x + \frac{B}{A}y\right)^2 + \frac{1}{A}(AC - B^2)y^2 > 0,$$

we know that the curve integral is a well-defined regular integral. The area of the region $\Omega$ is

$$\text{Area} = \int_\Omega \mathrm{d}\mu_{(x,y)}$$

By a change of variables $x = r\cos\theta, y = r\sin\theta$, since the curve is given by

$$r = (A\cos^2\theta + 2B\cos\theta\,\sin\theta + C\sin^2\theta)^{-1/2}, \qquad 0 \le \theta < 2\pi.$$

we have

$$\begin{aligned}
\text{Area} &= \int_0^{2\pi}\int_0^{(A\cos^2\theta+2B\cos\theta\,\sin\theta+C\sin^2\theta)^{-1/2}} r\,\mathrm{d}r\mathrm{d}\theta \\
&= \frac{1}{2}\int_0^{2\pi} \frac{\mathrm{d}\theta}{A\cos^2\theta + 2B\cos\theta\,\sin\theta + C\sin^2\theta}.
\end{aligned}$$

The fact that the integral gives the area of $\Omega$ can be shown by the Green's Theorem. In fact, since

$$\left(\frac{x}{Ax^2 + 2Bxy + Cy^2}\right)_x = \left(\frac{-y}{Ax^2 + 2Bxy + Cy^2}\right)_y$$

we know that the curve $x^2 + y^2 = R^2$ can be replaced by $Ax^2 + 2Bxy + Cy^2 = 1$. Hence,

$$\begin{aligned}
I &= \frac{1}{2}\oint_{Ax^2+2Bxy+Cy^2=1} \frac{x\,\mathrm{d}y - y\,\mathrm{d}x}{Ax^2 + 2Bxy + Cy^2} \\
&= \frac{1}{2}\oint_{Ax^2+2Bxy+Cy^2=1} x\,\mathrm{d}y - y\,\mathrm{d}x \\
&= \frac{1}{2}\int_\Omega (1+1)d\mu_{(x,y)} = \text{Area of } \Omega.
\end{aligned}$$

To find the area of $\Omega$, notice that the curve $Ax^2+2Bxy+Cy^2 = 1$ can be re-written as

$$A\left(x + \frac{B}{A}y\right)^2 + \left(C - \frac{B^2}{A}\right)y^2 = 1.$$

So, if we make a change of the variables:

$$u = \sqrt{A}\left(x + \frac{B}{A}y\right), \qquad v = \sqrt{C - \frac{B^2}{A}}y,$$

we have

$$\begin{aligned}\text{Area of } \Omega &= \int\int_{Ax^2+2Bxy+Cy^2\le 1} \mathrm{d}x\,\mathrm{d}y \\ &= \int\int_{u^2+v^2\le 1} \left|\det \frac{\partial(x,y)}{\partial(u,v)}\right| \mathrm{d}u\,\mathrm{d}v\end{aligned}$$

Since

$$\det\frac{\partial(u,v)}{\partial(x,y)} = \det\begin{pmatrix} \sqrt{A} & \sqrt{A}B \\ 0 & \sqrt{C - \frac{B^2}{A}} \end{pmatrix} = \sqrt{AC - B^2}$$

we know that

$$\det\frac{\partial(x,y)}{\partial(u,v)} = \left(\det\frac{\partial(u,v)}{\partial(x,y)}\right)^{-1} = \frac{1}{\sqrt{AC-B^2}} > 0.$$

We obtain the area of $\Omega$:

$$\text{Area of } \Omega = \frac{1}{\sqrt{AC-B^2}} \cdot \int\int_{u^2+v^2\le 1} \mathrm{d}u\,\mathrm{d}v = \frac{\pi}{\sqrt{AC-B^2}}.$$

**Math 204 Final, Spring 2010**

(1) (20 points) Consider functions

$$f(x,y) = \begin{cases} \frac{1}{p} & \text{if } x = \frac{k}{p}, y = \frac{l}{p}, p \text{ prime} \\ 0 & \text{otherwise} \end{cases}, \qquad g(x,y) = \begin{cases} 1 & \text{if } x - y \in \mathbb{Q} \\ 0 & \text{if } x - y \notin \mathbb{Q} \end{cases}.$$

Show that $f$ is Riemann integrable on $[0,1]\times[0,1]$. Show that $g$ is not Riemann integrable on $[0,1]\times[0,1]$ but is Lebesgue integrable on $[0,1]\times[0,1]$.

(2) (20 points) Find continuously differentiable $\phi(x)$, such that the integral $\int(x\phi(y)dx + y\phi(x)dy)$ depends only on the beginning and end of paths. Then compute the integral from $(x_0, y_0)$ to $(x_1, y_1)$.

(3) (20 points) Suppose a curve $C$ is implicitly given by $F(x,y) = 0$. Suppose the direction of the curve is chosen in such a way that $F > 0$ on the left of $C$ and $F < 0$ on the right of $C$. Express the (second type) integral $\int_C f dx + g dy$ as the (first type) integral $\int_C \phi ds$ for suitable $\phi$.

(4) (20 points) Suppose $f(x,y)$ is a function on $[a,b]\times[c,d]$ that is Riemann integrable in $x$ and is increasing in $y$.

1. Prove that $f$ is bounded and $\omega_{[a,b]\times[c,d]}f = \max_{[a,b]} f(x,d) - \min_{[a,b]} f(x,c)$.

2. Let $P$ and $Q$ be partitions of $[a,b]$ and $[c,d]$ with $\Delta x_i = \delta$ for $P$. Prove that

$$\sum_{I\in P\times Q} \omega_I(f)\mu(I) \le \left( U(P, f(x,d)) - L(P, f(x,c)) + \sum_{j=1}^{n-1}\sum_{i=1}^{m} \omega_{[x_{i-1},x_i]}(f(x,y_j))\Delta x_i \right) \|Q\|.$$

3. Prove that $f$ is Riemann integrable.

(5) (20 points) A subset of $\mathbb{R}$ (or of $\mathbb{R}^n$ in general) is a $G_\delta$-set if it is the intersection of countably many open subsets. A subset is an $F_\sigma$-set if it is the union of countably many closed subsets.

1. Show that $G_\delta$-sets and $F_\sigma$-sets are Lebesgue measurable.

2. Prove that a (bounded) subset $A$ is Lebesgue measurable if and only if there is a $G_\delta$-set $X$ and $F_\sigma$-set $Y$, such that $Y \subset A \subset X$, and $\mu(X - Y) = 0$.

**Answer to Math 204 Final, Spring 2010**

(1.1) For any $\epsilon > 0$, there are only finitely many $(x,y) \in A = [0,1]\times[0,1]$, such that $|f(x,y)| \ge \epsilon$. Let $N_\epsilon$ be the number of such $(x,y)$. Then for any partition $P$ of $A$ with $\|P\| < \sqrt{\dfrac{\epsilon}{N_\epsilon}}$, each $I \subset P$ satisfies

$$\mu(I) \le \|P\|^2 \le \frac{\epsilon}{N_\epsilon},$$

we always have $\omega_I(f) \le 1$, and we get

$$\begin{aligned}
\sum_{I\in P} \omega_I(f)\mu(I) &\le \sum_{|f(x,y)|\ge\epsilon \text{ somewhere on } I} \omega_I(f)\mu(I) + \sum_{|f(x,y)|<\epsilon \text{ everywhere on } I} \omega_I(f)\mu(I) \\
&\le N_\epsilon \cdot 1 \cdot \frac{\epsilon}{N_\epsilon} + \sum_{|f(x,y)|<\epsilon \text{ on } I} 2\epsilon \cdot \mu(I) \\
&\le \epsilon + 2\epsilon\mu(A) = 3\epsilon.
\end{aligned}$$

This proves that $f$ is Riemann integrable.
(1.2) Let

$$A = \{(x,y) \in [0,1]^2 \colon x - y \in \mathbb{Q}\}, \quad B = \{(x,y) \in [0,1]^2 \colon x - y \notin \mathbb{Q}\} = [0,1]^2 - A.$$

Then both $A$ and $B$ are dense in $[0,1]^2$. Therefore for any partition $P$ of $[0,1]^2$ by finitely many rectangles, any $I$ in the partition satisfies $A\cap I \ne \emptyset$ and $B \cap I \ne \emptyset$. Then $\omega_I(g) = 1$, and $\sum_{I\in P}\omega_I(g)\mu(I) = \sum_{I\in P}\mu(I) = \mu([0,1]^2) = 1$. This shows that $g$ is not Riemann integrable.

On the other hand,

$$A = \cup_{r\in\mathbb{Q}}\{(x,y) \in [0,1]^2 \colon x-y = r\} = \cup_{r\in\mathbb{Q}}\{(x,x-r) \in [0,1]^2 \colon x \in \mathbb{R}\} = \cup_{r\in\mathbb{Q}} L_r \cap [0,1]^2,$$

where $L_r$ is the straight line of angle 45 degrees and passing through $(r,0)$. Then $L_r$ is Lebesgue measurable with measure zero. As a countable union of parts of $L_r$, $A$ is also Lebesgue measurable with measure zero. Since $g=0$ outside $A$, $g$ is Lebesgue integrable with $\int g = 0$.

(2) The condition is $(x\phi(y))_y = (y\phi(x))_x$, which is the same as $\dfrac{\phi'(x)}{x}$ being a constant. Then $\phi'(x) = cx$ means exactly $\phi(x) = \dfrac{1}{2}cx^2 + b = ax^2 + b$, and the integral

$$\begin{aligned}\int_{(x_0,y_0)}^{(x_1,y_1)} (x\phi(y)dx + y\phi(x)dy) &= \int_{(x_0,y_0)}^{(x_1,y_1)} a(xy^2dx + x^2ydy) + b(xdx + ydy) \\ &= \int_{(x_0,y_0)}^{(x_1,y_1)} \frac{a}{2}d(x^2y^2) + \frac{b}{2}d(x^2+y^2) \\ &= \frac{a}{2}(x_1^2y_1^2 - x_0^2y_0^2) + \frac{b}{2}d(x_1^2 + y_1^2 - x_0^2 - y_0^2).\end{aligned}$$

(3) We have $\int_C fdx + gdy = \int_C (f,g)\cdot d\vec{x} = \int_C (f,g)\cdot \vec{x}'(s)ds = \int_C (f,g)\cdot Tds$, where $T$ is the unit length tangent vector of $C$.

Since $C$ is given by $F(x,y)=0$, the gradient $\nabla F = (F_x, F_y)$ is orthogonal to $C$. Moreover, the direction of $\nabla F$ is pointing towards the direction where $F$ is increasing, which is the left of $C$. Therefore the unit length vector $\dfrac{\nabla F}{\|\nabla F\|_2} = \dfrac{(F_x,F_y)}{\sqrt{F_x^2+F_y^2}}$ is obtained by rotating the tangent vector $T$ counterclockwise by 90 degrees. Equivalently, $T$ is obtained by rotating $\dfrac{\nabla F}{\|\nabla F\|_2}$ clockwise by 90 degrees. Therefore $T = \dfrac{(F_y, -F_x)}{\sqrt{F_x^2+F_y^2}}$, and we get

$$\int_C fdx + gdy = \int_C (f,g)\cdot Tds = \int_C \frac{fF_y - gF_x}{\sqrt{F_x^2+F_y^2}}ds.$$

(4.1) By the increasing property in $y$, we have

$$f(x,c) \le f(x,y) \le f(x,d)$$

for any $x \in [a.b]$. Since $f(x,c)$ and $f(x,d)$ are Riemann integrable on $[a,b]$. The two functions are bounded, and we have

$$\inf_{[a,b]} f(x,c) \le f(x,c) \le f(x,y) \le f(x,d) \le \sup_{[a,b]} f(x,d).$$

Therefore $f$ is bounded. Moreover, the inequality also shows $\omega_{[a,b]\times[c,d]}f = \sup_{[a,b]} f(x,d) - \inf_{[a,b]} f(x,c)$.

(4.2) By the first part, we have

$$\omega_{[x_{i-1},x_i]\times[y_{j-1},y_j]}(f) = \sup_{[x_{i-1},x_i]} f(x,y_j) - \inf_{[x_{i-1},x_i]} f(x,y_{j-1}).$$

Then

$$
\begin{aligned}
\sum_{I\in P\times Q}\omega_I(f)\mu(I) &= \sum_{i,j=1}^{m,n}\left(\sup_{[x_{i-1},x_i]} f(x,y_j) - \inf_{[x_{i-1},x_i]} f(x,y_{j-1})\right)\Delta x_i\Delta y_j \\
&\le \sum_{i=1}^{m}\left(\sum_{j=1}^{n}\sup_{[x_{i-1},x_i]} f(x,y_j) - \sum_{j=1}^{n}\inf_{[x_{i-1},x_i]} f(x,y_{j-1})\right)\delta\|Q\| \\
&\le \sum_{i=1}^{m}\left(\sum_{j=1}^{n}\sup_{[x_{i-1},x_i]} f(x,y_j) - \sum_{j=0}^{n-1}\inf_{[x_{i-1},x_i]} f(x,y_j)\right)\delta\|Q\| \\
&= \sum_{i=1}^{m}\left(\sup_{[x_{i-1},x_i]} f(x,d) + \sum_{j=1}^{n-1}\left(\sup_{[x_{i-1},x_i]} f(x,y_j) - \inf_{[x_{i-1},x_i]} f(x,y_j)\right) - \inf_{[x_{i-1},x_i]} f(x,c)\right)\delta\|Q\| \\
&= \left(U(P,f(x,d)) + \sum_{j=1}^{n-1}\sum_{i=1}^{m}\omega_{[x_{i-1},x_i]}(f(x,y_j))\Delta x_i - L(P,f(x,c))\right)\|Q\|.
\end{aligned}
$$

(4.3) Let $|f| < M$ on $[a,b]\times[c,d]$. Then $|U(P,f(x,d))| \le M(b-a)$ and $|L(P,f(x,c))| \le M(b-a)$. For any $\epsilon > 0$, we fix $Q$ satisfying $\|Q\| < \epsilon$. Moreover, for each of finitely many partition points $y_j$ of $Q$, the Riemann integrability of $f(x,y_j)$ means that there is sufficiently small partition $P$ of $[a,b]$, such that all $\Delta x_i = \delta$ are the same, and

$$
\sum_{i=1}^{m}\omega_{[x_{i-1},x_i]}(f(x,y_j))\Delta x_i < \frac{1}{n}, \quad j = 1,2,\dots,n-1.
$$

By the second part, we have

$$
\sum_{I\in P\times Q}\omega_I(f)\mu(I) \le (2M(b-a)+1)\|Q\| < (2M(b-a)+1)\epsilon.
$$

(5) We know open and closed subsets are Lebesgue measurable. Therefore their countable intersections and countable unions are also Lebesgue integrable. In particular, the $G_\delta$-sets and $F_\sigma$-sets are Lebesgue measurable.

Given any bounded subset $A$, by the definition of outer measure, we have open subsets $U_i \supset A$, such that $\mu^*(A) = \lim\mu(U_i)$. Then $X = \cap U_i$ is a $G_\delta$-set satisfying $A \subset X \subset U_i$ for any $i$. The inclusion and the fact that $X$ is measurable implies that $\mu^*(A) \le \mu(X) \le \mu(U_i)$, taking the limit as $i\to\infty$, we get $\mu^*(A) = \mu(X)$.

By similar argument, we can find an $F_\sigma$-set $Y$, such that $Y \subset A$ and $\mu(Y) = \mu_*(A)$. Then

$$
A \text{ measurable} \iff \mu_*(A) = \mu^*(A) \iff \mu(X) = \mu(Y) \iff \mu(Y - X) = 0.
$$

In the last step, we use $Y \subset X$.

Conversely, if there are $G_\delta$-set $X$ and $F_\sigma$-set $Y$, such that $Y \subset A \subset X$ and $\mu(X-Y) = 0$ ($X$ and $Y$ are not necessarily constructed as above), then $A - Y \subset X - Y$ and the completeness of the Lebesgue measure implies $A - Y$ is also Lebesgue measurable. Then by $Y$ measurable, we get $A- = Y \cup (A - Y)$ measurable.

**Math 3043 Final, Spring 2012**

(1) A sequence $A_i$ is almost increasing if for each $i$, $A_i - A_{i+1}$ is contained in a measurable subset of measure zero. Prove that if $A_i$ are measurable and almost increasing, then $\lim \mu(A_i) = \mu(\cup A_i)$.

(2) Suppose $f_n$ are measurable, $f_1 + \cdots + f_n \geq 0$, $\sum f_n$ converges, and $\sum \int_X f_n d\mu$ converges. Prove that $\int_X (\sum f_n) d\mu \leq \sum \int_X f_n d\mu$. Moreover, show that the strict inequality may happen.

(3) Suppose $\mu_1^*$ and $\mu_2^*$ are outer measures.

1. Prove that $\mu_1^* + \mu_2^*$ is an outer measure.
2. Prove that if a subset is $\mu_1^*$-measurable and $\mu_2^*$-measurable, then the subset is $(\mu_1^* + \mu_2^*)$-measurable.
3. Suppose $\mu_1^*(X), \mu_2^*(X) < +\infty$. Prove that if a subset is $(\mu_1^* + \mu_2^*)$-measurable, then the subset is $\mu_1^*$-measurable and $\mu_2^*$-measurable.
4. Show that the finite assumption in the third part is necessary.

(4) Let $f(x)$ be a non-negative function on $[0, +\infty)$. Let $\mathcal{C} = \{[a, b) \colon a \leq b\}$ and $\lambda[a, b) = f(b - a)$.

1. Prove that if $f(x) \geq x$, $f(0) = 0$, $f'_+(0) = 1$, then $\mathcal{C}$ and $\lambda$ induce the Lebesgue outer measure.
2. Prove that if $f(x) \geq c$ for a constant $c$ and all $x > 0$, then the only subsets measurable with respect to the outer measure induced by $\mathcal{C}$ and $\lambda$ are $\emptyset$ and $\mathbb{R}$.

**Answer to Math 3043 Final, Spring 2012**

(1)

$B_n = \cup_{i=1}^n A_i$ is an increasing sequence, such that $\cup_{n=1}^\infty B_n = \cup_{i=1}^\infty A_i$. Therefore by monotone limit property, we have $\mu(\cup_{i=1}^\infty A_i) = \lim_{n\to\infty} \mu(B_n)$.

On the other hand, we have $A_n \subset B_n = (A_1 - A_2) \cup (A_2 - A_3) \cup \cdots \cup (A_{n-1} - A_n) \cup A_n$. Therefore

$$B_n - A_n = (A_1 - A_2) \cup (A_2 - A_3) \cup \cdots \cup (A_{n-1} - A_n).$$

Since each $A_i - A_{i+1}$ is contained in a subset of measure 0, the right side has measure 0, and we get $\mu(B_n) = \mu(A_n)$. Therefore $\mu(\cup A_i) = \lim \mu(B_n) = \lim \mu(A_n)$.

(2)

Let $g_n = f_1 + \cdots + f_n$. Then $\lim_{n\to\infty} g_n = \lim_{n\to\infty} \sum_{i=1}^n f_i = \sum_{i=1}^\infty f_i$ converges, $g_n \geq 0$, and $\lim_{n\to\infty} \int_X g_n d\mu = \lim_{n\to\infty} \sum_{i=1}^n \int_X f_i d\mu = \sum_{i=1}^\infty \int_X f_i d\mu$ also converges. By Fatou Lemma, we have

$$\int_X (\sum f_i) d\mu = \int_X (\lim_{n\to\infty} g_n) d\mu = \int_X (\varliminf_{n\to\infty} g_n) d\mu \leq \varliminf_{n\to\infty} \int_X g_n d\mu = \lim_{n\to\infty} \int_X g_n d\mu = \sum \int_X f_i d\mu.$$

The equality may not hold. For example, let $f_1 = \chi_{[0,1)}$, $f_n = \chi_{[n-1,n)} - \chi_{[n-2,n-1)}$. Then $f_1 + \cdots + f_n = \chi_{[n-1,n)} \geq 0$, $\sum f_i = 0$, and

$$\int_{\mathbb{R}} (\sum f_i) d\mu = 0, \quad \sum \int_{\mathbb{R}} f_i d\mu = 1 + 0 + 0 + \cdots = 1.$$

(3.1)

Let $\mu^* = \mu_1^* + \mu_2^*$. Then

1. By $\mu_1^*(\emptyset) = \mu_2^*(\emptyset) = 0$, we get $\mu^*(\emptyset) = \mu_1^*(\emptyset) + \mu_2^*(\emptyset) = 0$.
2. For any $A$, we have $\mu_1^*(A) \geq 0$, $\mu_2^*(A) \geq 0$. Then we get $\mu^*(A) = \mu_1^*(A) + \mu_2^*(A) \geq 0$.
3. For a countable union $\cup_i A_i$, we have $\mu_1^*(\cup A_i) \leq \sum \mu_1^*(A_i)$, $\mu_2^*(\cup A_i) \leq \sum \mu_2^*(A_i)$. Then we get $\mu^*(\cup A_i) = \mu_1^*(\cup A_i) + \mu_2^*(\cup A_i) \leq \sum \mu_1^*(A_i) + \sum \mu_2^*(A_i) = \sum(\mu_1^*(A_i) + \mu_2^*(A_i)) = \sum \mu^*(A_i)$.

This verifies that $\mu^*$ is an outer measure.

(3.2)

For any $Y$, we have

$$\mu_1^*(Y) \leq \mu_1^*(Y \cap A) + \mu_1^*(Y - A), \quad \mu_2^*(Y) \leq \mu_2^*(Y \cap A) + \mu_2^*(Y - A).$$

This gives

$$\begin{aligned} \mu^*(Y) &= \mu_1^*(Y) + \mu_2^*(Y) \\ &\leq \mu_1^*(Y \cap A) + \mu_2^*(Y \cap A) + \mu_1^*(Y - A) + \mu_2^*(Y - A) \\ &= \mu^*(Y \cap A) + \mu^*(Y - A). \end{aligned}$$

In case $A$ is $\mu_1^*$-measurable and $\mu_2^*$-measurable, the two inequalities for $\mu_1^*$ and $\mu_2^*$ become equalities, and we get equality for $\mu^*$. Therefore $A$ is $\mu^*$-measurable.

(3.3)

In case $A$ is $\mu^*$-measurable, we have $\mu^*(Y) = \mu^*(Y \cap A) + \mu^*(Y - A)$ for all $Y$. By $\mu^*(Y) \leq \mu^*(X) = \mu_1^*(X) + \mu_2^*(X) < +\infty$ and the non-negativity of all the terms, we conclude that the inequalities for $\mu_1^*$ and $\mu_2^*$ must become equalities. Therefore $A$ is $\mu_1^*$-measurable and $\mu_2^*$-measurable.

(3.4)

If $\mu^*(X) = +\infty$, then the claim may not be true. For example, let $\mu_1^*(A) = +\infty$ for $A \neq \emptyset$ and $\mu_1^*(\emptyset) = 0$. Then for any outer measure $\mu_2^*$, we have $\mu_1^* + \mu_2^* = \mu_1^*$. Now any subset is measurable with respect to $\mu_1^*$. On the other hand, we have examples of outer measure $\mu_2^*$ such that not all subsets are measurable.

(4.1)

Let $\mu^*$ be the measure induced by $\mathcal{C} = \{[a,b)\}$ and $\lambda[a,b) = f(b-a)$. Let $\mu_L^*$ be the usual Lebesgue outer measure, which is induced by $\mathcal{C}_L = \{(a,b)\}$ and $\lambda_L(a,b) = b - a$. We only need to show that the two outer measures are the same. By Proposition 8.3.2, this means that $\mu^*(a,b) \leq \lambda_L(a,b) = b - a$ and $b - a = \mu_L^*[a,b) \leq \lambda[a,b) = f(b-a)$.

The inequality $b - a \le f(b-a)$ follows from the assumption $f(x) \ge x$. To prove $\mu^*(a,b) \le b - a$, we start with

$$(a,b) \subset [a,b) = \sqcup_{i=1}^n [a+(i-1)\epsilon, a+i\epsilon), \quad \epsilon = \frac{b-a}{n}.$$

The containment implies that

$$\mu^*(a,b) \le \sum_{i=1}^n \lambda[a+(i-1)\epsilon, a+i\epsilon) = nf(\epsilon) = (b-a)\frac{f(\epsilon)}{\epsilon}.$$

Letting $n \to \infty$, we have $\epsilon \to 0^+$, so that $\mu^*(a,b) \le (b-a)f'_+(0) = b - a$.
(4.2)

We have $\inf_{x>0} f(x) \ge c > 0$. So we may assume $c = \inf_{x>0} f(x)$. Then for any $A \ne \emptyset$, $A \subset \cup[a_i, b_i)$ implies $b_i - a_i > 0$ and $\sum \lambda[a_i, b_i) = \sum f(b_i - a_i) \ge \sum c \ge c$. Therefore $\mu^*(A) \ge c$ for any nonempty $A$.

Suppose $A \ne \emptyset, \mathbb{R}$. Then consider $Y = \{a, b\}$, with $a \in A$ and $b \notin A$. We have $Y \cap A = \{a\}$ and $Y - A = \{b\}$. If $A$ were measurable, then

$$\mu^*(Y) = \mu^*(Y \cap A) + \mu^*(Y - A) \ge 2c.$$

On the other hand, by $2c > \inf_{x>0} f(x)$, we can find $\epsilon > 0$, such that $f(\epsilon) < 2c$. Then for any $A \ne \emptyset, \mathbb{R}$, we can always find $a \in A, b \notin A$, such that $|b - a| < \epsilon$. In this case, we have $Y \subset [d, d+\epsilon)$, $d = \min\{a, b\}$. Therefore

$$\mu^*(Y) \le \lambda[d, d+\epsilon) = f(\epsilon) < 2c.$$

The two estimations of $\mu^*(Y)$ are contradictory. So we conclude that $A$ is not measurable.

**Math 3043 Final, Spring 2014**

(1) Suppose $f(x,y)$ is second order differentiable at $(0,0)$.

1. Prove that if $f = 0$ along the $x$-axis, the $y$-axis, and some direction $(u,v)$ with $uv \ne 0$, then the quadratic approximation is 0.

2. Prove that if $f = 0$ along three distinct straight lines passing through $(0,0)$, then the quadratic approximation is 0.

(2) Let $p, q > 0$ and

$$f(x,y) = \begin{cases} |x|^p|y|^q, & \text{if } |x| \ge |y|, \\ |x|^q|y|^p, & \text{if } |x| \le |y|. \end{cases}$$

1. Find the condition for $f$ to have all the second order derivatives at $(0,0)$.

2. Find the condition for $f$ to be second order differentiable at $(0,0)$.

(3) Suppose $\mu^*$ is an outer measure on $X$. Suppose $Y \subset X$ is a subset.

1. Prove that the restriction $\mu^*_Y$ of $\mu^*$ to subsets of $Y$ is an outer measure on $Y$.
2. Prove that if $Y$ is measurable with respect to $\mu^*$, then a subset of $Y$ is measurable with respect to $\mu^*_Y$ if and only if it is measurable with respect to $\mu^*$.
3. What may happen in the second part if $Y$ is not measurable with respect to $\mu^*$?

(4) Suppose $Z = Y \times X$ and $Y$ is countably infinite. Suppose $\mu$ is a measure on $Z$ satisfying

1. $0 < \mu(Z) < +\infty$.
2. If $y \times A$ is measurable for all $y \in Y$, then $\mu(y \times A) = \mu(y' \times A)$ for any $y, y' \in Y$.

Prove that $A$ in the second part must satisfy $\mu(y \times A) = 0$ for all $y$. Prove that $y \times X$ is not measurable for some $y$.

**Answer to Math 3043 Final, Spring 2014**
*not absolutely guaranteed to be correct*

(1) Suppose $p(x,y) = a + b_1x + b_2y + c_{11}x^2 + c_{22}y^2 + 2c_{12}xy$ is the quadratic approximation of $f(x,y)$ at $(0,0)$. Then for any $\epsilon > 0$, there is $\delta > 0$, such that

$$\|(x,y)\| < \delta \implies |f(x,y) - p(x,y)| < \epsilon\|(x,y)\|^2.$$

Let $(u,v)$ be one direction, such that $f(tu,tv) = 0$ for any $t$. Then by choosing $(u,v)$ to have length 1, we get

$$|t| < \delta \implies |a + t(b_1u + b_2v) + t^2(c_{11}u^2 + c_{22}v^2 + 2c_{12}uv)| < \epsilon t^2.$$

This implies that the quadratic function $a + (b_1u + b_2v)t + (c_{11}u^2 + c_{22}v^2 + 2c_{12}uv)t^2$ of $t$ is a quadratic approximation of the 0 function. Therefore the coefficients of this quadratic approximation vanish

$$a = 0, \quad b_1u + b_2v = 0, \quad c_{11}u^2 + c_{22}v^2 + 2c_{12}uv = 0.$$

(1.1) In this case, we have the equalities above for $(1,0)$, $(0,1)$ and $(u,v)$. From $(1,0)$ and $(0,1)$, we already get $a = b_1 = b_2 = c_{11} = c_{22} = 0$. Then we by $2c_{12}uv = c_{11}u^2 + c_{22}v^2 + 2c_{12}uv = 0$ and $uv \neq 0$, we get $c_{12} = 0$.

(1.2) Given three distinct directions, we can always find an invertible linear transform $L\colon \mathbb{R}^2 \to \mathbb{R}^2$, such that $L$ takes the $x$-axis and the $y$-axis to the first two directions. Correspondingly, $L$ takes a vector $(u,v)$ to the third directions. Since the third direction is distinct from the first and second directions, we get $u, v \neq 0$. Then $f \circ L$ is second order differentiable, with $p \circ L$ as the quadratic approximation. Note that $f \circ L$ and $p \circ L$ fits into the previous special case. Therefore we conclude that $p \circ L = 0$. Since $L$ is invertible, we get $p = 0$.

*Additional discussion*

To extend the first part to three variable functions, we note that the linear approximation (i.e., $b_i$) vanishes if we have vanishing in the three coordinate directions. The vanishing in the three coordinate directions also implies $c_{ii} = 0$. It remains to consider three cross terms, for which we expect three additional directions $(u_i, v_i, w_i)$, $i = 1, 2, 3$, are needed. In other words, we wish to have that

$$c_{12}u_iv_i + c_{13}u_iw_i + c_{23}v_iw_i = 0, \quad i = 1, 2, 3$$

implying $c_{12} = c_{13} = c_{23} = 0$. The condition for this is exactly

$$\det \begin{pmatrix} u_1v_1 & u_1w_1 & v_1w_1 \\ u_2v_2 & u_2w_2 & v_2w_2 \\ u_3v_3 & u_3w_3 & v_3w_3 \end{pmatrix} \neq 0.$$

One such example is $(1, 1, 0)$, $(1, 0, 0)$, $(0, 1, 1)$.

So we need six directions for the similar statement for three variables. The condition for the six directions is rather complicated.

(2.1) We have

$$f_x(x, y) = \begin{cases} \operatorname{sign}(x)p|x|^{p-1}|y|^q, & \text{if } |x| > |y|, \\ \operatorname{sign}(x)q|x|^{q-1}|y|^p, & \text{if } 0 < |x| < |y|, \\ 0, & \text{if } x = 0, y \neq 0, q > 1; \text{ or } x = y = 0, \\ \text{no}, & \text{if } x = 0, y \neq 0, q \leq 1. \end{cases}$$

Then

$$f_{xx}(0, 0) = \lim_{x \to 0} \frac{f_x(x, 0) - f_x(0, 0)}{x} = \lim_{x \to 0} \frac{p|x|^{p-1}|0|^q - 0}{x} = 0,$$
$$f_{xy}(0, 0) = \lim_{y \to 0} \frac{f_x(0, y) - f_x(0, 0)}{y} =_{(\text{if } q>1)} \lim_{y \to 0} \frac{0 - 0}{y} = 0.$$

Similarly, $f_{yy}(0, 0)$ and $f_{yx}(0, 0)$ exists if and only if $p > 1$.

We conclude that all second order derivatives at $(0, 0)$ exist if and only if $p > 1$ and $q > 1$.

(2.2) Since $f(x, 0) = f(0, y) = 0$, we have $f_x(0, 0) = f_y(0, 0) = 0$. Therefore the linear approximation at $(0, 0)$ is the zero function. This implies that, if $f$ is second order differentiable at $(0, 0)$, then the quadratic approximation is $p(x, y) = ax^2 + bxy + cy^2$, and for any $\epsilon > 0$, there is $\delta > 0$, such that

$$|y| \leq |x| < \delta \implies \left| |x|^p|y|^q - ax^2 - bxy - cy^2 \right| < \epsilon \max\{|x|, |y|\}^2,$$
$$|x| \leq |y| < \delta \implies \left| |x|^q|y|^p - ax^2 - bxy - cy^2 \right| < \epsilon \max\{|x|, |y|\}^2,$$

We restrict the first implication to $x = At, y = t$ with $|A| \geq 1$ and get

$$|t| < \frac{\delta}{|A|} \implies \left| |A|^p|t|^{p+q} - (aA^2 + bA + c)t^2 \right| < \epsilon(At)^2, \text{ for all } |A| \geq 1.$$

There are two possibilities for this to happen: (i) $p+q>2$, $a=b=c=0$; (ii) $p+q=2$, $p=2$, $a=1$, $b=c=0$. By restricting the second implication to $x=t, y=At$ with $|A|\ge 1$, we similarly get two possibilities: (i) $p+q>2$, $a=b=c=0$; (ii) $p+q=2$, $q=2$, $c=1$, $a=b=0$.

Since both implications must hold, and the two second possibilities are incompatible, we conclude that the second order differentiability implies $p+q>2$.

Conversely, if $p+q>2$, then

$$|x|^p|y|^q \le |x|^{p+q} \text{ for } |x|\ge|y|, \quad \lim_{x\to 0}\frac{|x|^{p+q}}{|x|^2}=0,$$

and

$$|x|^q|y|^p \le |y|^{p+q} \text{ for } |x|\le|y|, \quad \lim_{y\to 0}\frac{|y|^{p+q}}{|y|^2}=0,$$

imply that $f$ is second order differentiable at $(0,0)$, with the zero function as the second order approximation.

(3) We know the outer measure $\mu^*$ on $X$ satisfies

1. $\mu^*(\emptyset)=0$.
2. $A\subset B\subset X$ imply $\mu^*(A)\le\mu^*(B)$.
3. $\mu^*(\cup_{\text{countable}}A_i)\le\sum\mu^*(A_i)$.

If we restrict the properties to $A\subset B\subset Y$ and $A_i\subset Y$, then we get the conditions for $\mu_Y^*$ to be an pouter measure on $Y$.

A subset $A\subset X$ is measurable with respect to $\mu^*$ if and only if

$$\mu^*(Z)=\mu^*(Z\cap A)+\mu^*(Z-A) \text{ for all } Z\subset X.$$

For $A\subset Y$, if we restrict the property to all subsets $Z\subset Y$, then we conclude that $A$ is also measurable with respect to $\mu_Y^*$. Note that this direction does not require $Y$ to be measurable with respect to $\mu^*$.

On the other hand, suppose $Y$ is measurable with respect to $\mu^*$, and $A\subset Y$ is measurable with respect to $\mu_Y^*$. Then for any $Z\subset X$, we have

$$\begin{aligned}
\mu^*(Z) &= \mu^*(Z\cap Y)+\mu^*(Z-Y) && (Y \text{ is } \mu^*\text{-measurable})\\
&= \mu_Y^*(Z\cap Y)+\mu^*(Z-Y) && (Z\cap Y\subset Y)\\
&= \mu_Y^*(Z\cap Y\cap A)+\mu_Y^*(Z\cap Y-A)+\mu^*(Z-Y) && (A \text{ is } \mu_Y^*\text{-measurable})\\
&= \mu^*(Z\cap Y\cap A)+\mu^*(Z\cap Y-A)+\mu^*(Z-Y) && (Z\cap Y\cap A\subset Y,\ Z\cap Y-A\subset Y)\\
&= \mu^*(Z\cap A)+\mu^*((Z-A)\cap Y)+\mu^*((Z-A)-Y) && (A\subset Y)\\
&= \mu^*(Z\cap A)+\mu^*(Z-A). && (Y \text{ is } \mu^*\text{-measurable})
\end{aligned}$$

This shows that $A$ is measurable with respect to $\mu^*$.

If $Y$ is not measurable with respect to $\mu^*$, then by $Y$ automatically measurable with respect to $\mu_Y^*$, the $\Longrightarrow$ direction of the second part is not true.

(4) Let $y_1, \dots, y_n$ be distinct points in $Y$. By the second property and the countability of $Y$, we have

$$\begin{aligned}\mu(Z) = \mu(\sqcup_{y\in Y} y \times A) &= \sum_{y\in Y} \mu(y\times A) \\ &\ge \mu(y_1 \times A) + \cdots + \mu(y_n \times A) = n\mu(y\times A).\end{aligned}$$

Since $n$ can be arbitrarily large, we get $\mu(y \times A) = 0$.

If $y \times X$ is measurable for all $y \in Y$, then we already know $\mu(y \times X) = 0$ for all $y$. Therefore

$$\mu(Z) = \mu(\sqcup_{y\in Y} y \times X) = \sum_{y\in Y} \mu(y \times X) = 0.$$

Since this contradicts to the assumption $\mu(Z) > 0$, we conclude that $y \times X$ is not measurable for some $y \in Y$.

**Math 3043 Final, Spring 2016**

(1) Suppose $K$ is compact and $U$ is open.

1. Prove that $\lambda(K) \le \lambda(K - U) + \lambda(U)$, and the equality happens when $U \subset K$.
2. Prove that $\lambda(U) \le \lambda(U - K) + \lambda(K)$, and the equality happens when $K \subset U$.

Your argument can only use the material in Section 9.1.

(2) Suppose $f$ is a bounded integrable function on a measure space $(X, \Sigma, \mu)$ with $\mu(X) < +\infty$.

1. Prove that if $\displaystyle\int_A f d\mu = 0$ for all $A \subset \Sigma$, then $f = 0$ almost everywhere.
2. Let $\nu'$ be another measure on $(X, \Sigma)$. Find the necessary and sufficient condition for $\nu'$ to be mutually singular to the signed measure $\nu(A) = \displaystyle\int_A f d\mu$.

(3) Suppose $A \subset [a, b]$ satisfies $\mu([a, b] - A) = 0$. Prove that $A$ is dense in $[a, b]$.

(4) Suppose $A_n$, $n \in \mathbb{N}$, are measurable, and $\sum \mu(A_n) < +\infty$. Use the function $f = \sum_n \chi_{A_n}$ to prove that almost all $x \in X$ lie in finitely many $A_n$. Moreover, show that the conclusion fails if the condition is related to $\lim \mu(A_n) = 0$. [Borel-Cantelli Lemma]

**Answer to Math 3043 Final, Spring 2016**

(1.1) Let $K \subset V$, $V$ open. Then $K - U$ is compact, $K - U \subset V$, and the inequality becomes

$$\lambda(V) - \lambda(V - K) \le \lambda(V) - \lambda(V - (K - U)) + \lambda(U).$$

This is the same as

$$\lambda(V - (K - U)) \le \lambda(V - K) + \lambda(U).$$

This is a consequence of $V - (K - U) \subset (V - K) \cup U$ and Proposition 9.1.4. Moreover, if $U \subset K$, then $V - (K - U) = (V - K) \sqcup U$, and the inequality becomes equality.

(1.2) Let $U \cup K \subset V$, $V$ open. Then $V \supset (V-K) \cup U$, and $(V-K) \cap U = U - K$. By Proposition 9.1.4, we have

$$\lambda(V) \geq \lambda((V-K) \cup U) = \lambda(V-K) + \lambda(U) - \lambda(U-K).$$

This means

$$\lambda(K) = \lambda(V) - \lambda(V-K) \geq \lambda(U) - \lambda(U-K).$$

Moreover, the equality happens when $V = (V-K) \cup U$. This means $K \subset U$.

(2.1) By Theorem 10.2.3, we know there is a bounded and measurable $g$, such $f = g$ almost everywhere. We also have $\displaystyle\int_A g d\mu = \int_A f d\mu = 0$ for all $A \in \Sigma$.

Since $g$ is measurable, the subset $A_\epsilon = \{x \colon g(x) > \epsilon\}$ is measurable for any $\epsilon > 0$. Moreover, by Proposition 10.1.4, we have

$$0 = \int_{A_\epsilon} g dx \geq \epsilon \mu(A_\epsilon).$$

Since the right side is non-negative, we get $\mu(A_\epsilon) = 0$. This implies that $\{x \colon g(x) > 0\} = \cup_{n=1}^{\infty} A_{\frac{1}{n}}$ has measure 0. By the same reason, $\{x \colon g(x) < 0\}$ also has measure 0. Therefore $g = 0$ away from the the subset $\{x \colon f(x) \neq 0\} = \{x \colon f(x) > 0\} \cup \{x \colon f(x) < 0\}$ of measure 0. In other words, we ahve $g = 0$ almost everywhere. Since $f = g$ almost everywhere, we conclude that $f = 0$ almost everywhere.

(2.2) Suppose $\nu$ and $\nu'$ are mutually singular. Then we have measurable decomposition $X = Y \sqcup Y'$, such that $\nu(A) = 0$ for any measurable $A \subset Y'$ and $\nu'(A) = 0$ for any measurable $A \subset Y$. By the first part, $\nu(A) = \displaystyle\int_A f d\mu = 0$ for all measurable $A \subset Y'$ is equivalent to $f = 0$ almost everywhere on $Y'$. We conclude that there is $Y \in \Sigma$, such that $\nu'(Y) = 0$, and $f = 0$ almost everywhere on $X - Y$ (this is the same as $f\chi_Y = 0$ almost everywhere).

Conversely, suppose there is $Y \in \Sigma$, such that $\nu'(Y) = 0$, and $f = 0$ almost everywhere on $X - Y$. Then for any measurable $A \subset Y'$, we have $0 \leq \nu'(A) \leq \nu'(Y) = 0$ (since $\nu'$ is a measure), which implies $\nu'(A) = 0$. On the other hand, $f = 0$ almost everywhere on $X - Y$ implies $\mu(A) = \displaystyle\int_A f d\mu = 0$ for any $A \subset X - Y$. Therefore $X = Y \sqcup (X - Y)$ is a measurable decomposition that shows that $\nu$ and $\nu'$ are mutually singular.

(3) If $A$ is not dense in $[a, b]$, then there is an open interval $(c, d) \subset [a, b]$, such that $A$ and $(c, d)$ are disjoint. Then $(c, d) \subset [a, b] - A$, and $\mu^*([a, b] - A) \geq \mu^*(c, d) = d - c > 0$ (the inequality $d > c$ is due to open interval). This contradicts the assumption.

(4) By the monotone convergence theorem, we have $\displaystyle\int f d\mu = \sum \int \chi_{A_n} d\mu < +\infty$. Let $X_k$ be the set of those $x$ belong to at least $k$ subsets $A_n$. Then $f \geq k$ on $X_k$, and $X_k = \cup_{n_1 < n_2 < \cdots < n_k} A_{n_1} \cap A_{n_2} \cap \cdots \cap A_{n_k}$ is measurable. Let $M = \displaystyle\int f d\mu$. Then

$$k\mu(X_k) \leq \int_{X_k} f d\mu \leq \int f d\mu = M.$$

This implies $0 \le \mu(\cap_k X_k) \le \mu(X_k) \le \dfrac{M}{k}$ for all $k$. Therefore $\cap_k X_k$ has measure 0. The subset $\cap_k X_k$ is exactly the set of those $x$ belonging to infinitely many $A_n$.

For the counterexample in case $\lim \mu(A_n) = 0$, we take $A_n$ to be the sequence of "moving" intervals of length $\dfrac{1}{k}$:

$$[0,1],\ [0,\frac{1}{2}],\ [\frac{1}{2},1],\ [0,\frac{1}{3}],\ [\frac{1}{3},\frac{2}{3}],\ [\frac{2}{3},1],\dots,[0,\frac{1}{k}],\ [\frac{1}{k},\frac{2}{k}],\dots,[\frac{k-1}{k},1],\dots$$

Then the length of $A_n$ converges to 0, and any $x \in [0,1]$ is in inifnitely many $A_n$.

[*Alternative proof without using integration*]

Let $X_k$ be the set of those $x$ belong to at least $k$ subsets $A_n$. Then $X_1 \supset X_2 \supset \cdots \supset X_k \supset \cdots$, and $\cap_k X_k$ is the set of those $x$ belonging to infinitely many $A_n$.

We fix $k$ and consider the earliest $k$ subsets that any $x \in X_k$ belongs to. This means that, for any $I = \{n_1 < n_2 < \cdots < n_k\}$, we introduce

$$\begin{aligned} A_I &= \{x \in A_{n_i} \text{ for all } 1 \le i \le k \text{ and } x \notin \text{ other } A_n \text{ satisfying } n \le n_k\} \\ &= A_{n_1} \cap A_{n_2} \cap \cdots \cap A_{n_k} - \cup_{n \le n_k,\, n \notin I} A_n. \end{aligned}$$

Then $A_I$ are measurable, and we have

1. $A_I \subset A_n$ for any $n \in I$.

2. $A_I$ are disjoint for distinct sets $I$ of $k$ natural numbers.

3. $X_k = \cup_{|I|=k} A_I$ (the union is disjoint by the second property).

The properties imply that $A_n \supset \sqcup_{n \in I,\, |I|=k} A_I$, and we have

$$\mu(A_n) \ge \sum_{n \in I,\, |I|=k} \mu(A_I).$$

Let $M = \sum_n \mu(A_n) < +\infty$. Then we have

$$M \ge \sum_n \sum_{n \in I,\, |I|=k} \mu(A_I) \ge \sum_{|I|=k} \sum_{n \in I} \mu(A_I) = \sum_{|I|=k} k\mu(A_I) = k\mu(X_k).$$

Here the first equality is due to that each $I$ contains $k$ indices, and the second equality is due to $X_k = \sqcup_{|I|=k} A_I$. Since the inequality holds for all $k$, we conclude that $\mu(X_k) \le \dfrac{k}{M}$. This further implies that $\mu(\cap_k X_k) = 0$.