Exercise 9.19

The countable subadditivity is changed to the following stricter version: $\mu^*(\sqcup A_i) \le \sum \mu^*(A_i)$ for countable disjoint union $\sqcup A_i$.

For a general countable union $\cup A_i$, let $B_i = A_i - A_1 - \cdots - A_i$. Then $\cup A_i = \sqcup B_i$. By $B_i \subset A_i$ and the monotone property, we have $\mu^*(B_i) \le \mu^*(A_i)$. Therefore

$$\mu^*(\cup A_i) = \mu^*(\sqcup B_i) \le \sum \mu^*(B_i) \le \sum \mu^*(A_i),$$

where the stricter subadditivity is used int eh first inequality.

Exercise 9.20

Suppose μ_i^* are outer measures on X and $\mu^* = \sum \mu_i^*$. Then $\mu^*(\emptyset) = \sum \mu_i^*(\emptyset) = \sum 0 = 0$. Moreover, $A \subset B$ implies $\mu_i^*(A) \le \mu_i^*(B)$, so that $\mu^*(A) = \sum \mu_i^*(A) \le \sum \mu_i^*(B) = \mu^*(B)$. Finally, for a countable union $\cup_j A_j$, we have $\mu_i^*(\cup_j A_j) \le \sum_j \mu_i^*(A_j)$, so that $\mu^*(\cup_j A_j) = \sum_i \mu_i^*(\cup_j A_j) \le \sum_i \sum_j \mu_i^*(A_j) = \sum_j \sum_i \mu_i^*(A_j) = \sum_j \mu^*(A_j)$. The switching of summation in i and in j is due to the fact that the series are non-negative.

The proof for positive scalar multiple is similar (and even simpler).

Exercise 9.21

This is tautology, by simply restricting the three properties to subsets of Y.

Exercise 9.22

We have $\mu^*(\emptyset) = \mu_1^*(\emptyset \cap X_1) + \mu_2^*(\emptyset \cap X_2) = \mu_1^*(\emptyset) + \mu_2^*(\emptyset) = 0 + 0 = 0$.

For $A \subset B$, we have $A \cap X_i \subset B \cap X_i$, so that $\mu^*(A) = \mu_1^*(A \cap X_1) + \mu_2^*(A \cap X_2) \le \mu_1^*(B \cap X_1) + \mu_2^*(B \cap X_2) = \mu^*(B)$.

For a countable union $\cup A_j$, we have $(\cup A_j) \cap X_i = \cup(A_j \cap X_i)$, so that $\mu^*(\cup A_j) = \mu_1^*(\cup(A_j \cap X_1)) + \mu_2^*(\cup(A_j \cap X_2)) \le \sum[\mu_1^*(A_j \cap X_1) + \mu_2^*(A_j \cap X_2)] = \sum \mu^*(A_j)$.

Exercise 9.23

Let

$$\mu_1^*(A) = \lim_{r \to +\infty} \mu^*(A \cap [-r, r]) = \sup_{r>0} \mu^*(A \cap [-r, r]).$$

We have $\mu_1^*(\emptyset) = \sup_{r>0} \mu^*(\emptyset \cap [-r, r]) = \sup_{r>0} \mu^*(\emptyset) = \sup_{r>0} 0 = 0$.

For $A \subset B$, we have $A \cap [-r, r] \subset B \cap [-r, r]$. By the monotone property of μ^* for bounded subsets, we have $\mu^*(A \cap [-r, r]) \subset \mu^*(B \cap [-r, r])$. This implies $\mu_1^*(A) \le \mu_1^*(B)$.

For a countable union $\cup A_i$, we have $(\cup A_i) \cap [-r, r] = \cup(A_i \cap [-r, r])$. By the subadditivity of μ^* for bounded subsets, we have $\mu^*((\cup A_i) \cap [-r, r]) = \sum \mu^*(A_i \cap [-r, r])$. This implies $\mu_1^*(\cup A_i) \le \sum \mu_1^*(A_i)$.

Exercise 9.24

Let

$$\mu_2^*(A) = \inf\{\lambda(U) \colon A \subset U,\ U \text{ open}\}.$$

We have $\mu_2^*(\emptyset) = 0$ by taking $U = \emptyset$.

For $A \subset B$, we have $B \subset U$ implying $A \subset U$. This implies $\mu_2^*(A) \le \mu_2^*(B)$.

Like the bounded case, the countable additivity is reduced to $\lambda(\cup U_i) \le \lambda(U_i)$ for a countable union $\cup U_i$ of open subsets. By writing down the definition, this is reduced to $\lambda(\cup(a_i, b_i)) \le$

$\sum \lambda(a_i, b_i) = \sum(b_i - a_i)$. The proof for the first part of Proposition 9.1.4 can be adopted without any change.

In general, we need $\inf\{\lambda(C) \colon C \in \mathcal{C}\} = 0$ in order for the first condition for μ^* to be an outer measure satisfied. A sufficient condition is $\emptyset \in \mathcal{C}$ and $\lambda(\emptyset) = 0$. The definition always gives monotone μ^*. For μ^* to be countably additive, we need $C_i \in \mathcal{C}$ implying the countable union $\cup C_i \in \mathcal{C}$, and $\lambda(\cup C_i) \le \sum \lambda(C_i)$.

Exercise 9.25

By $A \cap [-r, r] \subset A \cap (-r-1, r+1)$, we get

$$\mu_1^*(A) = \sup_{r>0} \mu^*(A \cap [-r, r]) \le \sup_{r>0} \mu^*(A \cap (-r-1, r+1)) \le \mu_2^*(A).$$

On the other hand, we have

$$\mu_1^*(A) \ge \mu_1^*(A \cap (-n-1, n]) = \mu^*(A \cap (-n-1, n]) = \sum_{i=-n}^{n} \mu^*(A \cap (i-1, i]),$$

where the inequality is by monotone property, and the second equality is by the measurability of intervals. Therefore

$$\mu_1^*(A) \ge \sum_{i=-\infty}^{+\infty} \mu^*(A \cap (i-1, i]).$$

For any $\epsilon > 0$ and each i, we have open $U_i \supset A \cap (i-1, i]$, such that $\lambda(U_i) < \mu^*(A \cap (i-1, i]) + \frac{\epsilon}{2^{|i|}}$. Then

$$\mu_1^*(A) \ge \sum_{i=-\infty}^{+\infty} \lambda(U_i) - 3\epsilon \ge \lambda(\cup U_i) - 3\epsilon \ge \mu_2^*(A) - 3\epsilon,$$

where the second inequality is proved in Exercise 9.24.

Exercise 9.26

The equality we want to prove is

$$\mu^*(Y) = \mu^*(Y \cap A_1) + \cdots + \mu^*(Y \cap A_n) + \mu^*(Y - A_1 \sqcup \cdots \sqcup A_n)$$

for disjoint and measurable $A_1, \dots, A_n$ and any Y. The definition of measurability shows the equality holds for $n = 1$. Assume the equality holds for $n-1$. Then

$$\mu^*(Y) = \mu^*(Y \cap A_1) + \cdots + \mu^*(Y \cap A_{n-1}) + \mu^*(Y - A_1 \sqcup \cdots \sqcup A_{n-1}).$$

Applying the definition of measurability to $Y - A_1 \sqcup \cdots \sqcup A_{n-1}$ and A_n, we get

$$\begin{aligned}\mu^*(Y - A_1 \sqcup \cdots \sqcup A_{n-1}) &= \mu^*((Y - A_1 \sqcup \cdots \sqcup A_{n-1}) \cap A_n) + \mu^*(Y - A_1 \sqcup \cdots \sqcup A_{n-1} - A_n) \\ &= \mu^*(Y \cap A_n) + \mu^*(Y - A_1 \sqcup \cdots \sqcup A_{n-1} \sqcup A_n),\end{aligned}$$

where the second equality is due to the disjoint assumption among $A_1, \dots, A_n$. Combining the two equalities, we complete the inductive definition.

Exercise 9.27

(1) Let $\mu^* = \mu_1^* + \mu_2^*$. Then

1. By $\mu_1^*(\emptyset) = \mu_2^*(\emptyset) = 0$, we get $\mu^*(\emptyset) = \mu_1^*(\emptyset) + \mu_2^*(\emptyset) = 0$.

2. For any A, we have $\mu_1^*(A) \geq 0$, $\mu_2^*(A) \geq 0$. Then we get $\mu^*(A) = \mu_1^*(A) + \mu_2^*(A) \geq 0$.

3. For a countable union $\cup_i A_i$, we have $\mu_1^*(\cup A_i) \leq \sum \mu_1^*(A_i)$, $\mu_2^*(\cup A_i) \leq \sum \mu_2^*(A_i)$. Then we get $\mu^*(\cup A_i) = \mu_1^*(\cup A_i) + \mu_2^*(\cup A_i) \leq \sum \mu_1^*(A_i) + \sum \mu_2^*(A_i) = \sum(\mu_1^*(A_i) + \mu_2^*(A_i)) = \sum \mu^*(A_i)$.

This verifies that μ^* is an outer measure.

(2) For any Y, we have

$$\mu_1^*(Y) \leq \mu_1^*(Y \cap A) + \mu_1^*(Y - A), \quad \mu_2^*(Y) \leq \mu_2^*(Y \cap A) + \mu_2^*(Y - A).$$

This gives

$$\begin{aligned} \mu^*(Y) &= \mu_1^*(Y) + \mu_2^*(Y) \\ &\leq \mu_1^*(Y \cap A) + \mu_2^*(Y \cap A) + \mu_1^*(Y - A) + \mu_2^*(Y - A) \\ &= \mu^*(Y \cap A) + \mu^*(Y - A). \end{aligned}$$

In case A is μ_1^*-measurable and μ_2^*-measurable, the two inequalities for μ_1^* and μ_2^* become equalities, and we get equality for μ^*. Therefore A is μ^*-measurable.

(3) In case A is μ^*-measurable, we have $\mu^*(Y) = \mu^*(Y \cap A) + \mu^*(Y - A)$ for all Y. By $\mu^*(Y) \leq \mu^*(X) = \mu_1^*(X) + \mu_2^*(X) < +\infty$ and the non-negativity of all the terms, we conclude that the inequalities for μ_1^* and μ_2^* must become equalities. Therefore A is μ_1^*-measurable and μ_2^*-measurable.

(4) If $\mu^*(X) = +\infty$, then the claim may not be true. For example, let $\mu_1^*(A) = +\infty$ for $A \neq \emptyset$ and $\mu_1^*(\emptyset) = 0$. Then for any outer measure μ_2^*, we have $\mu_1^* + \mu_2^* = \mu_1^*$. Now any subset is measurable with respect to μ_1^*. On the other hand, we have examples of outer measure μ_2^* such that not all subsets are measurable.

Exercise 9.28

(1) We know the outer measure μ^* on X satisfies

1. $\mu^*(\emptyset) = 0$.

2. $A \subset B \subset X$ imply $\mu^*(A) \leq \mu^*(B)$.

3. $\mu^*(\cup_{\text{countable}} A_i) \leq \sum \mu^*(A_i)$.

If we restrict the properties to $A \subset B \subset Y$ and $A_i \subset Y$, then we get the conditions for μ_Y^* to be an pouter measure on Y.

A subset $A \subset X$ is measurable with respect to μ^* if and only if

$$\mu^*(Z) = \mu^*(Z \cap A) + \mu^*(Z - A) \text{ for all } Z \subset X.$$

For $A \subset Y$, if we restrict the property to all subsets $Z \subset Y$, then we conclude that A is also measurable with respect to μ_Y^*. Note that this direction does not require Y to be measurable with respect to μ^*.

On the other hand, suppose Y is measurable with respect to μ^*, and $A \subset Y$ is measurable with respect to μ^*_Y. Then for any $Z \subset X$, we have

$$
\begin{aligned}
\mu^*(Z) &= \mu^*(Z \cap Y) + \mu^*(Z - Y) && (Y \text{ is } \mu^*\text{-measurable}) \\
&= \mu^*_Y(Z \cap Y) + \mu^*(Z - Y) && (Z \cap Y \subset Y) \\
&= \mu^*_Y(Z \cap Y \cap A) + \mu^*_Y(Z \cap Y - A) + \mu^*(Z - Y) && (A \text{ is } \mu^*_Y\text{-measurable}) \\
&= \mu^*(Z \cap Y \cap A) + \mu^*(Z \cap Y - A) + \mu^*(Z - Y) && (Z \cap Y \cap A \subset Y,\ Z \cap Y - A \subset Y) \\
&= \mu^*(Z \cap A) + \mu^*((Z - A) \cap Y) + \mu^*((Z - A) - Y) && (A \subset Y) \\
&= \mu^*(Z \cap A) + \mu^*(Z - A). && (Y \text{ is } \mu^*\text{-measurable})
\end{aligned}
$$

This shows that A is measurable with respect to μ^*.

(2) If Y is not measurable with respect to μ^*, then by Y automatically measurable with respect to μ^*_Y, the $\implies$ direction of the second part is not true.

Exercise 9.29

Suppose A is measurable with respect to μ^*. Then for any $Y \subset X$, we have

$$\mu^*(Y) = \mu^*(Y \cap A) + \mu^*(Y - A) \text{ for any } Y \subset X.$$

For $Y \subset X_1$, we have $Y \cap A = Y \cap (A \cap X_1)$, $Y - A = Y - (A \cap X_1)$, and this means

$$\mu^*_1(Y) = \mu^*_1(Y \cap (A \cap X_1)) + \mu^*_1(Y - (A \cap X_1)) \text{ for any } Y \subset X_1.$$

Therefore $A \cap X_1$ is measurable with respect to μ^*_1. By the same reason, $A \cap X_2$ is measurable with respect to μ^*_2.

Suppose $A \cap X_1$ is measurable with respect to μ^*_1 and $A \cap X_2$ is measurable with respect to μ^*_2. Then any $Y \subset X$ is $Y = Y_1 \cup Y_2$, with $Y_1 = Y \cap X_1 \subset X_1$ and $Y_2 = Y \cap X_2 \subset X_2$. By the measurability of $A \cap X_1$ and $A \cap X_2$, we get

$$
\begin{aligned}
\mu^*(Y) &= \mu^*_1(Y \cap X_1) + \mu^*_1(Y \cap X_2) \\
&= \mu^*_1(Y \cap X_1 \cap A) + \mu^*_1(Y \cap X_1 - A) + \mu^*_2(Y \cap X_2 \cap A) + \mu^*_2(Y \cap X_2 - A) \\
&= \mu^*_1((Y \cap A) \cap X_1) + \mu^*_2((Y \cap A) \cap X_2) + \mu^*_1((Y - A) \cap X_1) + \mu^*_2((Y - A) \cap X_2) \\
&= \mu^*(Y \cap A) + \mu^*(Y - A).
\end{aligned}
$$

This proves that A is measurable with respect to μ^*.

Exercise 9.30

We have $\phi(Y \cap A) = \phi(Y) \cap f(A)$ and $\phi(Y - A) = \phi(Y) - \phi(A)$ for invertible ϕ. Therefore by $\mu^*(\phi(A)) = \mu^*(A)$, the equality $\mu^*(Y) = \mu^*(Y \cap A) + \mu^*(Y - A)$ is the same as $\mu^*(\phi(Y)) = \mu^*(\phi(Y) \cap f(A)) + \mu^*(\phi(Y) - \phi(A))$. Since any subset can be written as $\phi(Y)$, the second equality holds for any Y if and only if $\mu^*(Z) = \mu^*(Z \cap \phi(A)) + \mu^*(Z - \phi(A))$ for any subset Z. Thus proves that A is measurable if and only if $\phi(A)$ is measurable.

In case both A and $\phi(A)$ are measurable, we have $\mu(A) = \mu^*(A) = \mu^*(\phi(A)) = \mu(\phi(A))$.

Exercise 9.31

If A is measurable, then by the definition of σ-algebra, $A \cap X_i$ is measurable.

If $A \cap X_i$ are measurable, then by the definition of σ-algebra, the countable union $A = A \cap X = \cup(A \cap X_i)$ is measurable.

Exercise 9.32

For a general countable union $\cup A_i$, let $B_i = A_i - A_1 - \cdots - A_i$. Then $\cup A_i = \sqcup B_i$. By the second condition for σ-algebra, we have $B_i \in \Sigma$. Then by the restricted third property, we have $\cup A_i = \sqcup B_i \in \Sigma$.

Exercise 9.33

A σ-algebra has three properties. We need to show that the three properties imply the conditions in the definition of σ-algebra.

By the first two properties, we have $\emptyset = X - X \in \Sigma$.

The second property implies the second condition in the definition of σ-algebra.

For a general countable union $\cup A_i$, with $A_i \in \Sigma$, by the second condition, we have $B_i = A_1 \cup \cdots \cup A_i \in \Sigma$. Moreover, $B_i \subset B_{i+1}$. Therefore by the third property, $\cup A_i = \cup B_i \in \Sigma$.

Exercise 9.34

We know the Borel σ-algebra contains all these subsets. We need to show that containing any of these subsets implying containing any open interval.

(1) $(a, b) = \cup_{n=1}^{\infty}[a + \frac{1}{n}, b - \frac{1}{n}]$.

(2) $(a, b) = \cup_{n=1}^{\infty}(a, b - \frac{1}{n}]$.

(3) open intervals are intervals.

(4) $(a, b) = \cup_{n=1}^{\infty}(a, b - \frac{1}{n}] = \cup_{n=1}^{\infty}\left((a, +\infty) - (b - \frac{1}{n}, +\infty)\right)$.

(5) Any bounded open interval is union of finitely many open intervals of length ϵ for a sufficiently small ϵ. On the other hand, for small $1 > \epsilon > 0$, we have $(a, a+\epsilon) = (a - 1 + \epsilon, a + \epsilon) \cap (a, a+1)$.

(6) open intervals are open subsets.

(7) $(a, b) = \cup_{n=1}^{\infty}[a + \frac{1}{n}, b - \frac{1}{n}]$, and $[a + \frac{1}{n}, b - \frac{1}{n}]$ are closed.

(8) $(a, b) = \cup_{n=1}^{\infty}[a + \frac{1}{n}, b - \frac{1}{n}]$, and $[a + \frac{1}{n}, b - \frac{1}{n}]$ are compact.

Exercise 9.35

(1) $f^{-1}(\emptyset) = \emptyset \in \Sigma$.

If $f^{-1}(B), f^{-1}(B') \in \Sigma$, then $f^{-1}(B - B') = f^{-1}(B) - f^{-1}(B') \in \Sigma$.

If $f^{-1}(B_i) \in \Sigma$ for countably many B_i, then $f^{-1}(\cup B_i) = \cup f^{-1}(B_i) \in f^{-1}(\Sigma)$.

(2) The main problem is $f(A) \cap f(B)$ may not be equal to $f(A \cap B)$.

Specifically, let $f\colon X = \{x, y, z_1, z_2\} \to Y = \{x, y, z\}$ be given by $f(x) = x$, $f(y) = y$, $f(z_1) = f(z_2) = z$. Then for $A = \{x, z_1\}$ and $B = \{y, z_2\}$, we have $f(A) \cap f(B) = \{z\}$, while $f(A \cap B) = \emptyset$. Let $\Sigma = \{\emptyset, A, B, X\}$. Then Σ is a σ-algebra on X. However, $f(\Sigma) = \{\emptyset, \{x, z\}, \{y, z\}, Y\}$ is not a σ-algebra on Y.

Exercise 9.36

(1) $\emptyset = f^{-1}(\emptyset) \in f^{-1}(\Sigma)$.

If $B, B' \in \Sigma$, then $B - B' \in \Sigma$, so that $f^{-1}(B) - f^{-1}(B') = f^{-1}(B - B') \in f^{-1}(\Sigma)$.

If countably many $B_i \in \Sigma$, then $\cup f^{-1}(B_i) = f^{-1}(\cup B_i) \in f^{-1}(\Sigma)$.

(2) Take the map in the first part. Take the σ-algebra $\Sigma = \{\emptyset, \{x, z\}, \{y\}, Y\}$ on Y. Then $f^*(\Sigma) = \{\emptyset, \{x, z_1\}, \{x, z_2\}, \{x, z_1, z_2\}, \{y\}, \{x, z_1, y\}, \{x, z_2, y\}, Y\}$. The collection $f^*(\Sigma)$ is not a σ-algebra because it does not contain the complement of $\{x, z_1\}$.

Exercise 9.37

We clearly have $\mu(A) \geq 0$. Suppose we have countably many disjoint A_i. If for each i, we have $\mu_x \neq 0$ for countably many $x \in A_i$, then the number of $x \in \sqcup A_i$ with $\mu_x \neq 0$ is still countable, and we have

$$\mu(\sqcup A_i) = \sum_{x \in \sqcup A_i,\ \mu_x \neq 0} \mu_x = \sum_i \sum_{x \in A_i,\ \mu_x \neq 0} \mu_x = \sum_i \mu(A_i).$$

On the other hand, if we have $\mu_x \neq 0$ for uncountably many x in some A_i, then we have $\mu_x \neq 0$ for uncountably many x in some $\sqcup A_i$. Therefore $\mu(A) = \mu(A_i) = +\infty$, and we still have $\mu(\sqcup A_i) = \sum_i \mu(A_i)$.

The measure is σ-finite if and only if all μ_x is finite, and there are only countably many $\mu_x \neq 0$.

Exercise 9.38

Let $\mu = \sum \mu_j$. Then $\mu(\sqcup_i A_i) = \sum_j \mu_j(\sqcup_i A_i) = \sum_j \sum_i \mu_j(A_i) = \sum_i \sum_j \mu_j(A_i) = \sum_i \mu(A_i)$.

If each σ_j is finite, it does not imply $\sum \sigma_j$ is finite. A counterexample is given by taking σ_j to be the same nontrivial measure.

Exercise 9.39

Suppose $A_i \subset \cup_j C_{ij}$, with $C_{ij} \in \Sigma$ and $\mu(C_{ij}) < +\infty$. If the number of A_i is countable, and each union $\cup_j C_{ij}$ is countable, then $\cup_j A_i \subset \cup_{ij} C_{ij}$, and $\cup_{ij} C_{ij}$ is a union of countably many measurable subsets of finite measure.

Exercise 9.40

If $Y \in \Sigma$, then $Y \in \Sigma_Y$ by the definition of Σ_Y. The other two conditions for Σ_Y to be a σ-algebra comes directly from the corresponding conditions for Σ to be a σ-algebra. The conditions for the restriction μ_Y to be a measure comes directly from the corresponding conditions for μ to be a measure.

If $Y \notin \Sigma$, then $Y \notin \Sigma_Y$, so that Σ_Y is not a σ-algebra.

Exercise 9.41

(1) By $X_1 \in \Sigma_1$ and $X_2 \in \Sigma_2$, we have $X_1 \sqcup X_2 \in \Sigma_1 \sqcup \Sigma_2$.

Suppose $A_1 \sqcup A_2, B_1 \sqcup B_2 \in \Sigma_1 \sqcup \Sigma_2$, with $A_1, B_1 \in \Sigma_1$ and $A_2, B_2 \in \Sigma_2$. We have $A_1 - B_1 \in \Sigma_1$ and $A_2 - B_2 \in \Sigma_2$, so that $A_1 \sqcup A_2 - B_1 \sqcup B_2 = (A_1 - B_1) \sqcup (A_2 - B_2) \in \Sigma_1 \sqcup \Sigma_2$.

Suppose $A_{1i} \sqcup A_{2i} \in \Sigma_1 \sqcup \Sigma_2$, with $A_{1i} \in \Sigma_1$ and $A_{2i} \in \Sigma_2$, for countable indices i. Then $\cup_i A_{1i} \in \Sigma_1$ and $\cup_i A_{2i} \in \Sigma_2$, so that $\cup_i (A_{1i} \sqcup A_{2i}) = (\cup_i A_{1i}) \sqcup (\cup_i A_{2i}) \in \Sigma_1 \sqcup \Sigma_2$.

(2) We have $\mu(\emptyset) = \mu(\emptyset \sqcup \emptyset) = \mu_1(\emptyset) + \mu_2(\emptyset) = 0 + 0 = 0$.

We have $\mu(A_1 \sqcup A_2) = \mu_1(A_1) + \mu_2(A_2) \geq 0$, by $\mu_1(A_1) \geq 0$ and $\mu_2(A_2) \geq 0$.

Suppose countably many $A_{1i} \sqcup A_{2i} \in \Sigma_1 \sqcup \Sigma_2$ are disjoint. Then A_{1i} are disjoint, so that

$\mu_1(\sqcup_i A_{1i}) = \sum_i \mu_1(A_{1i})$. By the same reason, we have $\mu_2(\sqcup_i A_{2i}) = \sum_i \mu_2(A_{2i})$. Then

$$\begin{aligned}\mu(\sqcup_i(A_{1i} \sqcup A_{2i})) &= \mu((\sqcup_i A_{1i}) \sqcup (\sqcup_i A_{2i})) = \mu_1(\sqcup_i A_{1i}) + \mu_2(\sqcup_i A_{2i}) \\ &= \sum_i \mu_1(A_{1i}) + \sum_i \mu_2(A_{2i}) = \sum_i (\mu_1(A_{1i}) + \mu_2(A_{2i})) = \sum_i \mu(A_{1i} \sqcup A_{2i}).\end{aligned}$$

(3) For $A \in \Sigma$, we have $A \cap Y, A - Y \in \Sigma$. Therefore $A \cap Y \in \Sigma_Y$, $A - Y \in \Sigma_{X-Y}$, and $A = (A \cap Y) \sqcup (A - Y) \in \Sigma_Y \sqcup \Sigma_{X-Y}$. Moreover, we have

$$\mu(A) = \mu(A \cap Y) + \mu(A - Y) = \mu_Y(A \cap Y) + \mu_{X-Y}(A - Y).$$

This shows that $(X, \Sigma, \mu) = (Y, \Sigma_Y, \mu_Y) \sqcup (X - Y, \Sigma_{X-Y}, \mu_{X-Y})$.

Exercise 9.42

Let μ be the usual Lebesgue measure on $\mathbb{R}$. Let $A_i = (i, +\infty)$. Then $\mu(A_i) = +\infty$ and $\mu(\cap A_i) = \mu(\emptyset) = 0$.

Exercise 9.43

By Exercise 9.39, a union of countably many disjoint measurable subsets of finite measure is σ-finite. Conversely, suppose A is measurable, and $A \subset \cup_{i=1}^{\infty} C_i$, with $C_i \in \Sigma$ and $\mu(C_i) < +\infty$. Then

$$A = \sqcup_{i=1}^{\infty}[A \cap (C_i - C_1 - C_2 - \cdots - C_{i-1})].$$

By the first property in Proposition 9.4.4, $A \cap (C_i - C_1 - C_2 - \cdots - C_{i-1}) \subset C_i$ and has finite measure.

Exercise 9.44

By Exercise 9.39, a union of countably many disjoint measurable subsets of finite measure is σ-finite. Conversely, suppose A is measurable, and $A \subset \cup_{i=1}^{\infty} C_i$, with $C_i \in \Sigma$ and $\mu(C_i) < +\infty$. Then

$$A = \cup_{i=1}^{\infty}[A \cap (C_1 \cup C_2 \cup \cdots \cup C_i)].$$

By the second property in Proposition 9.4.4, $C_1 \cup C_2 \cup \cdots \cup C_i$ has finite measure. By the first property of the proposition, $A \cap (C_1 \cup C_2 \cup \cdots \cup C_i)$ has finite measure. We also note that $A \cap (C_1 \cup C_2 \cup \cdots \cup C_i)$ in increasing.

Exercise 9.45

Let $Z = \{x\colon f(x) \neq g(x)\}$. Then

$$\begin{aligned}f^{-1}(A) - g^{-1}(A) &= \{x\colon f(x) \in A,\ g(x) \notin A\} \subset Z, \\ g^{-1}(A) - f^{-1}(A) &= \{x\colon f(x) \notin A,\ g(x) \in A\} \subset Z.\end{aligned}$$

Since Z is contained in a subset of measure 0, we conclude that the difference $(f^{-1}(A) - g^{-1}(A)) \cup (g^{-1}(A) - f^{-1}(A))$ is contained in a subset of measure 0.

Exercise 9.46

(1) Let $f(x) = g(x)$ for $x \notin A$, $g(x) = h(x)$ for $x \notin B$, where $\mu(A) = \mu(B) = 0$. Then we have $f(x) = h(x)$ for $x \notin A \cup B$, and $\mu(A) = \mu(B) = 0$ impies $\mu(A \cup B) = 0$.

(2) Let $f(x)=g(x)$ for $x\notin A$, where $\mu(A)=0$. Then $h\circ f(x)=h\circ g(x)$ for $x\notin A$, and $\mu(A)=0$.

On the other hand, we only know $f\circ h(x)=g\circ h(x)$ for $x\notin h^{-1}(A)$. There is no guarantee that $h^{-1}(A)$ remains a set of measure 0. Specifically, let $f(x)=1$ for $x\neq 0$ and $f(0)=0$, $g(x)=1$ for all x, and $h(y)=0$ for all y. Then $f=g$ almost everywhere (with respect tot he usual Lebesgue measure), yet $f\circ h=0$ is never equal to $f\circ h=1$.

(3) Let $f_1(x)=g_1(x)$ for $x\notin A_1$, $f_2(x)=g_2(x)$ for $x\notin A_2$, where $\mu(A_1)=\mu(A_2)=0$. Then $f_1(x)+f_2(x)=g_1(x)+g_2(x)$ and $f_1(x)f_2(x)=g_1(x)g_2(x)$ for $x\notin A_1\cup A_2$, and $\mu(A_1)=\mu(A_2)=0$ implies $\mu(A_1\cup A_2)=0$.

(4) Let $f_i(x)=g_i(x)$ for $x\notin A_i$, where $\mu(A_i)=0$. Then $\sup f_i(x)=\sup g_i(x)$, $\inf f_i(x)=\inf g_i(x)$, $\overline{\lim}_{i\to\infty} f_i(x)=\overline{\lim}_{i\to\infty} g_i(x)$, and $\underline{\lim}_{i\to\infty} f_i(x)=\underline{\lim}_{i\to\infty} g_i(x)$ for $x\notin\cup A_i$. For the countable union, we have $\mu(A_i)=0$ implying $\mu(\cup A_i)=0$.

Exercise 9.47

(1) Let $A-B\subset X_1$, $B-A\subset X_2$, $B-C\subset Y_1$, $C-B\subset Y_2$, where $\mu(X_1)=\mu(X_2)=\mu(Y_1)=\mu(Y_2)=0$. Then $A-C\subset(A-B)\cup(B-C)\subset X_1\cup Y_1$, and $\mu(X_1)=\mu(Y_1)=0$ implies $\mu(X_1\cup Y_1)=0$. Similarly, $C-A\subset(B-A)\cup(C-B)\subset X_2\cup Y_2$, and $\mu(X_2)=\mu(Y_2)=0$ implies $\mu(X_2\cup Y_2)=0$.

(2) Let $A-B\subset X_1$, $B-A\subset X_2$, $C-D\subset Y_1$, $D-C\subset Y_2$, where $\mu(X_1)=\mu(X_2)=\mu(Y_1)=\mu(Y_2)=0$. Then $(A-C)-(B-D)=(A-B)\cup(D-C)\subset X_1\cup Y_2$, and $\mu(X_1)=\mu(Y_2)=0$ implies $\mu(X_1\cup Y_2)=0$.

(3) Let $A_i-B_i\subset X_i$, where $\mu(X_i)=0$. Then $\cup A_i-\cup B_i=\cup_i(A_i-\cup_j B_j)\subset\cup_i(A_i-B_i)\subset\cup_i X_i$, $\cap A_i-\cap B_i=\cup_j(\cap_i A_i-B_j)\subset\cup_j(A_j-B_j)\subset\cup_j X_j$, and $\mu(X_i)=0$ implies $\mu(\cup X_i)=0$.

Exercise 9.48

Suppose $A_i-A_{i+1}\subset X_i$, $\mu(X_i)=0$. Let $B_i=A_i\cup X_1\cup\cdots\cup X_{i-1}$.

By $A_i\subset B_i$ and subadditivity in Proposition 9.4.4, we have

$$\mu(A_i)\le\mu(B_i)\le\mu(A_i)+\mu(X_1)+\cdots+\mu(X_{i-1})=\mu(A_i).$$

Therefore $\mu(A_i)=\mu(B_i)$. Similarly, we have $\cup_i B_i=(\cup_i A_i)\cup(\cup_i X_i)$, and

$$\mu(\cup_i A_i)\le\mu(\cup_i B_i)\le\mu(\cup_i A_i)+\mu(\cup_i X_i)\le\mu(\cup_i A_i)+\sum_i\mu(X_i)=\mu(\cup_i A_i).$$

Therefore $\mu(\cup_i A_i)=\mu(\cup_i B_i)$.

On the other hand, we have $A_i-A_{i+1}\subset X_i$. This implies $A_i\subset A_{i+1}\cup X_i$, so that

$$B_i\subset(A_{i+1}\cup X_i)\cup X_1\cup\cdots\cup X_{i-1}=A_{i+1}\cup X_1\cup\cdots\cup X_{i-1}\cup X_i=B_{i+1}.$$

By the monotone limit property in Proposition 9.4.4, we have

$$\mu(\cup A_i)=\mu(\cup B_i)=\lim\mu(B_i)=\lim\mu(A_i).$$

Exercise 9.49

The set of those $x \in X$ lying in infinitely many A_n is $\cap_{k=1}^{n} \cup_{n\geq k} A_n$. The sequence $\cup_{n\geq k} A_n$ is decreasing, and by the assumption, we have

$$\mu(\cup_{n\geq 1} A_n) \leq \sum_n \mu(A_n) < +\infty.$$

Then we may apply the monotone limit property in Proposition 9.4.4 to get

$$\mu(\cap_{k=1}^{n} \cup_{n\geq k} A_n) = \lim_{k\to\infty} \mu(\cup_{n\geq k} A_n).$$

Moreover, we have

$$\mu(\cup_{n\geq k} A_n) \leq \sum_{n\geq k} \mu(A_n),$$

and the right side converges to 0 due tot he convergence of the series $\sum_n \mu(A_n)$. Therefore $\lim_{k\to\infty} \mu(\cup_{n\geq k} A_n) = 0$.

For the counterexample in case $\lim \mu(A_n) = 0$, we take A_n to be the sequence of "moving" intervals of length $\frac{1}{k}$:

$$[0,1],\ [0,\frac{1}{2}],\ [\frac{1}{2},1],\ [0,\frac{1}{3}],\ [\frac{1}{3},\frac{2}{3}],\ [\frac{2}{3},1],\dots,[0,\frac{1}{k}],\ [\frac{1}{k},\frac{2}{k}],\dots,[\frac{k-1}{k},1],\dots$$

Then the length of A_n converges to 0, and any $x \in [0,1]$ is in inifnitely many A_n.

[*A more complicated alternative proof*]

Let X_k be the set of those x belong to at least k subsets A_n. Then $X_1 \supset X_2 \supset \cdots \supset X_k \supset \cdots$, and $\cap_k X_k$ is the set of those x belonging to infinitely many A_n.

We fix k and consider the earliest k subsets that any $x \in X_k$ belongs to. This means that, for any $I = \{n_1 < n_2 < \cdots < n_k\}$, we introduce

$$\begin{aligned} A_I &= \{x \in A_{n_i} \text{ for all } 1 \leq i \leq k \text{ and } x \notin \text{other } A_n \text{ satisfying } n \leq n_k\} \\ &= A_{n_1} \cap A_{n_2} \cap \cdots \cap A_{n_k} - \cup_{n\leq n_k,\, n\notin I} A_n. \end{aligned}$$

Then A_I are measurable, and we have

1. $A_I \subset A_n$ for any $n \in I$.
2. A_I are disjoint for distinct sets I of k natural numbers.
3. $X_k = \cup_{|I|=k} A_I$ (the union is disjoint by the second property).

The properties imply that $A_n \supset \sqcup_{n\in I,\, |I|=k} A_I$, and we have

$$\mu(A_n) \geq \sum_{n\in I,\, |I|=k} \mu(A_I).$$

Let $M = \sum_n \mu(A_n) < +\infty$. Then we have

$$M \geq \sum_n \sum_{n\in I,\, |I|=k} \mu(A_I) \geq \sum_{|I|=k} \sum_{n\in I} \mu(A_I) = \sum_{|I|=k} k\mu(A_I) = k\mu(X_k).$$

Here the first equality is due to that each I contains k indices, and the second equality is due to $X_k = \sqcup_{|I|=k} A_I$. Since the inequality holds for all k, we conclude that $\mu(X_k) \leq \dfrac{k}{M}$. This further implies that $\mu(\cap_k X_k) = 0$.

Exercise 9.50

Since $A, B \in \Sigma$ are almost the same, we have $(A-B)\cup(B-A) \subset C$, with some $C \in \Sigma$ and $\mu(C) = 0$. By completeness of the measure space and $A-B, B-A \subset C$, we find $A-B, B-A \in \Sigma$ and $\mu(A-B) = \mu(B-A) = 0$. Therefore $A \in \Sigma$ implies $B = [A-(A-B)]\cup(B-A) \in \Sigma$.

Exercise 9.51

If A is measurable, then the property is satisfied by taking $B = C = A$.

Suppose the property is satisfied. Let $\epsilon_n \to 0$. We have $B_n \subset A \subset C_n$, B_n and C_n measurable, such that $\mu(C_n - B_n) < \epsilon_n$. Then $\cup B_n \subset A \subset \cap C_n$, and by $\mu(\cap C_n - \cup B_n) \leq \mu(C_n - B_n) < \epsilon_n$ for any n, we know $\mu(\cap C_n - \cup B_n) = 0$. Then $A - \cup B_n \subset \cap C_n - \cup B_n$ and the completeness implies that $A - \cup B_n$ is measurable. This implies that $A = (A - \cup B_n) \cup (\cup B_n)$ is measurable.

Exercise 9.52

We have a sequence of measurable subsets B_n and C_n, such that

$$(B_n - A)\cup(A - B_n) \subset C_n, \quad \mu(C_n) < \frac{1}{2^n}.$$

The subset $D = \cap_n(B_n \cup B_{n+1} \cup B_{n+2} \cup \cdots)$ is measurable. We claim that A is almost the same as D. We have

$$\begin{aligned} D - A &= \cap_n[(B_n - A)\cup(B_{n+1}-A)\cup(B_{n+2}-A)\cup\cdots] \subset \cap_n(C_n \cup C_{n+1}\cup C_{n+2}\cup\cdots), \\ A - D &= \cup_n[(A-B_n)\cap(A-B_{n+1})\cap(A-B_{n+2})\cup\cdots] \subset \cup_n(C_n\cap C_{n+1}\cap C_{n+2}\cap\cdots). \end{aligned}$$

By

$$\mu(C_n \cup C_{n+1}\cup C_{n+2}\cup\cdots) \leq \mu(C_n)+\mu(C_{n+1})+\mu(C_{n+2})+\cdots \leq \frac{1}{2^{n-1}},$$

we get $\mu(\cap_n(C_n\cup C_{n+1}\cup C_{n+2}\cup\cdots)) = 0$. By

$$\mu(C_n\cap C_{n+1}\cap C_{n+2}\cap\cdots) \leq \mu(C_k) < \frac{1}{2^k} \text{ for any } k \geq n,$$

we get $\mu(C_n\cap C_{n+1}\cap C_{n+2}\cap\cdots) = 0$ for all n, so that $\mu(\cup_n(C_n\cap C_{n+1}\cap C_{n+2}\cap\cdots)) = 0$. This shows that A and D are almost the same.

Exercise 9.53

By the construction in Proposition 9.4.8, a subset $B \in \bar{\Sigma}$ satisfies $(A-B)\cup(B-A) \subset C$ for some $A, C \in \Sigma$ satisfying $\mu(C) = 0$. Then $A - C - B = (A-B) - C = \emptyset$. Therefore B contains the measurable subset $A - C$. On the other hand, $B - (A - C) = (B - A) \cup (B \cap C) \subset C$. So B is the union of a subset $A - C \in \Sigma$ and another subset $B - (A - C)$ that is contained in a measurable subset of measure 0.

Exercise 9.54

(1) This follows from the remark in Exercise 9.24. Specifically, the only thing we need to verify is the subadditivity $\mu^*(\cup A_i) \leq \sum \mu^*(A_i)$. For any $\epsilon > 0$ and i, we have $A_i \subset C_i \in \Sigma$ satisfying $\mu^*(A_i) > \mu(C_i) - \frac{\epsilon}{2^i}$. Then

$$\sum \mu^*(A_i) > \sum \mu(C_i) - \epsilon \geq \mu(\cup C_i) - \epsilon \geq \mu^*(\cup A_i) - \epsilon,$$

where the second inequality is the subadditivity of the measure μ, and the third inequality follows from the definition of μ^* and $\cup A_i \subset \cup C_i \in \Sigma$. Since the inequality holds for any ϵ, we get $\mu^*(\cup A_i) \leq \sum \mu^*(A_i)$.

For $A \in \Sigma$, we have $\mu^*(A) = \mu(A)$ by the monotone property of μ.

(2) For any natural number n, there is $B_n \in \Sigma$, such that $A \subset B_n$ and $\mu(B_n) < \mu^*(A) + \frac{1}{n}$. Then $A \subset B = \cap B_n \in \Sigma$ and $\mu^*(A) \leq \mu(B) \leq \mu(B_n) < \mu^*(A) + \frac{1}{n}$ for any n. This implies $\mu^*(A) = \mu(B)$.

(3) Consider $B \in \Sigma$. For any subset Y, by part (2), there is $D \in \Sigma$ containing Y, such that $\mu(D) = \mu^*(Y)$. Then

$$\mu^*(Y) = \mu(D) = \mu(D \cap B) + \mu(D - B) \geq \mu^*(Y \cap B) + \mu^*(Y - B),$$

where the second equality is the additivity of μ, and the inequality is the definition of μ^*. The inequality shows that B is μ^*-measurable.

(4) Let A be μ^*-measurable, with $\mu^*(A) < \infty$. By part (2), there is $B \in \Sigma$ containing A, such that $\mu(B) = \mu^*(A)$. Then

$$\mu^*(B) = \mu^*(B \cap A) + \mu^*(B - A) = \mu^*(B) + \mu^*(B - A).$$

where the first equality is the definition of μ^*-measurability of A. Since $\mu(B) = \mu^*(A) < \infty$, we get $\mu^*(B - A) = 0$. Using part (2) again, there is $C \in \Sigma$ containing $B - A$, such that $\mu(C) = \mu^*(B - A) = 0$. Therefore we find $B \in \Sigma$, such that $A \Delta B = B - A$ is contained in $C \in \Sigma$ of measure 0. By the definition of completion, A is measurable with respect to the completion of μ, and $\bar{\mu}(A) = \mu(B) = \mu^*(A)$.

(5) We may use the σ-finite assumption to extend part (4) to the case $\mu^*(A) = \infty$. Specifically, we have $X = \cup X_n$ with $X_n \in \Sigma$ and $\mu(X_n) < \infty$. We have shown that X_n are also μ^*-measurable. Therefore $A \cap X_n$ are μ^*-measurable, with $\mu^*(A \cap X_n) \leq \mu^*(X_n) < \infty$. From the finite case, we know $A \cap X_n$ is measurable with respect to the completion of μ. This implies that $A = \cup(A \cap X_n)$ is measurable with respect to the completion of μ.

Finally, we consider the converse that the $\bar{\mu}$-measurability implies μ^*-measurability. Since the measure induced by any outer measure is complete, and the completion is the minimal complete measure space, the converse is true.

(6) Let $\Sigma = \{\emptyset, X\}$ and $\mu(X) = \infty$. We have $\mu^*(A) = \infty$ when $A \neq \emptyset$ and $\mu^*(\emptyset) = 0$. Then any subset is μ^*-measurable. However, if a subset $A \neq \emptyset, X$, then the differences between A and the subsets in Σ are not of measure 0.

Exercise 9.55

(1) Since A contains infinitely many subsets in Σ, we can find $B \subset A$, such that $\emptyset \neq B \neq A$ and $B \in \Sigma$. Then both $A - B$ and B are non-empty and belong to Σ. Now any subset C satisfying $C \subset A$ and $C \in \Sigma$ must be of the form $C = (C \cap B) \cup (C - B)$, where $C \cap B \subset B$, $C - B \subset A - B$, and both subsets are still in Σ. Since there are infinitely many such C, we conclude that either there are infinitely many such $C \cap B$ or there are infinitely many such $C - B$. Therefore either B contains infinitely many subsets in Σ or $A - B$ contains infinitely many subsets in Σ. Both B and $A - B$ are different from A.

(2) By the first part, we can find $X_1 \subset X$ such that $X_1 \in \Sigma$, $X_1 \neq X$, and X_1 contains infinitely many subsets on Σ. Applying the first part to X_1 again, we get $X_2 \subset X_1$ with the similar property. Keep going, we find a strictly decreasing sequence $X \supset X_1 \supset X_2 \supset \cdots$ of subsets in Σ. Then $A_i = X_{i-1} - X_i$ are non-empty and disjoint ($X_0 = X$), and $A_i \in \Sigma$. Moreover, we have $X = (\sqcup_{i=1}^{\infty} A_i) \sqcup (\cap_{i=1}^{\infty} A_i)$, where $\cap_{i=1}^{\infty} A_i$ is also in Σ. By adding $\cap_{i=1}^{\infty} A_i$ to the union in case the intersection is not empty, we express X as a disjoint union of countably many non-empty subsets in Σ.

(3) By second part, we have $X = \sqcup_{i=1}^{\infty} A_i$ for some nonempty and disjoint $A_i \in \Sigma$. Then for any subset $I \subset \mathbb{N}$, the countable union $X_I = \cup_{i \in I} A_i$ is in Σ. Since A_i are disjoint and non-empty, we have $A_I \neq A_J$ when $I \neq J$. Since the collection of subsets of $\mathbb{N}$ is not countable, we conclude that there are uncountably many subsets in Σ.

Exercise 9.56

When $\mu(A) = +\infty$, we need to show $\alpha = \sup\{\mu(B) \colon B \subset A,\ \mu(B) < \infty\} = +\infty$. If not, then α is finite and $\alpha = \lim \mu(B_n)$, $B_n \subset A$, $\mu(B_n) < \infty$. We also have $B_1 \cup \cdots \cup B_n \subset A$ and $\mu(B_1 \cup \cdots \cup B_n) < +\infty$. Therefore $\mu(B_n) \leq \mu(B_1 \cup \cdots \cup B_n) < \alpha$, which implies that $B = \cup B_n \subset A$ satisfies $\mu(B) = \alpha$.

Now $\mu(A - B) = \mu(A) - \mu(B) = +\infty - \alpha = +\infty$. By the semifinite assumption, there is measurable $C \subset A - B$, such that $+\infty > \mu(C) > 0$. Then $B \cup C \subset A$ and $+\infty > \mu(B \cup C) = \mu(B) + \mu(C) = \alpha + \mu(C) > \alpha$. This contradicts to the definition of α.

We conclude that $\alpha = +\infty$.

Exercise 9.57

Let $X = \sqcup X_n$, $\mu(X_n) < \infty$. Then $\mu(A) = \sum \mu(A \cap X_n)$. If $\mu(A) = +\infty$, then $\sum \mu(A \cap X_n)$ diverges. Therefore some $\mu(A \cap X_n) > 0$, while $\mu(A \cap X_n) \leq \mu(X_n) < +\infty$.

Exercise 9.58

The counting measure on an uncountable set is semifinite but not σ-finite.

Let $\Sigma = \{\emptyset, X\}$ and $\mu(X) = \infty$. The measure is not semifinite.

Exercise 9.59

If $\mu(A) < +\infty$, then it follows directly from the definition of μ_1 and the monotone property of μ that $\mu(A) = \mu_1(A) < +\infty$. On the other hand, the proof of Exercise 9.56 shows that if $\mu(A) = +\infty$, then $\mu_1(A) = +\infty$. This is the same as $\mu_1(A) < +\infty$ implying $\mu(A) < +\infty$.

Now we show μ_1 is a measure. The key is countable additivity: $\mu_1(\sqcup A_n) = \sum \mu_1(A_n)$. If some $\mu_1(A_n) = +\infty$, then both sides are $+\infty$. So we assume all $\mu_1(A_n) < +\infty$. By what we just proved, we have $\mu(A_n) = \mu_1(A_n)$. Therefore $\sum \mu_1(A_n) = \sum \mu(A_n) = \mu(\sqcup A_n)$. It remains to prove that the right side is $\mu_1(\sqcup A_n)$.

If $\mu(\sqcup A_n) < +\infty$, then by what we just proved, we have $\mu(\sqcup A_n) = \mu_1(\sqcup A_n)$, so that the countable additivity holds.

If $\mu(\sqcup A_n) = +\infty$, then $\sum \mu(A_n)$ diverges. This means that $\mu(A_1 \sqcup \cdots \sqcup A_n) = \mu(A_1) + \cdots + \mu(A_n)$ can get arbitrarily large. Since $A_1 \sqcup \cdots \sqcup A_n \subset \sqcup A_k$ and $\mu(A_1 \sqcup \cdots \sqcup A_n) < +\infty$, we conclude that $\mu_1(\sqcup A_k) = +\infty$ by the definition of μ_1. The countable additivity still holds.

Finally, the fact that $\mu_1(A) = \mu(A)$ for A with finite measure implies that μ_1 is semifinite.

Exercise 9.60

If $\mu(A) < \infty$, then $\mu_1(A) + \mu_2(A) = \mu(A) < \infty$, so that $\mu_2(A) = 0$. We conclude that $\mu_1(A) = \mu(A)$ whenever $\mu(A) < \infty$. This implies that $\mu_1(A) \geq \sup\{\mu(B)\colon B \subset A,\ \mu(B) < \infty\}$ for all measurable A. This shows that the semifinite measure μ_1 in Exercise 9.59 is the smallest in such decompositions.

Define $\mu_2(A) = 0$ when A is σ-finite (a countable union of measurable subsets with finite μ) and $\mu_2(A) = +\infty$ when A is not σ-finite. The countable additivity for μ_2 follows from the fact that a countable union is σ-finite if and only if each subset is σ-finite. We claim that, for μ_1 in Exercise 9.59, we have $\mu = \mu_1 + \mu_2$.

If A is σ-finite, then $A = \sqcup A_n$, $\mu(A_n) < +\infty$. This implies $\mu(A_n) = \mu_1(A_n)$. By the countable additivity of μ and μ_1, we get $\mu(A) = \mu_1(A)$. The equality $\mu(A) = \mu_1(A) + \mu_2(A)$ holds because $\mu_2(A) = 0$.

If A is not σ-finite, then we must have $\mu(A) = +\infty$. The equality $\mu(A) = \mu_1(A) + \mu_2(A)$ holds because $\mu_2(A) = +\infty$.

Exercise 9.61

Let $X = \cup X_n$, $\mu(X_n) < \infty$. If $A \cap B$ is measurable for any measurable B with $\mu(B) < +\infty$, then $A \cap X_n$ is measurable for any n. This implies that $A = \cup(A \cap X_n)$ is measurable.

Exercise 9.62

The μ^*-measurability of A is the same as

$$\mu^*(Y) = \mu^*(Y \cap A) + \mu^*(Y - A), \text{ for any } Y \text{ satisfying } \mu^*(Y) < +\infty.$$

In case $\mu^*(Y) < +\infty$, we have $Y \subset B$ for a μ^*-measurable B satisfying $\mu^*(B) < +\infty$. Then the equality is the same as $\mu^*(Y) = \mu^*(Y \cap (A \cap B)) + \mu^*(Y - (A \cap B))$. The observations shows that A is μ^*-measurable if and only if $A \cap B$ is μ^*-measurable for all μ^*-measurable B satisfying $\mu^*(B) < +\infty$.

Exercise 9.63

$X \cap B = B$ is measurable for any measurable B, whether $\mu(B)$ is finite or not.

If $A \cap B$ and $A' \cap B$ are measurable for any measurable B satisfying $\mu(B) < +\infty$, then $(A - A') \cap B = A \cap B - A' \cap B$ is measurable for any measurable B satisfying $\mu(B) < +\infty$.

If $A_n \cap B$ are measurable for any measurable B satisfying $\mu(B) < +\infty$, then $(\cup A_n) \cap B = \cup(A_n \cap B)$ is measurable for any measurable B satisfying $\mu(B) < +\infty$.

Exercise 9.64

We show countable additivity: $\mu'(\sqcup A_n) = \sum \mu'(A_n)$ for disjoint $A_n \in \Sigma'$. If all $A_n \in \Sigma$, then this is the countable additivity of μ. If some $A_n \notin \Sigma$, then $\sqcup A_n \notin \Sigma$, and the both sides are $+\infty$.

We show μ' is saturated. Suppose $A \cap B$ is measurable for any $B \in \Sigma'$ satisfying $\mu'(B) < +\infty$. By the definition of μ', this is the same as $A \cap B$ is measurable for any $B \in \Sigma$ satisfying $\mu(B) < +\infty$. By the definition of Σ', this means that $A \in \Sigma'$.

Exercise 9.65

Any extension μ' must satisfy

$$\mu'(A) \geq \mu''(A) = \sup\{\mu(B) \colon A \supset B \in \Sigma,\ \mu(B) < +\infty\}, \quad A \in \Sigma'.$$

As shown in Exercise 9.59, μ'' is a measure.

It remains to show that μ'' is saturated. Suppose $A \cap B \in \Sigma'$ for any $B \in \Sigma'$ satisfying $\mu''(B) < +\infty$. This means that $A \cap B \cap C \in \Sigma$ for any $B \in \Sigma'$ satisfying $\mu''(B) < +\infty$ and any $C \in \Sigma$ satisfying $\mu(C) < +\infty$. Note that we may take $B = C$, so that the property implies that $A \cap C \in \Sigma$ for any $C \in \Sigma$ satisfying $\mu(C) < +\infty$. This means $A \in \Sigma'$, and therefore μ'' is saturated.

Let $X = \{a, b\}$, $\Sigma = \{\emptyset, X\}$ and $\mu(X) = \infty$. Then $\{a\}$ and $\{b\}$ are locally measurable, and Σ' is all subsets of X. We have $\mu'(a) = \mu'(b) = 0$ and $\mu''(a) = \mu''(b) = 0$.

Exercise 10.1

Let $X = \mathbb{N}$. Let $|f| < B$. For any $\epsilon > 0$, there is N, such that $\sum_{x>N} \mu_x < \epsilon$. Take

$$\sqcup X_i = \{x_1\} \sqcup \{x_2\} \sqcup \cdots \sqcup \{x_N\} \sqcup \{x_n : n > N\}.$$

If $\sqcup Y_j$ refines $\sqcup X_i$, then

$$\sqcup Y_j = \{x_1\} \sqcup \{x_2\} \sqcup \cdots \sqcup \{x_N\} \sqcup Z_1 \sqcup Z_2 \sqcup \cdots,$$

where $Z_1 \sqcup Z_2 \sqcup \cdots$ is a partition of $\{x_n : n > N\}$. Thus

$$\begin{aligned}
\left| S(\sqcup Y_j, f) - \sum \mu_x f(x) \right| &= \left| \sum_{x=1}^{N} \mu_x f(x) + \sum_j \mu(Z_j) f(z_j^*) - \sum_{x=1}^{\infty} \mu_x f(x) \right| \\
&= \left| \sum_j \sum_{x \in Z_j} \mu_x f(z_j^*) - \sum_{x>N} \mu_x f(x) \right| \\
&= \left| \sum_{x>N} \mu_x (f(z_j^*) - f(x)) \right| \\
&\le 2B \sum_{x>N} \mu_x < 2B\epsilon.
\end{aligned}$$

Here for $x > N$, we have $x \in Z_j$, and z_j^* is the sample number from Z_j.

Exercise 10.2

If $Y_j \subset X_j$, then $\omega_{Y_j}(f) \le \omega_{X_i}(f)$. Therefore, if $\sqcup Y_j$ refines $\sqcup X_i$, then we have

$$\begin{aligned}
\sum_j \omega_{Y_j}(f)\mu(Y_j) &= \sum_i \sum_{Y_j \subset X_i} \omega_{Y_j}(f)\mu(Y_j) \le \sum_i \sum_{Y_j \subset X_i} \omega_{X_i}(f)\mu(Y_j) \\
&\le \sum_i \omega_{X_i}(f) \sum_{Y_j \subset X_i} \mu(Y_j) = \sum_i \omega_{X_i}(f)\mu(X_i).
\end{aligned}$$

Exercise 10.3

Partitions $\sqcup A_i$ of A are in one-to-one correspondence with partitions $(\sqcup A_i) \sqcup (X - A)$ of X, and $(\sqcup A_i) \sqcup (X - A)$ refines the partition $A \sqcup (X - A)$ of X. On the other hand, any partition that refines $A \sqcup (X - A)$ is $(\sqcup A_i) \sqcup (\sqcup B_j)$ with $A_i \subset A$ and $B_j \subset X - A$, and for sample points $x_i^* \in A_i$ and $x_j^* \in B_j$, we have

$$S((\sqcup A_i) \sqcup (\sqcup B_j), f\chi_A) = \sum_i f(x_i^*) \cdot \mu(A_i) + \sum_j 0 \cdot \mu(B_j) = \sum_i f(x_i^*) \cdot \mu(A_i) = S(\sqcup A_i, f|_A).$$

The equality implies that $f\chi_A$ is integrable if and only if $f\chi_A$ is integrable, and the two integrals are equal.