Exercise 10.4

Since f is measurable, we have $f^{-1}(a,b] \in \Sigma$. Then $(f \circ g)^{-1}(a,b] = g^{-1}(f^{-1}(a,b]) \in g^{-1}(\Sigma)$ by the definition of $g^{-1}(\Sigma)$.

Exercise 10.5

Since $f \circ g$ is measurable, we have $f^{-1}(a,b] \in \Sigma$. Then $g^{-1}(f^{-1}(a,b]) = (f \circ g)^{-1}(a,b] \in \Sigma$. By the definition of $g_*(\Sigma)$, this implies $f^{-1}(a,b] \in g_*(\Sigma)$.

Exercise 10.6

Let $\epsilon_n > 0$ converge to 0. Then

$$\begin{aligned}
f^{-1}[a,b] &= f^{-1}(\cap(a-\epsilon_n, b+\epsilon_n)) = \cap f^{-1}(a-\epsilon_n, b+\epsilon_n),\\
f^{-1}(a,b] &= f^{-1}(\cup[a+\epsilon_n, b]) = \cup f^{-1}(a+\epsilon_n, b],\\
f^{-1}[a,+\infty) &= f^{-1}(\cup(a, b+n]) = \cup f^{-1}(a, b+n],\\
f^{-1}(-\infty,a) &= f^{-1}(\mathbb{R} - [a,+\infty)) = X - f^{-1}[a,+\infty),\\
f^{-1}(-\infty,a] &= f^{-1}(\cap(-\infty, a+\epsilon_n]) = \cap f^{-1}(-\infty, a+\epsilon_n],\\
f^{-1}(a,b) &= f^{-1}(\cup(-\infty, b-\epsilon_n] - (-\infty,a]) = \cup f^{-1}(-\infty, b-\epsilon_n] - f^{-1}(-\infty,a].
\end{aligned}$$

This shows that 1, 2, 3, 7, 8, 9 are equivalent.

Note that 4 and 5 are special case of 1. Conversely, for any a, b, we have rational decreasing sequence $r_n \to a$ and rational increasing sequence $s_n \to b$. Then $f^{-1}(a,b) = f^{-1}(\cup(r_n, s_n)) = \cup f^{-1}(r_n, s_n)$ shows that 4 implies 1. Moreover, if $b - a < n$, then we can find n intervals (a_i, b_i) of length $b_i - a_i < 1$, such that $(a,b) = (a_1,b_1) \cup \cdots \cup (a_n,b_n)$. Thus $f^{-1}(a,b) = f^{-1}(a_1,b_1) \cup \cdots \cup f^{-1}(a_n,b_n)$, and this shows that 5 implies 1.

1 is a special case of 10. The converse is due to the fact that any open subset is a countable union of open intervals, and $f^{-1}(\cup(a_i,b_i)) = \cup f^{-1}(a_i,b_i)$.

10 and 11 are equivalent because for $C = \mathbb{R} - U$, U is open if and only if C is closed, and $f^{-1}(U) = X - f^{-1}(C)$, $f^{-1}(C) = X - f^{-1}(U)$.

12 is a special case of 11. Conversely, if C is closed, then $C \cap [-n,n]$ is compact, and $f^{-1}(C) = \cup f^{-1}(C \cap [-n,n])$. This shows that 12 implies 11.

Exercise 10.7

Consider $f_*(\Sigma) = \{B \subset \mathbb{R} \colon f^{-1}(B) \in \Sigma\}$. If f is measurable, then $(a,b] \in f_*(\Sigma)$ for all $a < b$. By Exercise 9.35, $f_*(\Sigma)$ is a σ-algebra of subsets of $\mathbb{R}$. Since the Borel σ-algebra is the smallest σ-algebra containing all $(a,b]$, we conclude that $f_*(\Sigma)$ contains the Borel σ-algebra. In other words, any Borel subset B lies in $f_*(\Sigma)$. By the definition of $f_*(\Sigma)$, this means that $f^{-1}(B) \in \Sigma$, i.e., $f^{-1}(B)$ is measurable for any Borel subset B.

Exercise 10.8

We know $f^{-1}[n,+\infty)$ is decreasing, and

$$\cap f^{-1}[n,+\infty) = f^{-1}(\cap[n,+\infty)) = f^{-1}(\emptyset) = \emptyset.$$

By $\mu(X) < +\infty$ and the monotone limit property (Proposition 9.4.4), we get

$$\lim_{n\to\infty} \mu(f^{-1}[n,+\infty)) = 0.$$

The proof of $\lim_{n\to\infty}\mu(f^{-1}(-\infty,-n])=0$ is similar.

So for any $\epsilon>0$, there is n, such that $\mu(f^{-1}[n,+\infty))<\epsilon$ and $\mu(f^{-1}(-\infty,-n])<\epsilon$. This means that, if $x\notin A=f^{-1}[n,+\infty)\cup f^{-1}(-\infty,-n]$, then $|f(x)|<n$. In other words, f is bounded on $X-A$, where $\mu(A)=\mu(f^{-1}[n,+\infty))+\mu(f^{-1}(-\infty,-n])<2\epsilon$.

Exercise 10.9

Suppose $f(x)=g(x)$ for $x\in X-A$, $\mu(A)=0$. Then

$$(X-A)\cap f^{-1}(a,b]=(f|_{X-A})^{-1}(a,b]=(g|_{X-A})^{-1}(a,b]=(X-A)\cap g^{-1}(a,b].$$

This implies that the difference $f^{-1}(a,b]\Delta g^{-1}(a,b]\subset A$. Since $\mu(A)=0$ and the measure is complete, by Exercise 9.50, $f^{-1}(a,b]$ is measurable if and only if $g^{-1}(a,b]$ is measurable.

Exercise 10.10

Suppose $A\subset\mathbb{R}$ is not Lebesgue measurable. For each $a\in A$, the function $f_a(x)=1$ for $x\neq a$ and $f_a(a)=0$ is lower semicontinuous. However, the infimum function $f=\inf_{a\in A}f_a$ is given by $f(x)=1$ for $x\notin A$ and $f(x)=0$ for $x\in A$. Since $f^{-1}(-\infty,1)=A$ is not measurable, f is not measurable.

The key here is that A has to be uncountable. So the infimum of uncountably many measurable functions is not necessarily measurable.

Exercise 10.11

(1) Suppose f and g are lower semicontinuous. For any x_0 and $\epsilon>0$, there is $\delta>0$, such that

$$|x-x_0|<\delta \implies f(x)>f(x_0)-\epsilon,\ g(x)>g(x_0)-\epsilon.$$

Then

$$|x-x_0|<\delta \implies f(x)+g(x)>f(x_0)+g(x_0)-2\epsilon,\ cf(x)>cf(x_0)-c\epsilon.$$

This proves that $f+g$ and cf are lower semicontinuous.

(2) Suppose f is lower semicontinuous. For any x_0 and $\epsilon>0$, there is $\delta>0$, such that

$$|x-x_0|<\delta \implies f(x)>f(x_0)-\epsilon.$$

Then

$$|x-x_0|<\delta \implies -f(x)<-f(x_0)+\epsilon.$$

this shows that $-f$ is upper semicontinuous.

The converse is similar.

(3) Suppose f is lower semicontinuous and g is continuous. For any x_0 and $\epsilon>0$, there is $\lambda>0$, such that

$$|y-g(x_0)|<\lambda \implies f(y)>f(g(x_0))-\epsilon.$$

By the continuity of g, for this $\lambda>0$, there is $\delta>0$, such that

$$|x-x_0|<\delta \implies |g(x)-g(x_0)|<\lambda.$$

Combining the implications, we have

$$|x-x_0|<\delta \implies f(g(x))>f(g(x_0))-\epsilon.$$

This proves that $f \circ g$ is lower semicontinuous.

(4) Suppose f is lower semicontinuous and g is increasing and left continuous. For any x_0 and $\epsilon > 0$, there is $\lambda > 0$, such that

$$f(x_0) - \lambda < y \le f(x_0) \implies g(f(x_0)) - \epsilon < g(y) \le g(f(x_0)).$$

By the lower semicontinuity of f, for this $\lambda > 0$, there is $\delta > 0$, such that

$$|x - x_0| < \delta \implies f(x) > f(x_0) - \lambda.$$

Now if $|x - x_0| < \delta$, then we have either $f(x_0) \ge f(x) > f(x_0) - \lambda$ or $f(x) > f(x_0)$. In the first case, we have $g(f(x)) > g(f(x_0)) - \epsilon$. In the second case, by g increasing, we have $g(f(x)) \ge g(f(x_0)) > g(f(x_0)) - \epsilon$. Therefore we always get $g(f(x)) > g(f(x_0)) - \epsilon$.

Exercise 10.12

Suppose $f \ge g$ and g is lower semicontinuous. For any x_0 and $\epsilon > 0$, there is $\delta > 0$, such that

$$|x - x_0| < \delta \implies g(x) > g(x_0) - \epsilon \implies f(x) > g(x_0) - \epsilon.$$

Then $\inf_{(x_0-\delta', x_0+\delta')} f \ge g(x_0) - \epsilon$ for any $0 < \delta' \le \delta$, so that the lower envelop

$$f_*(x_0) = \lim_{\delta \to 0^+} \inf_{(x_0-\delta, x_0+\delta)} f \ge g(x_0) - \epsilon$$

for any $\epsilon > 0$. Since ϵ is arbitrary, we get $f_*(x_0) \ge g(x_0)$ for any x_0.

It remains to show that the lower envelop f_* is lower semicontinuous. By the definition of $f_*(x_0)$, for any x_0 and $\epsilon > 0$, there is $\delta > 0$, such that

$$\inf_{(x_0-\delta, x_0+\delta)} f > f_*(x_0) - \epsilon.$$

Now if $|x - x_0| < \delta$, then we have $(x - \delta', x + \delta') \subset (x_0 - \delta, x_0 + \delta)$ for some $\delta' > 0$. Then $0 < \delta'' < \delta'$ implies $(x - \delta'', x + \delta'') \subset (x - \delta', x + \delta')$, so that

$$\inf_{(x-\delta'', x+\delta'')} f \ge \inf_{(x_0-\delta, x_0+\delta)} f > f_*(x_0) - \epsilon.$$

This implies that $f_*(x) = \lim_{\delta'' \to 0} \inf_{(x-\delta'', x+\delta'')} f \ge f_*(x_0) - \epsilon$. So we have shown

$$|x - x_0| < \delta \implies f_*(x) \ge f_*(x_0) - \epsilon,$$

which means that f_* is lower semicontinuous.

Exercise 10.13

By Theorem 10.2.3, the Lebesgue integrability of f implies that f is equal to a measurable function almost everywhere. Then the conclusion follows from Exercise 9.45.

Exercise 10.14

For the case f is measurable, the subset Y_ϵ is measurable, and $f \ge \epsilon \chi_{Y_\epsilon}$. Then

$$0 \le \epsilon \mu(Y_\epsilon) = \int_X \epsilon \chi_{Y_\epsilon} d\mu \le \int_X f d\mu = 0.$$

This implies $\mu(Y_\epsilon) = 0$. Pick a decreasing sequence $\epsilon_n \to 0$. Then

$$\cup Y_{\epsilon_n} = \{x \in X : f(x) > 0\} = \{x \in X : f(x) \neq 0\},$$

and $\mu(Y_{\epsilon_n}) = 0$ for each n implies $\mu(\cup Y_{\epsilon_n}) = 0$.

By Theorem 10.2.3, a Lebesgue integrable f equals a measurable function g almost everywhere. If $f \geq 0$, then we may fuether assume that $g \geq 0$. Moreover, $\int_X f d\mu = 0$ implies $\int_X g d\mu = 0$. Then we get $g = 0$ almost everywhere. This implies $f = 0$ almost everywhere.

Exercise 10.15

By Theorem 10.2.3, we know there is a bounded and measurable g, such $f = g$ almost everywhere. We also have $\int_A g d\mu = \int_A f d\mu = 0$ for all $A \in \Sigma$.

Since g is measurable, the subset $Y_\epsilon = \{x : g(x) \geq \epsilon\}$ is measurable for any $\epsilon > 0$. Moreover, by Proposition 10.1.4, we have

$$0 = \int_{Y_\epsilon} g dx \geq \epsilon \mu(A_\epsilon).$$

Since the right side is non-negative, we get $\mu(Y_\epsilon) = 0$. Pick a decreasing sequence $\epsilon_n \to 0$. Then $\{x : g(x) > 0\} = \cup_{n=1}^{\infty} Y_{\epsilon_n}$ also has measure 0. By the same reason, $\{x : g(x) < 0\}$ also has measure 0. Therefore $g = 0$ away from the the subset $\{x : g(x) \neq 0\} = \{x : g(x) > 0\} \cup \{x : g(x) < 0\}$ of measure 0. In other words, we have $g = 0$ almost everywhere. Since $f = g$ almost everywhere, we conclude that $f = 0$ almost everywhere.

Exercise 10.16

If $\int_a^b f dx = 0$ on any bounded interval (a, b), then by Proposition 10.1.4(5), we have $\int_U f dx = 0$ on any finite disjoint union U of open intervals.

Let $|f| < M$. By Exercise 9.17(3), for the Lebesgue measurable subset A with $\mu(A) < +\infty$, there is a finite disjoint union U of open intervals, such that $\mu((A - U) \cup (U - A)) < \epsilon$. Then

$$\left| \int_A f dx \right| = \left| \int_A f dx - \int_U f dx \right| \leq M\mu((A - U) \cup (U - A)) < M\epsilon.$$

Since ϵ is arbitrary, we get $\int_A f dx = 0$. By Exercise 10.15, this implies that $f = 0$ almost everywhere.

Strictly speaking, we only know that $f = 0$ almost everywhere on any bounded interval $(-n, n)$. Then $\{x : f(x) \neq 0\} \cap (-n, n)$ has 0 measure implies that $\{x : f(x) \neq 0\} = \cup_n \{x : f(x) \neq 0\} \cap (-n, n)$ also has 0 measure.

Exercise 10.17

(1) Let

$$X_i = f^{-1}(c_{i-1}, c_i] = f^{-1}(-\infty, c_i] - f^{-1}(-\infty, c_{i-1}].$$

Then by $a < f < b$, we have $X = \sqcup_{i=1}^n X_i$. Moreover,

$$\begin{aligned}\Delta\alpha_i &= \alpha(c_i) - \alpha(c_{i-1}) = \mu(f^{-1}(-\infty, c_i]) - \mu(f^{-1}(-\infty, c_{i-1}]) \\ &= \mu(f^{-1}(-\infty, c_i] - f^{-1}(-\infty, c_{i-1}]) = \mu(X_i).\end{aligned}$$

Therefore by $c_{i-1} < f \le c_i$ on X_i, we have

$$\sum_{i=1}^n c_{i-1}\Delta\alpha_i = \sum_{i=1}^n c_{i-1}\mu(X_i) \le \int_X f d\mu = \sum_{i=1}^n \int_{X_i} f d\mu \le \sum_{i=1}^n c_i\mu(X_i) = \sum_{i=1}^n c_i\Delta\alpha_i.$$

(2) The Riemann-Stieltjes integral $\int_a^b x d\alpha$ is the limit of the Riemann-Stieltjes sum

$$S(\{c_i\}, x, \alpha) = \sum_{i=1}^n c_i^*\Delta\alpha_i, \quad c_i^* \in [c_{i-1}, c_i].$$

Since α is increasing, we have

$$\sum_{i=1}^n c_{i-1}\Delta\alpha_i \le S(\{c_i\}, x, \alpha) \le \sum_{i=1}^n c_i\Delta\alpha_i.$$

Then by the first part, we have

$$\left| S(\{c_i\}, x, \alpha) - \int_X f d\mu \right| \le \sum_{i=1}^n (c_i - c_{i-1})\Delta\alpha_i \le (\max\{c_i - c_{i-1}\})\,(\alpha(b) - \alpha(a)).$$

This implies that, as the side of the partition $\{c_i\}$ approaches 0,

$$\int_a^b x d\alpha = \lim_{\max\{c_i - c_{i-1}\}\to 0} S(\{c_i\}, x, \alpha) = \int_X f d\mu.$$

Exercise 10.18

For any $B \in \Sigma$, $\mu(B) < +\infty$, $a < 0 < b$, we have

$$\int_B (f\chi_A)_{[a,b]} d\mu = \int_B f_{[a,b]}\chi_A d\mu = \int_{B\cap A} f_{[a,b]} d\mu = \int_{B\cap A} f_{[a,b]} d\mu_A.$$

We also know that $B \cap A$ are all the possible measurable subsets of A with finite measure. By taking the limit of both sides, therefore, we get $\int_A f d\mu_A = \int_X f\chi_A d\mu$.

Exercise 10.19

By Proposition 10.3.3, the integrability is the same as boundedness. If $|f| < g$ and the integral of g is bounded, then the integral of f is bounded.

Exercise 10.20

If f is integrable, then for any finite subset A, we have $a < f < b$ on A for some a, b, so that

$$\left|\sum_{x\in A} \mu_x f(x)\right| = \left|\int_A f_{[a,b]} d\mu\right| < M.$$

If we further know $\epsilon^{-1} > f > \epsilon$ and $\mu_x > \epsilon$ on A, then

$$\epsilon^2 |A| \le \epsilon \sum_{x\in A} \mu_x \le \sum_{x\in A} \mu_x f(x) = \left|\int_A f_{[\epsilon,\epsilon^{-1}]} d\mu\right| < M.$$

Therefore the number of elements in A has upper bound $\frac{M}{\epsilon^2}$. This implies that the subset $\{x\colon \epsilon^{-1} > f(x) > \epsilon,\ \mu_x > \epsilon\}$ is finite for any $\epsilon > 0$. (If this is infinite, then we can find a finite subset of more than $\frac{M}{\epsilon^2}$ elements.) By the same reason, the subset $\{x\colon -\epsilon^{-1} < f(x) < -\epsilon,\ \mu_x > \epsilon\}$ is also finite. Therefore

$$X_\epsilon = \{x\colon \epsilon^{-1} > |f(x)| > \epsilon,\ \mu_x > \epsilon\}$$

is finite for any $\epsilon > 0$. Note that $\cup X_{\frac{1}{n}}$ is exactly those x such that $\mu_x f(x) \neq 0$. Therefore the sum $\sum \mu_x f(x)$ contains only countably many nonzero terms. Since for any finite A, the partial sum $\sum_{x\in A} \mu_x f(x)$ is bounded, the series $\sum \mu_x f(x)$ converges absolutely. This implies that for any $\epsilon > 0$, there is a finite Y, such that $\sum_{x\notin Y} \mu_x |f(x)| < \epsilon$. Then $A \supset Y$ implies

$$\begin{aligned}\left|\int_A f(x)d\mu - \sum \mu_x f(x)\right| &= \left|\sum_{x\in A} \mu_x f(x) - \sum \mu_x f(x)\right| \\ &= \left|\sum_{x\notin A} \mu_x f(x)\right| \le \sum_{x\notin A} \mu_x |f(x)| \le \sum_{x\notin Y} \mu_x |f(x)| < \epsilon.\end{aligned}$$

By the definition of unbounded integral, we get $\int f(x)d\mu = \sum \mu_x f(x)$.

Exercise 10.21

(1) By Proposition 10.3.3, the integrability of f implies that $I = \int_X f^+ d\mu$ and $J = \int_X f^- d\mu$ are finite numbers. Then

$$\{x\colon |f(x)| \ge b\} = A_b \sqcup B_b, \quad A_b = \{x\colon f(x) \ge b\}, \quad B_b = \{x\colon f(x) \le -b\}.$$

We also have $f^+ \ge b\chi_{A_b}$ and $f^- \ge b\chi_{B_b}$. Since X is σ-finite (or semifinite), by the third property of Proposition 10.3.4 and Example 10.3.2, we have

$$\int_X f^+ d\mu \ge \int_X b\chi_{A_b} d\mu = b\mu(A_b).$$

Therefore $\mu(A_b) \le b^{-1}I$. We also have $\mu(B_b) \le b^{-1}J$ by the same reason. Then

$$0 \le \mu\{x\colon |f(x)| \ge b\} = \mu(A_b) + \mu(B_b) \le b^{-1}(I + J).$$

This implies $\lim_{b\to+\infty} \mu\{x\colon |f(x)| \ge b\} = 0$.

(2) This follows from $\{x\colon f(x) = \pm\infty\} \subset \{x\colon |f(x)| \ge b\}$ for any b and the first part.

(3) Let $g = 0$ on $\{x\colon f(x) = \pm\infty\}$ and $g = f$ away from $\{x\colon f(x) = \pm\infty\}$. Then g does not take extended value. Moreover, by the second part, we have $f = g$ almost everywhere.

Exercise 10.22

For any $B \in \Sigma$, $\mu(B) < +\infty$, $a < 0 < b$, we have

$$\int_B (f\chi_A)_{[a,b]} d\mu = \int_B f_{[a,b]}\chi_A d\mu = \int_{B\cap A} f_{[a,b]} d\mu = \int_{B\cap A} f_{[a,b]} d\mu_A.$$

We also know that $B \cap A$ are all the possible measurable subsets of A with finite measure. Therefore by Definition 10.3.5, we have $\int_X f\chi_A d\mu = +\infty$ if and only if $\int_A f d\mu_A = +\infty$.

Exercise 10.23

The proof of the third part of Proposition 10.3.4 directly applies here.

Exercise 10.24

(1) By Proposition 10.3.6, we have $\int_A f^+ d\mu = +\infty$, and f^- is integrable on A. By the fourth property of Proposition 10.3.4, f^- is integrable on B. Then $\int_B f d\mu$ is finite or $+\infty$, depending whether $\int_B f^+ d\mu$ is finite or $+\infty$.

(2) By Proposition 10.3.6, we have $\int_A f^+ d\mu = +\infty$, and f^- is integrable on A and on B. By the fourth property of Proposition 10.3.4, f^- is integrable on $A\cup B$. In other words, $\int_{A\cup B} f^- d\mu$ is bounded. On the other hand, we have $f^+ \ge f^+\chi_A$ on $A \cup B$. By Exercise 10.22, we know $\int_{A\cup B} f^+\chi_A d\mu = \int_A f^+ d\mu = +\infty$. By Exercise 10.23, we have $\int_{A\cup B} f^+ d\mu = +\infty$. Then by Proposition 10.3.6, we get $\int_{A\cup B} f d\mu = +\infty$.

Remark: The extended valued version of

$$\int_{A\cup B} f d\mu = \int_A f d\mu + \int_B f d\mu - \int_{A\cap B} f d\mu$$

can presumably be $+\infty = +\infty + (I \text{ or } +\infty) - (J \text{ or } -\infty)$, or $+\infty = I + J - (-\infty)$. The statement of the exercise is $+\infty = +\infty + (+\infty \text{ or } I) - *$ (actually the first part says that $*$ cannot be $-\infty$), which includes the first version. The second version cannot happen because the integrability on A implies the integrability on $A \cap B$.

Exercise 10.25

(1) By Proposition 10.3.6, the assumption tells us that $\int_X f^+ d\mu$ is unbounded, and $\int_X f^- d\mu$ and $\int_X g^- d\mu$ are bounded. We have $f + g \geq f - g^- = f^+ - (f^- + g^-)$. By Exercise 10.23, to prove $\int_X (f+g)d\mu = +\infty$, it is sufficient to prove $\int_X (f^+ - (f^- + g^-))d\mu = +\infty$.

By the second property of Proposition 10.3.4, $\int_X (f^- + g^-)d\mu$ is bounded. Therefore the problem is reduced to the following: If $f, g \geq 0$, $\int_X f d\mu = +\infty$ and g is integrable, then $\int_X (f-g)d\mu = +\infty$.

Since $(f-g)^- \leq g$, and g is integrable, by Proposition 10.3.3 or Exercise 10.19, we know $(f-g)^-$ is integrable. Then by Proposition 10.3.6, we have either $\int_X (f-g)d\mu = +\infty$ or $f - g$ is integrable. If $f - g$ were integrable, then by the second property of Proposition 10.3.4, the integrability of g implies the integrability of $f = (f-g) + g$. Since we know $\int_X f d\mu = +\infty$, we conclude that $f - g$ cannot be integrable. Therefore we conclude that $\int_X (f-g)d\mu = +\infty$.

(2) We have $(cf)_{[a,b]} = cf_{[c^{-1}a, c^{-1}b]}$ for $c > 0$ and $(cf)_{[a,b]} = cf_{[c^{-1}b, c^{-1}a]}$ for $c < 0$. Fitting the equality into Definition 10.3.5 proves the claim.

Exercise 10.26

We first show that the subsets $f^{-1}(+\infty)$ and $g^{-1}(+\infty)$ are almost the same. If not, then there is A with $\mu(A) > 0$, such that $f = +\infty$ and $g < +\infty$ on A (or $f < +\infty$ and $g = +\infty$ on A). Let $A_n = \{x \in A \colon g(x) < n\}$. Then $A = \cup_{n=1}^{\infty} A_n$, and $\mu(A) > 0$ implies $\mu(A_n) > 0$ for some n. Since μ is semifinite, we can find $B \subset A_n$, satisfying $0 < \mu(B) < +\infty$.

We have $f = +\infty$ and $g < n$ on B. Since $\mu(B) < +\infty$, we have $\int_B f d\mu = +\infty$ and $\int_B g d\mu \leq n\mu(B) < +\infty$. We get a contradiction $\int_B f d\mu \neq \int_B g d\mu$. The contradiction shows that $f^{-1}(+\infty)$ and $g^{-1}(+\infty)$ are almost the same. By the same reason, $f^{-1}(-\infty)$ and $g^{-1}(-\infty)$ are almost the same.

By focus on $X - f^{-1}(+\infty) \cup g^{-1}(+\infty) - f^{-1}(-\infty) \cup g^{-1}(-\infty)$, we may assume that f and g do not take extended values on X. This implies that, if we introduce

$$X_n = \{x \colon |f(y)| < n,\ |g(y)| < n,\ f(y) - g(y) > \frac{1}{n}\},$$
$$Y_n = \{x \colon |f(y)| < n,\ |g(y)| < n,\ g(y) - f(y) > \frac{1}{n}\},$$

then the places f and g are different is $\cup_{n=1}^{\infty}(X_n \cup Y_n)$.

For any measurable subset $A \subset X_n$ with $\mu(A) < +\infty$, we have

$$0 = \int_A f d\mu - \int_A g d\mu = \int_A (f - g) d\mu \geq \frac{1}{n}\mu(A).$$

Therefore $\mu(A) = 0$. By the semifiniteness of μ, we get $\mu(X_n) = 0$. By the same reason, we also get $\mu(Y_n) = 0$. Therefore $\cup_{n=1}^{\infty}(X_n \cup Y_n)$ has measure 0.

Exercise 10.27

By Proposition 10.3.3, we may assume $f \geq 0$.

For any $\epsilon > 0$, there is Y and N, such that $\mu(Y) < +\infty$ and

$$Y \subset A,\ \mu(A) < +\infty,\ b \geq N \implies \left| \int_A f_{[0,b]} d\mu - \int_X f d\mu \right| < \epsilon.$$

By $\int_A f_{[0,b]} d\mu \leq \int_A f d\mu \leq \int_X f d\mu$, we get

$$Y \subset A,\ \mu(A) < +\infty \implies \int_A f d\mu \leq \int_A f_{[0,N]} d\mu + \epsilon.$$

Then for any measurable A with $\mu(A) < +\infty$, we apply the above to $A \cup Y$ and get

$$\begin{aligned} \int_A f d\mu &= \int_{A\cup Y} f d\mu - \int_{Y-A} f d\mu \leq \int_{A\cup Y} f_{[0,N]} d\mu + \epsilon - \int_{Y-A} f_{[0,N]} d\mu \\ &= \int_A f_{[0,N]} d\mu + \epsilon \leq N\mu(A) + \epsilon. \end{aligned}$$

The estimation tells us

$$\mu(A) < \frac{\epsilon}{N} \implies \int_A f d\mu \leq 2\epsilon.$$

[*Alternative proof that uses Theorem 10.4.2*]

Again we may assume $f \geq 0$.

Suppose the claim is not true. Then there is $\epsilon > 0$ and measurable subsets A_n, such that $\mu(A_n) < \frac{1}{2^n}$ and $\int_{A_n} f d\mu > \epsilon$. Let $B_n = A_n \cup A_{n+1} \cup \cdots$. Then $B_n \supset B_{n+1}$ and $\mu(B_n) < \sum_{k \geq n} \frac{1}{2^k} = \frac{1}{2^{n-1}}$. Moreover, by $A_n \subset B_n$ and $f \geq 0$, we have

$$\int_X f\chi_{B_n} d\mu = \int_{B_n} f d\mu \geq \int_{A_n} f d\mu > \epsilon,$$

so that $\int_X f\chi_{B_n} d\mu$ does not converge to 0. On the other hand, $f\chi_{B_n}$ is a decreasing sequence of integrable functions satisfying $\lim_{n\to\infty} f\chi_{B_n} = f\chi_{\cap B_n}$, with $\mu(\cap B_n) = 0$ (because $\mu(\cap B_n) \le \mu(B_n) < \frac{1}{2^{n-1}}$ for any n). By Monotone Convergence Theorem, we get

$$\lim_{n\to\infty} \int_X f\chi_{B_n} d\mu = \int_X f\chi_{\cap B_n} d\mu = 0.$$

This contradicts the earlier assertion.

Exercise 10.28

By Exercise 10.26, we basically need to show that $\int_A f d\mu = 0$ for any measurable subset $A \subset (a, b)$.

By Exercise 10.27, for any $\epsilon > 0$, there is $\delta > 0$, such that $\mu(A) < \delta$ implies $\left|\int_A f d\mu\right| < \epsilon$. For any bounded Lebesgue measurable A, there is a finite union of open intervals U, such that $\mu(U - A) < \delta$ and $\mu(A - U) < \delta$. Then we have $\left|\int_{U-A} f d\mu\right| < \epsilon$ and $\left|\int_{A-U} f d\mu\right| < \epsilon$. The assumption that the integration of f on any interval vanishes implies that $\int_U f d\mu = 0$. Therefore

$$\left|\int_A f d\mu\right| \le \left|\int_U f d\mu\right| + \left|\int_{U-A} f d\mu\right| + \left|\int_{A-U} f d\mu\right| < 2\epsilon.$$

Since this is true for any ϵ, we get $\int_A f d\mu = 0$.

Exercise 10.29

By Exercise 10.27, for any $\epsilon > 0$, there is $\delta > 0$, such that $\mu(A) < \delta$ implies $\left|\int_A f d\mu\right| < \epsilon$. For any x, y satisfying $|x - y| < \delta$, we take $A = [x, y]$ and get

$$|F(y) - F(x)| = \left|\int_{[x,y]} f dx\right| < \epsilon.$$

This shows that $F(x)$ is uniformly continuous.