Exercise 11.23

By Lemma 10.4.1, there is an increasing sequence of non-negative measurable simple functions $\phi_n$ converging to $f$. This means that $G(f) = \cup G(\phi_n)$, and $G(\phi_n)$ is increasing. For a simple function of the form 12.4.1, we have $G(\phi) = (\sqcup X_i \times [0, c_i)) \sqcup (X - \sqcup X_i) \times 0$. $G(\phi)$ is measurable and $\mu(G(\phi)) = \sum \mu(X_i)c_i = \int_X \phi d\mu$. Therefore $G(f)$ is measurable, and we get $\mu(G(f)) = \lim \mu(G(\phi_n)) = \lim \int_X \phi_n d\mu = \int_X f d\mu$. The first equality is monotone limit property, and the third equality is Monotone Convergence Theorem.

Exercise 11.24

Let $\mu_i^*$ be induced outer measures. Since bigger collection has smaller outer measure, we have $\mu_1^* \le \mu_3^* \le \mu_2^* \le \mu_5^*$, $\mu_1^* \le \mu_6^*$, $\mu_3^* \le \mu_4^*$. Let $\mu_7^*$ be induced from closed cubes of side length $< 1$. Then we further have $\mu_5^* \le \mu_7^*$ and $\mu_6^* \le \mu_7^*$. It remains to show $\mu_7^* \le \mu_1^*$ and $\mu_4^* \le \mu_3^*$. We will do this for $n = 2$.

By Exercise 11.1.3, $\mu_4^* \le \mu_3^*$ is the same as $\mu_4^*([a,b]\times[c,d]) \le (b-a)(d-c)$. For any $\epsilon > 0$, we have

$$[a,b]\times[c,d] \subset (a-\epsilon, b]\times(c-\epsilon, d].$$

By the definition of $\mu_4^*$, we have $\mu_4^*([a,b]\times[c,d]) \le (b-a+\epsilon)(d-c+\epsilon)$. Since $\epsilon$ is arbitrary, we get $\mu_4^*([a,b]\times[c,d]) \le (b-a)(d-c)$.

By Exercise 11.1.3, $\mu_7^* \le \mu_1^*$ is the same as $\mu_7^*(A\times B) \le \mu(A)\mu(B)$ for any Borel subsets $A, B \subset \mathbb{R}$. Since $A, B$ are Lebesgue measurable, for any $\epsilon > 0$, there are open subsets $U = \sqcup(a_i, b_i) \supset A$ and $V = \sqcup(c_j, d_j) \supset B$, such that

$$\lambda(U) = \sum(b_i - a_i) < \mu(A) + \epsilon, \quad \lambda(V) = \sum(d_j - c_j) < \mu(B) + \epsilon.$$

By enlarging each $(a_i, b_i)$ and $(c_j, d_j)$ slightly, we may further assume that $a_i, b_i, c_j, d_j$ are rational, and the two inequalities above still hold. Now for each rectangle $(a_i, b_i)\times(c_j, d_j)$, since all numbers are rational, we have $b_i - a_i = \dfrac{k}{n}$ and $d_j - c_j = \dfrac{l}{n}$ for some natural numbers $n, k, l$. Then

$$(a_i, b_i)\times(c_j, d_j) \subset [a_i, b_i]\times[c_j, d_j] = \cup_{p,q=1}^{k,l}[a_i + \frac{p-1}{n}, a_i + \frac{p}{n}]\times[c_j + \frac{q-1}{n}, d_j + \frac{q}{n}].$$

The right side is a union of closed cubes of side length $\frac{1}{n} < 1$. Denote the union on the right side by $\cup_k I_{ijk}$. Then

$$\sum_k \lambda(I_{ijk}) = \sum_{p,q=1}^{k,l} \lambda([a_i + \frac{p-1}{n}, a_i + \frac{p}{n}]\times[c_j + \frac{q-1}{n}, d_j + \frac{q}{n}]) = kl\frac{1}{n^2} = (b_i - a_i)(d_j - c_j).$$

Thus we have $A\times B \subset \cup_{ij}(a_i, b_i)\times(c_j, d_j) \cup_{ijk} I_{ijk}$. By the definition of $\mu_7^*$, we have

$$\begin{aligned}\mu_7^*(A\times B) &\le \sum_{ijk}\lambda(I_{ijk}) = \sum_{ij}(b_i - a_i)(d_j - c_j)\\ &= \left(\sum_i (b_i - a_i)\right)\left(\sum_j (d_j - c_j)\right) < (\mu(A)+\epsilon)(\mu(B)+\epsilon).\end{aligned}$$

Since $\epsilon$ is arbitrary, we get $\mu_7^*(A\times B) \le \mu(A)\mu(B)$.

Exercise 11.25

If $A \ne \emptyset$ and $A \subset \cup I_i$, then $\sum\lambda(I_i) \ge 1$. Therefore $\mu^*(A) \ge 1$. However, there are many measurable subsets with Lebesgue measure between 0 and 1.

Exercise 11.26

The continuous function $f$ on the closed bounded rectangle $I$ is uniformly continuous. This means that for any $\epsilon > 0$, there is $\delta > 0$, such that $\|\vec{x}-\vec{y}\| < \delta$ implies $|f(\vec{x})-f(\vec{y})| < \epsilon$. Then we divide $I$ into the union of finitely closed rectangles $I_i$ of diameter $< \delta$, such that the intersection among $I_i$ lies in the boundary of the rectangle. This implies that $\mu_{n-1}(I) = \sum \mu_{n-1}(I_i)$. Moreover, $\vec{x}, \vec{y} \in i_i$ implies $\|\vec{x} - \vec{y}\| < \delta$, which further implies $|f(\vec{x}) - f(\vec{y})| < \epsilon$. Therefore we get $\omega_{I_i}(f) \leq \epsilon$.

$A$ is the collections of values of $f$. Since $\omega_{I_i}(f) \leq \epsilon$, the values of $f$ on $I_i$ lies in an interval $[a_i, a_i + \epsilon]$. This means $A \cap I_i \times \mathbb{R} \subset I_i \times [a_i, a_i + \epsilon]$, and implies $A \subset \cup I_i \times [a_i, a_i + \epsilon]$.

We have

$$\mu^*(A) \leq \sum \mu_n(I_i \times [a_i, a_i + \epsilon]) \leq \sum \mu(I_{n-1})\epsilon = \mu_{n-1}(I)\epsilon.$$

Since $\epsilon$ is arbitrary, we conclude that $A$ is measurable with measure 0.

Exercise 11.27

A submanifold of $\mathbb{R}^n$ is a union of countably (even finitely) many pieces, such that in each piece, some coordinates are continuous (even differentiable) functions of some other coordinates, on a closed and bounded rectangle. If we consider only one such function, the piece is contained in the graph of one such function. By Exercise 11.26, the piece has measure 0. Then the whole submanifold also has measure 0.

Exercise 11.28

The Borel $\sigma$-algebra contains all closed subsets and rectangles of the form $(a_1, b_1] \times \cdots \times (a_n, b_n]$. Therefore the $\sigma$-algebra generated by any of the listed collection is no bigger than the Borel $\sigma$-algebra.

Conversely, since any open subset is the union of countably many open balls with respect to any norm, and the balls can be as small as possible, any $\sigma$-algebra containing all rectangles (which include all open $L^\infty$-balls), or all open $L^1$-balls, or all open cubes of length $< 1$ (which include all small open $L^\infty$-balls) must also contain the Borel $\sigma$-algebra.

Since any open cube of length $< 1$ is the intersection of two open cubes of length $= 1$, the Borel $\sigma$-algebra, as the smallest $\sigma$-algebra containing all open cubes of length $< 1$, is also the smallest $\sigma$-algebra containing all open cubes of length $< 1$.

Since the definition of $\sigma$-algebra is symmetric with respect to the complement, the Borel $\sigma$-algebra is also the smallest $\sigma$-algebra containing the complements of all open subsets, which are all the closed subsets.

Since open cubes are the unions of countably many closed rectangles, or countably many rectangles of the form $(a_1, b_1] \times \cdots \times (a_n, b_n]$, or countably many closed cubes, the Borel $\sigma$-algebra is also the smallest $\sigma$-algebra containing all closed rectangles, or rectangles of the form $(a_1, b_1] \times \cdots \times (a_n, b_n]$, or all closed cubes.

Exercise 11.29

The Borel $\sigma$-algebra contains all the open triangles. Conversely, any open rectangle is the union of four open triangles (two more triangles are needed to cover the "diagonal"). Since by Exercise 11.28, the Borel $\sigma$-algebra is the smallest $\sigma$-algebra containing all open rectangles, it is also no bigger than the smallest $\sigma$-algebra containing all the open triangles.

Exercise 11.30

The product Borel $\sigma$-algebra $\mathcal{B}_m \times \mathcal{B}_n$ is the smallest $\sigma$-algebra containing all the Borel measurable rectangles $A \times B$, $A \in \mathcal{B}_m$, $B \in \mathcal{B}_n$. In particular, it contains all open rectangles in $\mathbb{R}^{m+n}$, which is the product of an open rectangle in $\mathbb{R}^m$ and an open rectangle in $\mathbb{R}^n$. Since by Exercise 11.28, the Borel $\sigma$-algebra $\mathcal{B}_{m+n}$ is the smallest $\sigma$-algebra contaning all the open rectangles, we get $\mathcal{B}_{m+n} \subset \mathcal{B}_m \times \mathcal{B}_n$.

Conversely, we need to explain that all Borel measurable rectangles $A \times B \in \mathcal{B}_{m+n}$. First we prove that, for any open $V \subset \mathbb{R}^n$, we have

$$\mathcal{B}_m \times V = \{A \times V \colon A \in \mathcal{B}_m\} \subset \mathcal{B}_{m+n}.$$

To prove this, we consider

$$\Sigma_V = \{A \subset \mathbb{R}^m \colon A \times V \in \mathcal{B}_{m+n}\}.$$

It is easy to verify that $\Sigma_V$ is a $\sigma$-algebra on $\mathbb{R}^m$. Since for any open $U \subset \mathbb{R}^m$, $U \times V$ is open in $\mathbb{R}^{m+n}$ and therefore belongs to $\mathcal{B}_{m+n}$, we also know that $\Sigma$ contains all open subsets of $\mathbb{R}^m$. Then by the definition of the Borel $\sigma$-algebra, we get $\mathcal{B}_m \subset \Sigma_V$. This inclusion can be translated into $\mathcal{B}_m \times V \subset \mathcal{B}_{m+n}$.

Next, for any $A \in \mathcal{B}_m$, we consider

$$\Sigma_A = \{B \subset \mathbb{R}^n \colon A \times B \in \mathcal{B}_{m+n}\}.$$

It is easy to verify that $\Sigma_A$ is a $\sigma$-algebra on $\mathbb{R}^m$. By $\mathcal{B}_m \times V \subset \mathcal{B}_{m+n}$ (or more exactly, $\mathcal{B}_V \subset \Sigma_m$), we also know that $\Sigma_A$ contains any open subset $V$. Then by the definition of the Borel $\sigma$-algebra, we get $\mathcal{B}_n \subset \Sigma_A$. This inclusion can be translated into that $\mathcal{B}_{m+n}$ contains the Borel measurable rectangle $A \times B$. Since $\mathcal{B}_m \times \mathcal{B}_n$ is the smallest $\sigma$-algebra containing all the Borel measurable rectangles, we conclude that $\mathcal{B}_{m+n} \supset \mathcal{B}_m \times \mathcal{B}_n$.

Exercise 11.31

The $\epsilon$-neighborhood $A^\epsilon$ in Exercise 6.80 gets smaller when $\epsilon$ gets smaller. Therefore for closed $A$, we have

$$A = \cap_{\epsilon>0} A^\epsilon = \cap_n A^{\frac{1}{n}}.$$

This shows that $A$ is a $G_\delta$-set.

Exercise 11.32

We know $f^{-1}(a, +\infty)$ and $g^{-1}(-\infty, a)$ are open for all $a$. We also know $f^{-1}(-\infty, a]$ and $g^{-1}[a, +\infty)$ are closed for all $a$.

$f^{-1}(-\infty, a) = \cup_n f^{-1}(-\infty, a - \frac{1}{n}]$ is an $F_\sigma$-set.

$f^{-1}(a, b) = f^{-1}(a, +\infty) \cap f^{-1}(-\infty, b)$. We know $f^{-1}(a, +\infty)$ is open, which is an $F_\sigma$-set by Exercise 11.31. We also know $f^{-1}(-\infty, b)$ is an $F_\sigma$-set. As the intersection of two $F_\sigma$-sets, $f^{-1}(a, b)$ is an $F_\sigma$-set.

$g^{-1}(a, b)$ behaves like the complement of $f^{-1}(a, b)$ and is therefore a $G_\delta$-set.

$\{x \colon f(x) > g(x)\} = (f - g)^{-1}(0, +\infty)$. Since $f - g$ is lower semicontinuous, the subset is an open subset.

$\{x \colon g(x) > f(x)\} = (g - f)^{-1}(0, +\infty)$. Since $f - g$ is upper semicontinuous (see $g^{-1}(a, b)$ above), the subset is an $G_\delta$-set.

$\{x\colon f(x) \geq g(x)\} = (f-g)^{-1}[0,+\infty) = \cap_n (f-g)^{-1}(-\frac{1}{n},+\infty)$. Since $f-g$ is lower semicontinuous, the subset is an $F_\sigma$-set.

Exercise 11.33

By Exercise 9.36, the the push forward of the Borel $\sigma$-algebra $\mathcal{B}_n$ is a $\sigma$-algebra

$$F_*(\mathcal{B}_n) = \{A \in \mathbb{R}^m\colon F^{-1}(A) \in \mathcal{B}_n\}$$

Moreover, for open $A$, the continuity of $F$ implies that $F^{-1}(A)$ is also open and therefore belongs to $\mathcal{B}_n$. Therefore the $\sigma$-algebra $F_*(\mathcal{B}_n)$ contains all open subsets. By the definition of the Borel $\sigma$-algebra, we get $F_*(\mathcal{B}_n) \supset \mathcal{B}_m$. The inclusion means exactly the conclusion of the exercise.

Exercise 11.34

Suppose $f$ is an increasing function on domain $A$. Then $f^{-1}(a,b) = \langle \alpha,\beta\rangle \cap A$ for some interval $\langle \alpha,\beta\rangle$ (the biggest interval satisfying $f\langle \alpha,\beta\rangle \subset (a,b)$). If $A$ is Borel measurable, then $f^{-1}(a,b)$ is always Borel measurable. If $A$ is Lebesgue measurable, then $f^{-1}(a,b)$ is always Lebesgue measurable.

Exercise 11.35

By Example 11.4.3, the set $X(f)$ of continuous points of any function $f$ on $\mathbb{R}^n$ is a $G_\delta$-set.

For countably many $f_n$, the points where all $f_n$ are continuous form $\cap_n X(f_n)$, which is again a $G_\delta$-set. The points where at least one $f_n$ is continuous form $\cup_n X(f_n)$, which is a $G_{\delta\sigma}$-set.

The points where infinitely many $f_n$ are continuous is the complement of the points where only finitely many $f_n$ are not continuous. The later subset is $\cup_{\text{finite } I\subset\mathbb{N}} \cap_{n\in I} (\mathbb{R}^n - X(f_n))$. This is a countable union of finite intersection of $F_\sigma$-set, and is therefore still an $F_\sigma$-set. As the complement of this, we conclude that the points where infinitely many $f_n$ are continuous form a $G_\delta$-set.

Exercise 11.36

Let $y_1,\dots,y_n$ be distinct points in $Y$. By the second property and the countability of $Y$, we have

$$\begin{aligned}\mu(Z) = \mu(\sqcup_{y\in Y} y\times A) &= \sum_{y\in Y}\mu(y\times A)\\ &\geq \mu(y_1\times A)+\cdots+\mu(y_n\times A) = n\mu(y\times A).\end{aligned}$$

Since $n$ can be arbitrarily large, we get $\mu(y\times A)=0$.

If $y\times X$ is measurable for all $y\in Y$, then we already know $\mu(y\times X)=0$ for all $y$. Therefore

$$\mu(Z) = \mu(\sqcup_{y\in Y} y\times X) = \sum_{y\in Y}\mu(y\times X) = 0.$$

Since this contradicts to the assumption $\mu(Z)>0$, we conclude that $y\times X$ is not measurable for some $y\in Y$.

Exercise 11.37

The definition of the Cantor function is

$$\kappa = 0.b_1b_2\cdots b_n1_{[2]} \text{ on } (0.a_1a_2\cdots a_n1_{[3]}, 0.a_1a_2\cdots a_n2_{[3]}),$$

and

$$\kappa(0.a_1a_2\cdots a_n\cdots_{[3]}) = 0.b_1b_2\cdots b_n\cdots_{[2]} \text{ on } K.$$

We always assume $a_i = 0$ or 2, and $b_i = \dfrac{a_i}{2}$.

The formula for $\kappa$ on $K$ appears to be one-to-one. The only possible cause of not being one-to-one is the ambiguity in the expression of $x$ and $\kappa(x)$.

The end points of deleted intervals are

$$x_- = 0.a_1a_2\cdots a_n1_{[3]} = 0.a_1a_2\cdots a_n0\bar{2}_{[3]}, \quad x_+ = 0.a_1a_2\cdots a_n2_{[3]}.$$

These are in $K$ and

$$\kappa(x_-) = 0.b_1b_2\cdots b_n0\bar{1}_{[2]} = 0.b_1b_2\cdots b_n1_{[2]}, \quad \kappa(x_+) = 0.b_1b_2\cdots b_n1_{[3]}.$$

The set $A$ is the collection of such $x_-$ and $x_+$, and the formula for $\kappa(x_-) = \kappa(x_+)$ show that $\kappa|_A$ is two-to-one.

For $x = 0.a_1a_2\cdots a_n\cdots_{[3]} \in K - A$, there are infinitely many $a_n = 0$ and infinitely many $a_n = 2$. This implies that the base 3 expression of $x$ is not ambiguous (i.e., not of the form $0.\cdots\bar{2}_{[3]}$). Consequently, the base 2 expression of $\kappa(x)$ is also not ambiguous. This and the formula for $\kappa$ on $K$ imply that $\kappa|_{K-A}$ is one-to-one.

The values of $\kappa|_{K-A}$ are $0.b_1b_2\cdots b_n\cdots_{[2]}$, with infinitely many $b_n = 0$ and infinitely many $b_n = 1$. The $\kappa|_A$ are $0.b_1b_2\cdots b_n\cdots_{[2]}$, with finitely many $b_n = 0$ or finitely many $b_n = 1$. Therefore $\kappa(A) \cap \kappa(K - A) = \emptyset$.

Exercise 11.38

We adopt the discussion in Exercise 11.37. The discussion shows that, if the first $n$ digits of the base 3 expressions of $x, y \in [0, 1]$ are the same, then the first $n$ digits of the base 2 expressions of $\kappa(x)$ and $\kappa(y)$ are the same.

For the continuity of $\kappa$ at $x_0 \in [0, 1]$, we consider three cases: $x_0 \in [0, 1] - K$, $x_0 \in K - A$ and $x_0 \in A$.

Since the function is constant on the deleted (open) intervals, we find that $\kappa$ is continuous on $[0, 1] - K$.

For $x_0 = 0.a_1a_2\cdots a_n\cdots_{[3]} \in K - A$, there are infinitely many $a_n = 0$ and infinitely many $a_n = 2$. Therefore there are infinitely many $a_n = 0$ and $a_{n+1} = 2$. If $|x - x_0| < \dfrac{1}{3^{n+1}}$, then $x = 0.a_1a_2\cdots a_{n-1} * *c_{n+2}c_{n+3}\cdots_{[3]}$, where $** = 01$, $02$, or $10$. In particular, the base 3 expressions of $x$ and $x_0$ have the same first $n-1$ digits. This implies that the base 2 expressions of $\kappa(x)$ and $\kappa(x_0)$ have the same first $n - 1$ digits, and we have $|\kappa(x) - \kappa(x_0)| \le \dfrac{1}{2^{n-1}}$. Since $n$ can be arbitrarily big, this proves the continuity of $\kappa$ at $x_0$.

Finally, we consider $x_0 \in K-A$. The first case is $x_0 = x_- = 0.a_1a_2\cdots a_n1_{[3]} = 0.a_1a_2\cdots a_n0\bar{2}_{[3]}$. Since $\kappa = \kappa(x_0)$ on the (deleted) interval $(x_-, x_+)$ on the right of $x_0$, $\kappa$ is right continuous at $x_0$.

For the left continuity, we note that $-\dfrac{1}{3^{n+1}} < x-x_0 < 0$ implies $x = 0.a_1a_2\cdots a_{n-1}0c_{n+1}c_{n+2}\cdots_{[3]}$. Then the base 2 expressions of $\kappa(x) = 0.b_1b_2\cdots b_n0\cdots$ and $\kappa(x_0) = 0.b_1b_2\cdots b_n1_{[2]}$ have the same first $n$ digits, and we have $|\kappa(x)-\kappa(x_0)| \le \dfrac{1}{2^n}$. This implies the left continuity of $\kappa$ at $x_0$.

It remains to consider the second case that $x_0 = x_+ = 0.a_1a_2\cdots a_n2_{[3]}$. Again we have $\kappa = \kappa(x_0)$ on the (deleted) interval $(x_-, x_+)$ on the left of $x_0$, and therefore $\kappa$ is left continuous at $x_0$. For the right continuity, we note that $0 < x - x_0 < \dfrac{1}{3^{n+1}}$ implies $x = 0.a_1a_2\cdots a_{n-1}2c_{n+1}c_{n+2}\cdots_{[3]}$. Then the base 2 expressions of $\kappa(x) = 0.b_1b_2\cdots b_n1\cdots$ and $\kappa(x_0) = 0.b_1b_2\cdots b_n1_{[2]}$ have the same first $n+1$ digits, and we have $|\kappa(x)-\kappa(x_0)| \le \dfrac{1}{2^{n+1}}$. This implies the right continuity of $\kappa$ at $x_0$.

[*Alternative using Exercise 11.39*]

We construct $\kappa$ by successively elevating the intervals that we delete to construct the Cantor set $K$. After $n$ steps, we get an increasing function defined on some intervals in $[0,1]$. Moreover, the "gap" on any remaining interval is $\frac{1}{2^n}$. Since $\kappa$ is increasing, this implies that we always have $\kappa(x^+)-\kappa(x^-) \le \frac{1}{2^n}$. Since $n$ is arbitrary, we get $\kappa(x^+) = \kappa(x^-)$ for all $x$. This shows that $\kappa$ is continuous.

The idea is provided by Exercise 2.38.

EXERCISE 11.39

Suppose $x = 0.a_1a_2\cdots a_n\cdots_{[3]} \in K$, which means $a_i = 0$ or 2. If $a_n = 2$ and $a_{n+1} = 0$, then $x$ is on the right of the deleted interval $(0.a_1a_2\cdots a_{n-1}1_{[3]}, 0.a_1a_2\cdots a_{n-1}2_{[3]})$, and is on the left of the deleted interval $(0.a_1a_2\cdots a_{n-1}21_{[3]}, 0.a_1a_2\cdots a_{n-1}22_{[3]})$. By the increasing property, $\kappa(x)$ lies between the values on the two intervals

$$0.b_1b_2\cdots b_{n-1}b_{n[2]} = 0.b_1b_2\cdots b_{n-1}1_{[2]} \le \kappa(x) \le 0.b_1b_2\cdots b_{n-1}11_{[2]} = 0.b_1b_2\cdots b_{n-1}b_n1_{[2]}.$$

This implies the formula for $\kappa(x)$ given for $x \in K$ in Example 11.4.5, in case there are infinitely many 0 and 2 in the base 3 expression of $x$.

It remains to consider $x = 0.a_1a_2\cdots a_n2_{[3]}$ and $x = 0.a_1a_2\cdots a_n0\bar{2}_{[3]} = 0.a_1a_2\cdots a_n1$, with $a_i = 0$ or 2. In the first case, $x$ is on the right of $(0.a_1a_2\cdots a_n1_{[3]}, 0.a_1a_2\cdots a_n2_{[3]})$ and the left of $(0.a_1a_2\cdots a_n20\cdots01_{[3]}, 0.a_1a_2\cdots a_n20\cdots02_{[3]})$. The increasing property then implies

$$0.b_1b_2\cdots b_n1_{[2]} \le \kappa(x) \le 0.b_1b_2\cdots b_n10\cdots01_{[2]}.$$

In the second case, $x$ is on the right of $(0.a_1a_2\cdots a_n02\cdots21_{[3]}, 0.a_1a_2\cdots a_n02\cdots22_{[3]})$ and the left of $(0.a_1a_2\cdots a_n1_{[3]}, 0.a_1a_2\cdots a_n2_{[3]})$. The increasing property then implies

$$0.b_1b_2\cdots b_n01\cdots11_{[2]} \le \kappa(x) \le 0.b_1b_2\cdots b_n1_{[2]}.$$

Both estimations imply the unique value of $\kappa(x)$ given for $x \in K$ in Example 11.4.5.

Although we used the increasing property to determine $\kappa|_K$. We still need to verify that $\kappa$ is increasing on $[0,1]$.

Let

$$x = 0.a_1a_2\cdots a_n\cdots_{[3]} \in K, \quad y = 0.a_1'a_2'\cdots a_n'\cdots_{[3]} \in K, \quad a_n, a_n' = 0 \text{ or } 2.$$

If $x < y$, then there is $N$, such that $a_n = a_n'$ for $n < N$, and $a_N = 0 < a_N' = 2$. This implies

$$\kappa(x) = 0.b_1b_2\cdots b_{N-1}0\cdots_{[2]} < \kappa(y) = 0.b_1b_2\cdots b_{N-1}1\cdots_{[2]}.$$

Let

$$x = 0.a_1a_2\cdots a_n\cdots_{[3]} \in K,\ a_n = 0 \text{ or } 2, \quad y = 0.a_1'a_2'\cdots a_n'\cdots_{[3]} \in [0,1] - K.$$

If $x < y$, then there is $N$, such that $a_n = a_n'$ for $n < N$, and $a_N = 0 < a_N' = 1$ or $a_N = 0 < a_N' = 2$. This implies

$$\kappa(x) = 0.b_1b_2\cdots b_{N-1}0\cdots_{[2]} \le \kappa(y) = 0.b_1b_2\cdots b_{N-1}1\cdots_{[2]}.$$

The equality happens if and only if $x = 0.a_1a_2\cdots a_{N-1}0\bar{2}_{[3]}$ is the left end of a deleted interval, and $y = 0.a_1a_2\cdots a_{N-1}1\cdots$ belongs to the deleted interval.

The increasing property of $\kappa$ on $[0,1] - K$ follows from the construction of $\kappa$.

[*Alternative to increasing property using Exercise 11.38*]

By the continuity, $\kappa$ is the limit of $\kappa|_{[0,1]-K}$. Since $\kappa|_{[0,1]-K}$ is increasing, the limit is also increasing.

EXERCISE 11.40

(1) Since $\kappa(x)$ is increasing and continuous, $\phi(x)$ is strictly increasing and continuous. By Theorem 2.5.3, $\phi$ is a continuous invertible map from $[0,1]$ to $[0,2]$.

(2) We have $[0,1] - K = \sqcup(a_i, b_i)$. Since $\kappa|_{(a_i,b_i)} = C_i$ is constant, we have $\phi(x) = C_i + x$ on $(a_i, b_i)$. In particular, this implies $\phi(a_i, b_i) = (\alpha_i, \beta_i)$ with $\beta_i - \alpha_i = b_i - a_i$. Since $\phi$ is strictly increasing, we have

$$\phi([0,1] - K) = \sqcup\phi(a_i, b_i) = \sqcup(\alpha_i, \beta_i),$$

so that

$$\mu(\phi([0,1] - K)) = \sum(\beta_i - \alpha_i) = \sum(b_i - a_i) = \mu([0,1] - K) = 1.$$

Since $\phi$ is invertible, this further implies

$$\mu(\phi(K)) = \mu([0,2] - \phi([0,1] - K)) = \mu([0,2]) - \mu(\phi([0,1] - K)) = 1.$$

(3) By $\mu(\phi(K)) > 0$ and Theorem 11.4.6, there is a non-Lebesgue measurable $B \subset \phi(K)$. Then $B = \phi(A)$ for $A = \phi^{-1}(B)$. We know $A$ is Lebesgue measurable because $A \subset K$ and $K$ has zero Lebesgue measure.

EXERCISE 11.41

The subset $A$ in Exercise 11.40 is Lebesgue measurable. If it is Borel measurable, then by applying Exercise 11.33 to the continuous map $\phi^{-1}$, we find that $B = \phi(A) = (\phi^{-1})^{-1}(A)$ (the inside $-1$ means inverse, the outside $-1$ means preimage) is also Borel measurable. Since Boreal measurable implies Lebesgue measurable, this contradicts to the fact that $B$ is not Lebesgue measurable.

Exercise 11.42

The function $\phi^{-1}$ is continuous, and $\chi_A$ is Lebesgue measurable because $A$ is. However, for the composition

$$f = \chi_A \circ \phi^{-1} \colon [0,2] \to [0,1] \to \mathbb{R},$$

the preimage $f^{-1}(\frac{1}{2}, +\infty) = (\phi^{-1})^{-1}(A) = \phi(A)$ is not Lebesgue measurable.

Exercise 12.1

For any measurable $A \subset X^+ \cap Y^-$, we have $A \subset X^+$ implying $\nu(A) \geq 0$ and $A \subset Y^-$ implying $\nu(A) \leq 0$. Therefore $\nu(A) = 0$. The argument for measurable subsets of $X^- \cap Y^+$ is the same.

Exercise 12.2

Suppose $X = X^+ \sqcup X^- = X'^+ \sqcup X'^-$ are the Hahn decompositions for $\nu$ and $\nu'$. For any measurable $A \subset X'^+ - X^+$, $A \subset X'^+$ implies $\nu'(A) \geq 0$ and $A \subset X^-$ implies $\nu(A) \leq 0$. Combined with $\nu(A) \geq \nu'(A)$, we get $\nu(A) = \nu'(A) = 0$. Similarly, any measurable $A \subset X^- - X^-$ satisfies $\nu(A) = \nu'(A) = 0$.

Exercise 12.3

(1) For any measurable $A \subset X^+ \cap Y^-$, $A \subset X^+$ implies $\nu(A) \geq 0$ and $A \subset Y^-$ implies $\nu^+(A) = 0$ and $\nu(A) = -\nu^-(A) \leq 0$. Therefore $\nu(A) = \nu^+(A) = \nu^-(A) = 0$.

(2) By the decomposition $X = Y^+ \sqcup Y^-$ for the mutually singular property, we have $\nu^+(A - Y^+) = \nu^-(A \cap Y^+) = 0$.

By $A \cap X^+ - A \cap X^+ \cap Y^+ \subset X^+ \cap Y^-$, $A \cap Y^+ - A \cap X^+ \cap Y^+ \subset X^- \cap Y^+$ and the first part, we have $\nu(A \cap Y^+) = \nu(A \cap X^+ \cap Y^+) = \nu(A \cap X^+)$. Then

$$\begin{aligned}\nu^+(A) &= \nu^+(A \cap Y^+) + \nu^+(A - Y^+) = \nu^+(A \cap Y^+) \\ &= \nu^+(A \cap Y^+) - \nu^-(A \cap Y^+) = \nu(A \cap Y^+) = \nu(A \cap X^+).\end{aligned}$$

The equality $\nu^-(A) = \nu(A \cap X^-)$ is similar.

Exercise 12.4

(1) For $A \subset X^+$, we have

$$\nu^+(A) = \nu(A) = \mu^+(A) - \mu^-(A) \leq \mu^+(A).$$

The proof of $\nu^-(A) \leq \mu^-(A)$ for $A \subset X^-$ is similar.

(2) From (1), we have

$$\begin{aligned}|\nu|(A) &= \nu^+(A) + \nu^-(A) = \nu^+(A \cap X^+) + \nu^-(A \cap A^-) \\ &\leq \mu^+(A \cap X^+) + \mu^-(A \cap A^-) \leq \mu^+(A) + \mu^-(A).\end{aligned}$$

(3) From (2), we have

$$\begin{aligned}|\nu|(B) &= \nu^+(B) + \nu^-(B) \leq \mu^+(B) + \mu^-(B), \\ |\nu|(A - B) &\leq \nu^+(A - B) + \nu^-(A - B) \leq \mu^+(A - B) + \mu^-(A - B).\end{aligned}$$

Adding up gives

$$\begin{aligned}|\nu|(A) &= |\nu|(B) + |\nu|(A - B) \\ &\leq \mu^+(B) + \mu^-(B) + \mu^+(A - B) + \mu^-(A - B) = \mu^+(A) + \mu^-(A).\end{aligned}$$

Therefore $|\nu|(A) = \mu^+(A) + \mu^-(A) < \infty$ implies that the two earlier inequalities must be equalities. The first earlier inequality becomes equality if and only if $\nu^+(B) = \mu^+(B)$ and $\nu^-(B) = \mu^-(B)$.

Exercise 12.6

(1)

$$|\mu|(A) = \mu^+(A) + \mu^-(A) = 0 \implies \mu^+(A) = \mu^-(A) = 0\mu(A) = \mu^+(A) - \mu^-(A) = 0.$$

(2)

$$\lambda(A) = 0 \overset{\mu \ll \lambda}{\implies} \mu(A) = 0 \overset{\nu \ll \mu}{\implies} \nu(A) = 0.$$

(3) By (1), we have $\nu \ll |\nu|$. Then the property follows from (2).

(4) By (1), we have $\mu \ll |\mu|$. Then the property follows from (2).

(5)

$$\mu(A) = 0 \implies \nu(A) = 0, \nu'(A) = 0 \implies (\nu + \nu')(A) = \nu(A) + \nu'(A) = 0.$$

Exercise 12.7

(1) Let $X = Y \sqcup Y'$ be a decomposition for $\nu \perp \mu$. Then

$$\text{measurable } A \subset Y' \implies \nu(A) = 0,$$

and

$$\text{measurable } A \subset Y \implies \mu(A) = 0 \overset{\nu \ll \mu}{\implies} \nu(A) = 0.$$

Then for general measurable $A$, we have $\nu(A) = \nu(A \cap Y) + \nu(A \cap Y') = 0$.

(2) Let $X = Y \sqcup Y'$ be a decomposition for $\nu \perp \mu$. Let $X = Z \sqcup Z'$ be a decomposition for $\nu' \perp \mu$. Then consider the decomposition $X = (Y \cup Z) \sqcup (Y' \cap Z')$.

We have

$$\begin{aligned} A \subset Y' \cap Z' &\implies A \subset Y', \nu(A) = 0 \text{ and } A \subset Z', \nu'(A) = 0 \\ &\implies (\nu + \nu')(A) = \nu(A) + \nu'(A) = 0. \end{aligned}$$

We also have

$$\begin{aligned} A \subset Y \cup Z &\implies A = A_1 \sqcup A_2,\ A_1 = A \cap Y \subset Y,\ A_2 = A \cap (Z - Y) \subset Z \\ &\implies \mu(A) = \mu(A_1) + \mu(A_2) = 0. \end{aligned}$$

(3) Let $X = X^+ \sqcup X^-$ be a Hahn decomposition for $\mu$. Then $|\mu|(A) = \mu(A \cap X^+) - \mu(A \cap X^-)$.

Suppose $\nu \perp \mu$, with corresponding measurable decomposition $X = Y \sqcup Y'$. Then

$$A \subset Y' \implies \nu(A) = 0,$$

and

$$A \subset Y \implies A \cap X^+, A \cap X^- \subset Y \implies \mu(A \cap X^+) = \mu(A \cap X^-) = 0 \implies |\mu|(A) = 0.$$

This verifies $\nu \perp |\mu|$.

Suppose $\nu \perp |\mu|$, with corresponding measurable decomposition $X = Y \sqcup Y'$. Then

$$A \subset Y' \implies \nu(A) = 0,$$

and

$$\begin{aligned} A \subset Y &\implies |\mu|(A) = \mu(A \cap X^+) - \mu(A \cap X^-) = 0 \\ &\implies \mu(A \cap X^+) = \mu(A \cap X^-) = 0 \\ &\implies \mu(A) = \mu(A \cap X^+) + \mu(A \cap X^-) = 0. \end{aligned}$$

This verifies $\nu \perp \mu$.

We have proved $\nu \perp \mu \iff \nu \perp |\mu|$. Replacing $\nu$ by $|\mu|$ and replacing $\mu$ by $\nu$, we further get $|\mu| \perp \nu \iff |\mu| \perp |\nu|$.

(4) Let $X = Y \sqcup Y'$ be a measurable decomposition for $\mu \perp \mu'$. Then

$$A \subset Y' \implies \mu(A) = 0 \overset{\nu \ll \mu}{\implies} \nu(A) = 0.$$

Moreover,

$$A \subset Y \implies \mu'(A) = 0.$$

This verifies $\nu \perp \mu'$.

Exercise 12.8

Let $\nu(A) = \displaystyle\int_A |x| d\mu$ be the usual Lebesgue integral of $|x|$. Then $\nu$ is a $\sigma$-finite measure. In fact, $\nu$ is the Lebesgue-Stiltjes measure induced by $\text{sign}(x)x^2$.

We clearly have $\mu(A) = 0$ implying $\nu(A) = 0$. However, if $A \cap [-N, N] = \emptyset$, then $|x| > N$ on $A$, and we have $\nu(A) \geq N\mu(A)$. In particular, for any fixed $\delta > 0$, we have $\mu(N, N+\delta) = \delta$, but $\nu(N, N+\delta) > N\delta$ can be as large as we want (say $\epsilon = 1$ and $N = \delta^{-1}$). This shows that Proposition 12.1.5 fails.

Exercise 12.1

Exercise 12.29

(1) Since $\alpha$ is increasing, we have $\alpha' \geq 0$. Therefore $\alpha_1$ is increasing.

For $a < b$, we also have $\alpha_0(b) - \alpha_0(a) = \alpha(b) - \alpha(a) - \int_a^b \alpha' dx$. By Theorem 12.3.3, this is $\geq 0$. Therefore $\alpha_0$ is also increasing.

(2) From Exercise 12.28, we know $\alpha_1' = \alpha'$ almost everywhere. This means that $\alpha' = 0$ almost everywhere. Therefore $X_1$ is almost the whole $\mathbb{R}$ as measured by $\mu$, and we get $\mu(X_0) = 0$. To prove $\mu_{\alpha_0} \perp \mu$, it remains to show $\mu_{\alpha_0}(X_1) = 0$.

We only need to show $\mu_{\alpha_0}(A) = 0$ for any bounded measurable $A \subset X_1$. By the regularity property of the Lebesgue-Stieltjes measure $\mu_{\alpha_0}$ (see Proposition 11.2.1 and the subsequent remark), for any $\epsilon > 0$, there is a compact $K \subset A$, such that $\mu_{\alpha_0}(A - K) < \epsilon$. For any $x \in K$, by $\alpha_0'(x) = 0$, we have an interval $(a_x, b_x)$ around $x$, such that $\mu_{\alpha_0}(c, d) \leq |\alpha_0(c) - \alpha_0(d)| \leq \epsilon|c - d|$ for any $c, d \in [a_x, b_x]$. Then all $(a_x, b_x)$ form an open cover of $K$. The compactness of $K$ implies that $K$ is contained in finitely may such open intervals $(a_1, b_1), \dots, (a_n, b_n)$. Suppose $A \subset [-R, R]$. Then we may shrink these intervals, such that we still have $K \subset (a_1, b_1) \cup \cdots \cup (a_n, b_n)$, and we additionally have $\sum_{i=1}^n (b_i - a_i) < 3R$. we have

$$\mu_{\alpha_0}(K) \leq \sum_{i=1}^n \mu_{\alpha_0}(a_i, b_i) \leq \epsilon \sum_{i=1}^n (b_i - a_i) < 3R\epsilon.$$

Therefore $\mu_{\alpha_0}(A) = \mu_{\alpha_0}(A - K) + \mu_{\alpha_0}(K) \leq (3R + 1)\epsilon$. Since $\epsilon$ is arbitrary, we get $\mu_{\alpha_0}(A) = 0$.

(3) By $\alpha = \alpha_0 + \alpha_1$, we have $\mu_\alpha = \mu_{\alpha_0} + \mu_{\alpha_1}$. We already know $\mu_{\alpha_0} \perp \mu$. By the definition of $\mu_{\alpha_1}$, we have $\mu_{\alpha_1} \ll \mu$. Therefore this is the Lebbesgue decomposition.

Exercise 12.30

If $f$ has bounded variation, then $f = v^+ - v^-$ for increasing $v^+$ and $v^-$. Let $v_1^\pm(x) = \int_0^x v^{\pm\prime} dx$ and $v_0^\pm = v^\pm - v_1^\pm$. Then we get Lebesgue decompositions $\mu_{v^\pm} = \mu_{v_0^\pm} + \mu_{v_1^\pm}$. this induces the Lebesgue decomposition (see Exercise 12.19)

$$\mu_f = \mu_{v^+} - \mu_{v^-} = (\mu_{v_0^+} - \mu_{v_0^-}) + (\mu_{v_1^+} - \mu_{v_1^-}) = \mu_{v_0^+ - v_0^-} + \mu_{v_1^+ - v_1^-}.$$

We note that $v_1^+(x) - v_1^-(x) = \int_0^x (v^+ - v^-)' dx = \int_0^x f' dx$. Therefore the Lebesgue decomposition is

$$\mu_f = \mu_{f_0} + \mu_{f_1}, \quad f_1(x) = \int_0^x f' dx, \quad f_0 = f - f_1,$$

just like the increasing case.

Exercise 12.29

(1) Since $\alpha$ is increasing, we have $\alpha' \geq 0$. Therefore $\alpha_1$ is increasing.

For $a < b$, we also have $\alpha_0(b) - \alpha_0(a) = \alpha(b) - \alpha(a) - \int_a^b \alpha' dx$. By Theorem 12.3.3, this is $\geq 0$. Therefore $\alpha_0$ is also increasing.

(2) From Exercise 12.28, we know $\alpha_1' = \alpha'$ almost everywhere. This means that $\alpha' = 0$ almost everywhere. Therefore $X_1$ is almost the whole $\mathbb{R}$ as measured by $\mu$, and we get $\mu(X_0) = 0$. To prove $\mu_{\alpha_0} \perp \mu$, it remains to show $\mu_{\alpha_0}(X_1) = 0$.

We only need to show $\mu_{\alpha_0}(A) = 0$ for any bounded measurable $A \subset X_1$. By the regularity property of the Lebesgue-Stieltjes measure $\mu_{\alpha_0}$ (see Proposition 11.2.1 and the subsequent remark), for any $\epsilon > 0$, there is a compact $K \subset A$, such that $\mu_{\alpha_0}(A - K) < \epsilon$. For any $x \in K$, by $\alpha_0'(x) = 0$, we have an interval $(a_x, b_x)$ around $x$, such that $\mu_{\alpha_0}(c, d) \leq |\alpha_0(c) - \alpha_0(d)| \leq \epsilon|c - d|$ for any $c, d \in [a_x, b_x]$. Then all $(a_x, b_x)$ form an open cover of $K$. The compactness of $K$ implies that $K$ is contained in finitely may such open intervals $(a_1, b_1), \dots, (a_n, b_n)$. Suppose $A \subset [-R, R]$. Then we may shrink these intervals, such that we still have $K \subset (a_1, b_1) \cup \dots \cup (a_n, b_n)$, and we additionally have $\sum_{i=1}^n (b_i - a_i) < 3R$. we have

$$\mu_{\alpha_0}(K) \leq \sum_{i=1}^n \mu_{\alpha_0}(a_i, b_i) \leq \epsilon \sum_{i=1}^n (b_i - a_i) < 3R\epsilon.$$

Therefore $\mu_{\alpha_0}(A) = \mu_{\alpha_0}(A - K) + \mu_{\alpha_0}(K) \leq (3R + 1)\epsilon$. Since $\epsilon$ is arbitrary, we get $\mu_{\alpha_0}(A) = 0$.

(3) By $\alpha = \alpha_0 + \alpha_1$, we have $\mu_\alpha = \mu_{\alpha_0} + \mu_{\alpha_1}$. We already know $\mu_{\alpha_0} \perp \mu$. By the definition of $\mu_{\alpha_1}$, we have $\mu_{\alpha_1} \ll \mu$. Therefore this is the Lebbesgue decomposition.

Exercise 12.30

If $f$ has bounded variation, then $f = v^+ - v^-$ for increasing $v^+$ and $v^-$. Let $v_1^\pm(x) = \int_0^x v^{\pm\prime} dx$ and $v_0^\pm = v^\pm - v_1^\pm$. Then we get Lebesgue decompositions $\mu_{v^\pm} = \mu_{v_0^\pm} + \mu_{v_1^\pm}$. this induces the Lebesgue decomposition (see Exercise 12.19)

$$\mu_f = \mu_{v^+} - \mu_{v^-} = (\mu_{v_0^+} - \mu_{v_0^-}) + (\mu_{v_1^+} - \mu_{v_1^-}) = \mu_{v_0^+ - v_0^-} + \mu_{v_1^+ - v_1^-}.$$

We note that $v_1^+(x) - v_1^-(x) = \int_0^x (v^+ - v^-)' dx = \int_0^x f' dx$. Therefore the Lebesgue decomposition is

$$\mu_f = \mu_{f_0} + \mu_{f_1}, \quad f_1(x) = \int_0^x f' dx, \quad f_0 = f - f_1,$$

just like the increasing case.

Exercise 12.35

For any measurable $A$, let $B = \{x \in: f(x) \geq g(x)\}$ and $B = \{x \in: f(x) < g(x)\}$. Then $A = B \sqcup C$. Since $f, g$ satisfy (12.5.1) for arbitrary measurable $A$, we get

$$\nu(B) \geq \int_B f d\mu, \quad \nu(C) \geq \int_C g d\mu.$$

By $\max\{f, g\} = f$ on $B$ and $\max\{f, g\} = g$ on $C$, we get

$$\nu(A) = \nu(B) + \nu(C) \geq \int_B f d\mu + \int_C g d\mu = \int_B \max\{f, g\} d\mu + \int_C \max\{f, g\} d\mu = \int_A \max\{f, g\} d\mu.$$

Exercise 12.36

We have $g_n \geq 0$ satisfying (12.5.1), such that

$$\lim_{n\to\infty} \int_X g_n d\mu = \sup \left\{ \int_X g d\mu \colon g \text{ satisfies (12.5.1)} \right\}.$$

Then $f_n = \max\{g_1, \dots, g_n\}$ is increasing and converges to $f$. By Exercise 12.35, $f_n$ also satisfies (12.5.1), and by Monotone Convergence Theorem, we further have

$$\int_A f d\mu = \lim \int_A f_n d\mu \leq \nu(A).$$

This shows that $f$ satisfies (12.5.1).

Since $\nu$ is a finite measure, (12.5.1) implies that the value of $\int f d\mu$ is always finite. Then (12.5.1) further implies that $\lambda(A) = \nu(A) - \int_A f d\mu$ is a measure.

Exercise 12.37

The function $f + \epsilon\chi_B$ satisfying (12.5.1) means

$$0 \leq \nu(A) - \int_A (f + \epsilon\chi_B) d\mu = \lambda(A) - \epsilon\mu(A) \text{ for } A \subset B,$$

and

$$0 \leq \nu(A) - \int_A (f + \epsilon\chi_B) d\mu = \lambda(A) \text{ for } A \cap B = \emptyset.$$

The second inequality always holds, and the first inequality means that $B$ is a positive subset for the signed measure $\lambda - \epsilon\mu$.

Under the assumption $\lambda(X) > 0$, therefore, we find $\epsilon > 0$ satisfying $\lambda(X) > \epsilon\mu(X)$ (finiteness of $\mu(X)$ used here). We get a Hahn decomposition $X = X^+ \sqcup X^-$ for the signed measure $\lambda - \epsilon\mu$. Presumably we can take $B$ to be $X^+$. It only remains to show that $\mu(X^+) > 0$.

If $\mu(X^+) = 0$, then by $\nu$ absolutely continuous with respect to $\mu$, we get $\nu(X^+) = 0$. Then $\mu(X^+) = 0$ and $\nu(X^+) = 0$ further implies

$$(\lambda - \epsilon\mu)(X^+) = \nu(X^+) - \int_{X^+} f d\mu - \epsilon\mu(X^+) = 0.$$

This contracts

$$(\lambda - \epsilon\mu)(X^+) \geq (\lambda - \epsilon\mu)(X) = \lambda(X) > \epsilon\mu(X) \geq 0,$$

and completes the proof that $\mu(X^+) > 0$.

Exercise 12.38

If $\lambda(X) > 0$, then we get $f + \epsilon\chi_B$ satisfying (12.5.1). On the other hand, we have (the second equality is by Monotone Convergence Theorem)

$$\int_X (f+\epsilon\chi_B)d\mu = \int_X fd\mu + \epsilon\mu(B) > \int_X fd\mu = \lim \int_X f_n d\mu = \sup\left\{\int_X gd\mu \colon g \text{ satisfies (12.5.1)}\right\}.$$

This contradicts the assumption that $f + \epsilon\chi_B$ satisfies (12.5.1).

Therefore $\lambda(X) = 0$. Since $\lambda$ is a measure, we get $\lambda(A) = 0$ for any measurable $A$. This means $\nu(A) = \int_A fd\mu$ for any measurable $A$.