1. Consider the system:

$$x' = 1 - 4x + x^2 y$$
$$y' = 3x - x^2 y$$

Show that the trapezoidal region with vertices $(\frac{1}{4}, 0)$, $(13, 0)$, $(1, 12)$ and $(\frac{1}{4}, 12)$ is a trapping region of the system. Hence, show that there is a (non-trivial) periodic solution to the system using the Poincaré-Bendixson's Theorem.

[Remark: Do verify ALL conditions required by the Poincaré-Bendixson Theorem. You may use the following fact from MATH 4051: an equilibrium point $\mathbf{x}^*$ to an ODE system $\mathbf{x}' = \mathbf{F}(\mathbf{x})$ (where $\mathbf{F}$ is C^1) is unstable if all eigenvalues of the Jacobian matrix $D\mathbf{F}_{\mathbf{x}^*}$ are positive (in case it is real), or has positive real part (in case it is a complex number).]

2. Let $\omega(\mathbf{x})$ be the ω-limit set from a point $\mathbf{x} \in \mathbb{R}^n$ of an ODE system which is C^1 on $\mathbb{R}^n$. Show that $\omega(\mathbf{x})$ must be a closed set. [Reminder: A set K is closed if every converging sequence $\mathbf{x}_n \in K$ must have its limit $\mathbf{x}_\infty$ in K.]

3. Discuss:
   (a) In the proof of the Poincaré-Bendixson's Theorem, why do we require the trapping region K to be closed and bounded? Explain by pointing out where this condition is used in the proof.
   (b) In the proof of the Poincaré-Bendixson's Theorem, why do we require the trapping region K contains no equilibrium point of the system? Again, explain by pointing out where this condition is used in the proof.
   (c) The proof of the Poincaré-Bendixson's Theorem is not valid when the system is defined on $\mathbb{R}^3$. Explain why.

4. The purpose of this structured problem is to disprove (using a counter-example) the Poincaré-Bendixson's Theorem for systems in $\mathbb{R}^4$.
   (a) Find the real general solution of the linear system:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\omega^2 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

   where $\omega \neq 0$ is a real constant. [Hint: Review your course material in MATH 2351/2352]
   (b) Show that if $T > 0$ is the period of any non-trivial solution to the system in (a), then $\omega T = 2\pi N$ for some integer N.
   (c) Label the coordinates of $\mathbb{R}^4$ by (x_1, y_1, x_2, y_2). Consider the following four-dimensional linear system:

$$\begin{bmatrix} x_1' \\ y_1' \\ x_2' \\ y_2' \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\omega_1^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -\omega_2^2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \end{bmatrix}$$

   where ω_1 and ω_2 are two non-zero real constants. Show that the real general solution of this four-dimensional system is given by:

$$\begin{bmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ \omega_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & \omega_2 & 0 \end{bmatrix} \left(c_1 \begin{bmatrix} \cos \omega_1 t \\ -\sin \omega_1 t \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} \sin \omega_1 t \\ \cos \omega_1 t \\ 0 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 0 \\ \cos \omega_2 t \\ -\sin \omega_2 t \end{bmatrix} + c_4 \begin{bmatrix} 0 \\ 0 \\ \sin \omega_2 t \\ \cos \omega_2 t \end{bmatrix} \right),$$

   where $c_1, \ldots, c_4$ are any real numbers.

TO BE CONTINUED ON NEXT PAGE...

(d) Show that if the ratio $\frac{\omega_1}{\omega_2}$ is irrational, then the four-dimensional system in (c) does not have any (non-equilibrium) periodic solution. [Hint: use (b)]

(e) Consider the following subset K of $\mathbb{R}^4$:

$$K = \left\{ (x_1, y_1, x_2, y_2) \in \mathbb{R}^4 : \frac{1}{2} \leq \omega_1^2 x_1^2 + y_1^2 + \omega_2^2 x_2^2 + y_2^2 \leq 2 \right\}.$$

By carefully picking $c_1, \ldots, c_4$ in the general solution formula proved in (c), find one solution curve to the four-dimensional system which is completely inside K at any time.

[Remark: Note that K is a closed and bounded set. From (e), the set K traps an entire solution curve of the system. It is easy to see that the only equilibrium point of the system is the origin, which is not inside K. Therefore, the set K and the system fulfill almost all conditions of the Poincaré-Bendixson Theorem except that the system is on $\mathbb{R}^4$. From (d), we know that the system does not have a periodic solution whenever $\frac{\omega_1}{\omega_2}$ is irrational. Therefore, it serves as a counter-example to the Poincaré-Bendixson Theorem in $\mathbb{R}^4$.]

Possible project directions:

- In class we proved that if $\varphi_t(\mathbf{x})$ is a trajectory trapped inside the closed and bounded set K, then take any $\mathbf{y} \in \omega(\mathbf{x})$, the trajectory $\varphi_t(\mathbf{y})$ is periodic. Since $\varphi_t(\mathbf{y}) \in \omega(\mathbf{x})$ provided that $\mathbf{y} \in \omega(\mathbf{x})$, the trajectory $\{\varphi_t(\mathbf{y})\}_{t\geq 0}$ from $\mathbf{y}$ is then a **subset** of $\omega(\mathbf{x})$.

  Through searching for references, write down and present the proof that the trajectory $\{\varphi_t(\mathbf{y})\}_{t\geq 0}$ is in fact **equal** to $\omega(\mathbf{x})$.

  [For the sake of coherence, your report may first include the proof (or sketch of proof) of the Poincaré-Bendixson's Theorem – digest the proof and write in your own style and do not copy words-by-words – then present why $\omega(\mathbf{x}) \subset \{\varphi_t(\mathbf{x})\}_{t\geq 0}$ from there.]

- Disprove the Poincaré-Bendixson's Theorem in $\mathbb{R}^4$ by working through Problem #4. However, present your work in *report* format, not *homework* format. Investigate that system furthermore. For instance, show that if $\frac{\omega_1}{\omega_2}$ is irrational, then $\omega(\mathbf{x}_0)$ is topologically a 2-dimensional torus in $\mathbb{R}^4$ for any non-zero $\mathbf{x}_0 \in \mathbb{R}^4$.

- In the examples we used to demonstrate the applications of the Poincaré-Bendixson's Theorem, the trapping region K are all *annular* regions. There is an explanation to this – that is because any periodic trajectory (if it really exists) must enclose an equilibrium point of the vector field $\mathbf{F}$. Therefore, in order for the trapping region K to fulfill all conditions of the Poincaré-Bendixson, one must drill a hole near the equilibrium point so that K becomes an annular region.

  The following lecture video, delivered by Professor Steve Strogatz from Cornell University, explains why it is so using Poincaré index theory:

  `https://www.youtube.com/watch?v=O2fcpxLT5wk`

  Watch the video and write a report on why "any periodic trajectory in an ODE system in $\mathbb{R}^2$ must enclose an equilibrium point of the system".