Question: How to describe the Horizon and $\mathbb{R}^2 \cup$ Horizon

Horizon $\triangleq$ the set of lines in $\mathbb{R}^2$ passing though $\vec{o}$

Define $\mathbb{RP}^1 = \{(x,y) \in \mathbb{R}^2 - \vec{o}\} / \sim$ $\sim$: equivalence relation.

$(x,y) \sim (x',y')$ iff $\exists \lambda \neq 0$, $\lambda \in \mathbb{R}^1$ such that $(x,y) = \lambda(x',y')$.

$[x,y] \triangleq$ the equivalence class of $(x,y) \in \mathbb{R}^2 - \{\vec{o}\}$

So $\mathbb{RP}^1 = \{[x,y] \mid (x,y) \neq \vec{o},\ [x,y] = [\lambda x, \lambda y],\ \forall \lambda \neq 0\}$.

$\mathbb{RP}^1 \cong S^1$ 

Generalization: $\mathbb{RP}^n$ = the set of lines in $\mathbb{R}^{n+1}$ passing $\vec{o}$

$\mathbb{RP}^n = \{(x_0, x_1, \dots, x_n) \in \mathbb{R}^{n+1} - \{\vec{o}\}\} / \sim$

$(x_0, \dots, x_n) \sim (x_0', \dots, x_n')$ iff $\exists \lambda \neq 0$.
$(x_0, \dots, x_n) = (\lambda x_0', \dots, \lambda x_n')$.

$= \{[x_0, x_1, \dots, x_n] \mid (x_0, x_1, \dots, x_n) \neq \vec{0},\ [x_0, \dots, x_n] = [\lambda x_0, \dots, \lambda x_n],\ \lambda \neq 0\}$

Look at $\mathbb{RP}^2$. Let $U_0 \triangleq \{[x_0, x_1, x_2] \in \mathbb{RP}^2 \mid x_0 \neq 0\}$

$= \{[1, \frac{x_1}{x_0}, \frac{x_2}{x_0}] \in \mathbb{RP}^2\} \xrightarrow{\varphi_0} (\frac{x_1}{x_0}, \frac{x_2}{x_0}) \in \mathbb{R}^2$.

φ_0 has an inverse map: $\mathbb{R}^2 \xrightarrow{\psi_0} U_0$
$(x,y) \longrightarrow [1, x, y]$

Thus we get an identification: $U_0 \cong \mathbb{R}^2$.

The complement of U_0 in $\mathbb{RP}^2$ is $H_0 = \{[0, x_1, x_2] \in \mathbb{RP}^2\}$ $\cong \mathbb{RP}^1$: Horizon.

So $\mathbb{RP}^2 = U_0 \sqcup H_0 = \mathbb{R}^2 \sqcup$ Horizon.

$\mathbb{R}^2 \to U_0 \hookrightarrow \mathbb{RP}^2$ $(\lambda a, \lambda b) \longrightarrow [1, \lambda a, \lambda b] = [\frac{1}{\lambda}, a, b]$

$(a,b) \to [1, a, b]$ when $\lambda \longrightarrow \infty$. $\downarrow$ $[0, a, b]$

Thus $[a, b] \in \mathbb{RP}^1 \longleftrightarrow [0, a, b] \in \mathbb{RP}^1 \subset \mathbb{RP}^2$
is the "vanishing point" of lines parallel $\| (a, b)$ d
We can also consider $U_1 = \{[x_0, x_1, x_2] \mid x_1 \neq 0\} \; (\subset \mathbb{RP}^2) \cong \mathbb{R}^2$
and $U_2 = \{[x_0, x_1, x_2] \mid x_2 \neq 0\} \; (\subset \mathbb{RP}^2) \cong \mathbb{R}^2$.
In general, $\mathbb{RP}^n \supset U_i = \{[x_0, x_1, \cdots, x_n] \mid x_i \neq 0\} \cong \mathbb{R}^n$
and $\mathbb{RP}^n - U_i = H_i = \{[x_0, x_1, \ldots x_n] \mid x_i = 0\} \cong \mathbb{RP}^{n-1}$.

$\mathbb{RP}^n$ is compact:

$$\mathbb{R}^{n+1} \supset S^n = \{x_0^2 + \cdots + x_n^2 = 1\} \xrightarrow{\pi} \mathbb{RP}^n$$

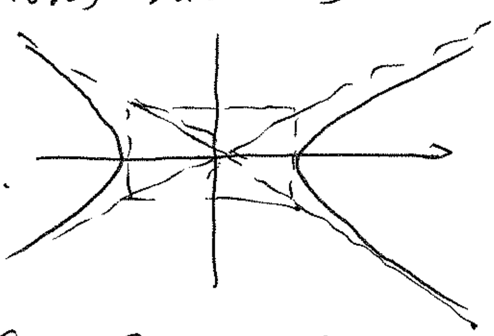
$\boxed{x_0^2 + \cdots + x_n^2 = 1,}$ $\quad (x_0, \ldots, x_n) \xrightarrow{\pi} [x_0, \ldots, x_n]$

$$\pi^{-1}([x_0, \ldots, x_n]) = \{(\lambda x_0, \ldots, \lambda x_n) \mid (\lambda x_0)^2 + \cdots + (\lambda x_n)^2 = 1\}$$

$\Rightarrow \lambda^2 = 1 \Rightarrow \lambda = \pm 1$. Thus S^n is a double cover of $\mathbb{RP}^n$. Since S^n is compact, $\mathbb{RP}^n$ is compact.

Algebraic Aspect:

In $\mathbb{R}^2$, we have lines, conic sections such as ellipses, hyperbola, and parabola.



Consider the hyperbola $C: \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

Consider $\mathbb{R}^2 \longrightarrow U_0 \subset \mathbb{RP}^2$

$[x_0, x_1, x_2] (\in U_0) = [1, \frac{x_1}{x_0}, \frac{x_2}{x_0}]$. So C in U_0 can be written as $\frac{\left(\frac{x_1}{x_0}\right)^2}{a^2} - \frac{\left(\frac{x_2}{x_0}\right)^2}{b^2} = 1$ or equivalently
$\frac{x_1^2}{a^2} - \frac{x_2^2}{b^2} - x_0^2 = 0$. The polynomial $\frac{x_1^2}{a^2} - \frac{x_2^2}{b^2} - x_0^2$ is not a function on $\mathbb{RP}^2$, but the zero locus $\left\{\frac{x_1^2}{a^2} - \frac{x_2^2}{b^2} - x_0^2 = 0\right\}$ in $\mathbb{RP}^2$ is well-defined.

$\overline{C} \triangleq \{[x_0, x_1, x_2] \subset \mathbb{RP}^2 \mid \frac{x_1^2}{a^2} - \frac{x_2^2}{b^2} - x_0^2 = 0\}$.

Then $C \subset \overline{C}$, $\overline{C} \cap U_0 = C$, and

$$\overline{C} - C = \begin{cases} \frac{x_1^2}{a^2} - \frac{x_2^2}{b^2} - x_0^2 = 0 \\ x_0 = 0 \end{cases} \Rightarrow \begin{cases} x_1 = \pm \frac{a}{b} x_2 \\ x_0 = 0 \end{cases}$$

$\Rightarrow \overline{C} - C$ consists of two points $[0, a, b]$, $[0, a, -b]$. $[0, a, b]$ is the "infinity point" corresponding to the line $bx - ay = 0$ in $\mathbb{R}^2$, which is one of the asymptotic of the hyperbola; $[0, a, -b]$ corresponds to the other asymptotic line $bx + ay = 0$ of th hyperbola.

So $\overline{C} \cong S^1$.

Homogenization and de-homogenization.

- $f(y_1, \cdots, y_n)$: a polynomial of degree d.

$F(x_0, \cdots, x_n) \triangleq x_0^d f(\frac{x_1}{x_0}, \cdots, \frac{x_n}{x_0})$ is a homogeneous polynomial of degree d.

Let $X = \{f = 0\} \subset \mathbb{R}^n$, $\overline{X} = \{F = 0\} \subset \mathbb{RP}^n$. Then $\overline{X} \cap U_0 = X$.

$$\mathbb{R}^n \ni (y_1, \cdots, y_n) \longrightarrow [1, y_1, \cdots, y_n] \in U_0 \subset \mathbb{RP}^n$$

- For a homogeneous polynomial $F(x_0, \cdots, x_n)$, we can "de-homogenize" F to get a polynomial (assume $\deg F = d$)

$$f(y_1, \cdots, y_n) = F(1, y_1, \cdots, y_n).$$

It is clear, that $x_0^d f(\frac{x_1}{x_0}, \cdots \frac{x_n}{x_0}) = x_0^d F(1, \frac{x_1}{x_0} \cdots \frac{x_n}{x_0}) = F(x_0, \cdots x_n)$

Blowup: Consider a subset $X \subset \mathbb{R}^2 \times \mathbb{RP}^1$:

$$X = \{(x,y)\times[l_1,l_2] \in \mathbb{R}^2\times\mathbb{RP}^1 \mid (x,y) = \lambda(l_1,l_2) \text{ for } \lambda\in\mathbb{R}^1\}$$

$$\begin{array}{ccc} X & & (x,y)\times[l_1,l_2] \\ \pi\downarrow & & \downarrow \\ \mathbb{R}^2 & & (x,y). \end{array}$$

$$\pi^{-1}((x,y)) = \begin{cases} \vec{0}\times\mathbb{RP}^1 & \text{if } (x,y)=\vec{0} \\ (x,y)\times[l_1,l_2] & \text{if } (x,y)\neq\vec{0} \end{cases}$$

$$= (x,y)\times\left[\tfrac{x}{\lambda},\tfrac{y}{\lambda}\right] = (x,y)\times[x,y]$$

So away from $\vec{0}$, $X - \pi^{-1}(\vec{0}) \cong \mathbb{R}^2 - \vec{0}$. a point.

X is called the blowup of $\mathbb{R}^2$ at $\vec{0}$.

Ex: C: $x^2 - y^2 + x^3 = 0$ in $\mathbb{R}^2$

$$\pi^{-1}(C) = \left\{ (x,y)\times[l_1,l_2] \;\middle|\; \begin{array}{l} x^2-y^2+x^3=0 \\ (x,y)=(\lambda l_1,\lambda l_2) \end{array} \right\}$$

Let $X_1 = \{(x,y)\times[l_1,l_2]\in X \mid l_1\neq 0\}$.

$$\pi^{-1}(C)\cap X_1 = \left\{ (x,y)\times[l_1,l_2] \;\middle|\; \begin{array}{l} x^2-y^2+x^3=0 \\ \lambda=\frac{x}{l_1},\ y=\lambda l_2 \Rightarrow y = x\frac{l_2}{l_1} \end{array} \right\}$$

Let $u = \frac{l_2}{l_1}$, so $X_1 \cong \mathbb{R}^2$ with coordinates (x,u)

Let $\overline{C} \triangleq \overline{\pi^{-1}(C-\{\vec{0}\})}$, the closure

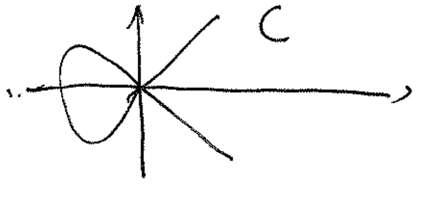
$$\pi^{-1}(C)\cap X_1 = \{x^2-(xu)^2+x^3=0\} = \{x^2(1-u^2+x)=0\}$$

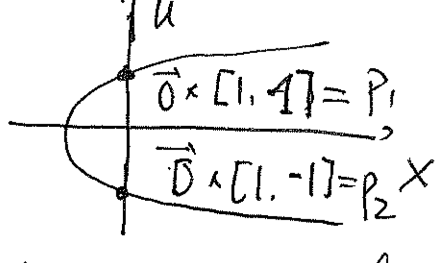
$$= \{x^2=0\} \cup \{1-u^2+x=0\} = (\pi^{-1}(\vec{0})\cap X_1)\cup(\overline{C}\cap X_1)$$

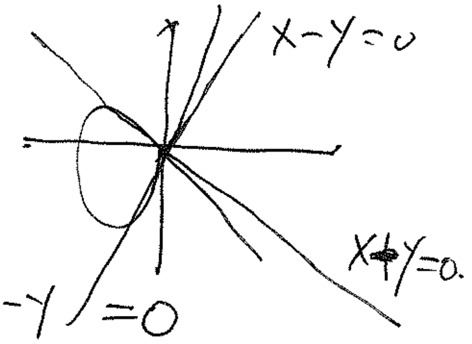
So $\overline{C}\cap X_1 = \{1-u^2+x=0\}$ is a smooth curve

Intersection of $\overline{C}$ with $\pi^{-1}(\vec{0})$ is $\begin{cases} 1-u^2+x=0 \\ x=0 \end{cases}$

$\Rightarrow \begin{cases} u=\pm 1 \\ x=0 \end{cases} \Rightarrow$ two intersection points $\vec{0}\times[1,1]$ and $\vec{0}\times[1,-1]$.







P_1 represents the tangent line $x-y=0$

and P_2 represents the tangent line $x+y=0$

Homework:

① Let $C: y=x^2$ be a parabola in $\mathbb{R}^2$.

Find the equation of $\overline{C}$ in $\mathbb{R}\mathbb{P}^2$ such that $\overline{C} \cap U_0 = C$. How many points added to C to get $\overline{C}$?

② Use the blowup to find the intersection of $\overline{C}$ with $\pi^{-1}\vec{0}$!

(1) $C: xy - x^6 - y^6 = 0$

(2) $C: x^3 = y^2 + x^4 + y^4$