Random sorting networks

Eric Marberg

Math 4991: Capstone Project in Pure Mathematics

Primary reference for this expository talk:

- **Random sorting networks**
  by Angel, Holroyd, Ramik, and Virág (2006)

All results due to these authors, as are most of the pictures.

Angel

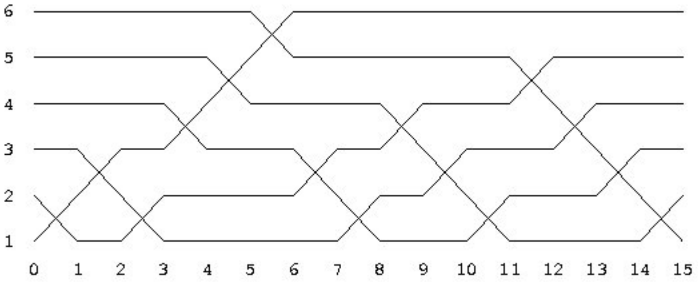
Romik

Holroyd

Virág

Sorting networks

A sorting network refers to a certain kind of "wiring diagram" like this:



- Consider the numbers $1, 2, \ldots, n$ listed (vertically) in decreasing order
- We want to rearrange these numbers to be in increasing order, by performing a minimal sequence of *swaps*, where a swap exchanges two numbers in adjacent positions and does nothing to remaining numbers

Sorting networks (continued)

Call the swap exchanging numbers in i^{th} and $(i+1)^{\text{th}}$ positions s_i.

Definition

An n-element sorting network is a sequence of swaps

$$w = (s_{i_1}, s_{i_2}, \ldots, s_{i_N})$$

with length N minimal which transforms

$$\begin{bmatrix} n \\ \vdots \\ 3 \\ 2 \\ 1 \end{bmatrix} \quad \rightsquigarrow \quad \begin{bmatrix} 1 \\ 2 \\ 3 \\ \vdots \\ n \end{bmatrix}$$

Sorting networks (continued)

Equivalently, an n-element sorting network is a reduced word of the reverse permutation $w_0 = n \cdots 321 \in S_n$ in symmetric group on n letters (i.e., a minimal factorization into product of the transpositions $s_i = (i, i+1)$).

Fact

Length of any n-element sorting network is $N = \binom{n}{2}$

Proof.

Each pair $\{i, j\} \in \binom{[n]}{2}$ must exchange somewhere.

Each swap exchanges one such pair, so number of swaps $N \geq \binom{n}{2}$.

Equality follows since there exist sorting networks of size $\binom{n}{2}$. □

Counting sorting networks

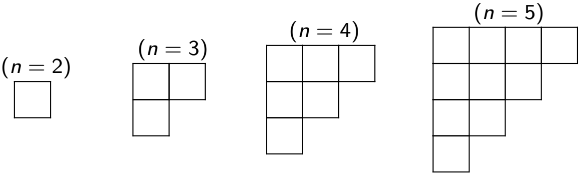
- Numbers of n-element sorting networks are

$$1 \ (n = 2) \qquad 2 \ (n = 3) \qquad 16 \ (n = 4) \qquad 768 \ (n = 5).$$

- These are also number of standard Young tableaux (SYT's) of shape

$$(n-1, n-2, \ldots, 3, 2, 1),$$

that is, labelings of the boxes in the arrays



by $1, 2, \ldots, N = \binom{n}{2}$ so that rows / columns are strictly increasing.

Counting sorting networks (continued)

Theorem (Stanley 1984)

$$\# \text{ n-elem sorting networks} = \# \text{ SYT's of shape } (n-1, \ldots, 2, 1)$$

Theorem (Edelman, Greene 1984)

There is an explicit bijection

$$\left\{ \begin{array}{l} \text{n-element sorting} \\ \text{networks} \end{array} \right\} \xrightarrow{EG} \left\{ \begin{array}{l} \text{SYT's of shape} \\ (n-1, \ldots, 2, 1) \end{array} \right\}$$

Such results bring us to the study of random sorting networks.

Random sorting networks

Define

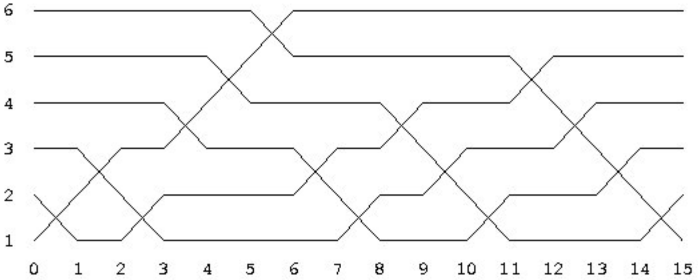
$$\underline{\text{random sorting network}} = n\text{-element sorting network selected at random from uniform distribution.}$$

- While sorting networks are *a priori* somewhat mysterious objects, SYT's are easy to construct, and there is a simple algorithm to sample an SYT of a given shape from uniform distribution
- Via EG bijection, this gives algorithm to sample n-element sorting networks from uniform distribution
- Simulations of random sorting networks for large n reveal several striking phenomena, some proved, some still conjectural

Swap locations

Consider the 6-element sorting network

$$w = (s_1, s_2, s_1, s_3, s_4, s_5, s_2, s_1, s_3, s_2, s_1, s_4, s_3, s_2, s_1)$$

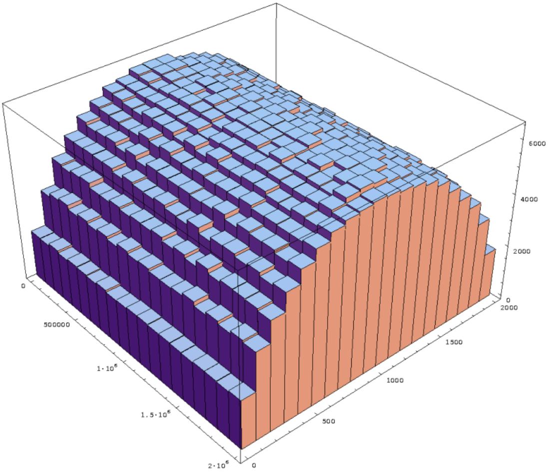


Define $swap_k = swap_k(w)$ as index of k^{th} swap in w.

- $swap_k(w) = 1$ for $k = 1, 3, 8, 11, 15$.
- $swap_k(w) = 3$ for $k = 4, 9, 13$.

Simulated swap locations

Histogram of swap locations in a 2000-element sorting network:



Theorem (Angel, Holroyd, Romik, Virág 2006)

Let w be a random n-element sorting network. Define

$$X_n = \Phi_n\left(swap_1(w)\right) \in [-1, 1]$$

where $\Phi_n(x) = \frac{2}{n}x - 1$ is rescaling map $[0, n] \to [-1, 1]$.

Then X_n converges in distribution as $n \to \infty$ to random variable with <u>semicircle law</u>, i.e., with probability density function $\frac{2}{\pi}\sqrt{1 - x^2}$.

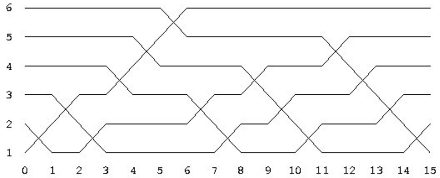
Semicircle laws (continued)

- Semicircle law is also limiting distribution of eigenvalues of a random symmetric matrix with entries chosen from normal distribution.
- More generally, AHRM prove that a semicircle law governs limiting distribution of $swap_k(w)$ for every k, as suggested by histogram.

Trajectories

Consider the 6-element sorting network

$$w = (s_1, s_2, s_1, s_3, s_4, s_5, s_2, s_1, s_3, s_2, s_1, s_4, s_3, s_2, s_1)$$



Define $\sigma_k = \sigma_k(w) \in S_n$ for $k = 0, 1, \ldots, N = \binom{n}{2}$ as permutation with

$$\sigma_k(n+1-i) = \text{ number in } i^{\text{th}} \text{ position after } k^{\text{th}} \text{ swap}$$
$$\sigma_k = \text{“numbers after swap } k \text{ read from top to bottom”}$$

In example: $\sigma_0 = 654321, \quad \sigma_1 = 654312, \quad \sigma_6 = 165423.$

Trajectories (continued)

Let $w = (s_{i_1}, s_{i_2}, \ldots, s_{i_N})$ be an n-element sorting network.
Define $\sigma_k = \sigma_k(w) : n+1-i \mapsto$ number in i^{th} position after k^{th} swap.

Definition

Scaled trajectory of $i \in [n]$ under w is map $[0,1] \to [-1,1]$ given by

$$T_i = T_i(w) = \Phi_n \circ t_i \circ \Psi_N$$

where $t_i(x)$ is piecewise linear function $[0,N] \to [0,n]$ interpolating

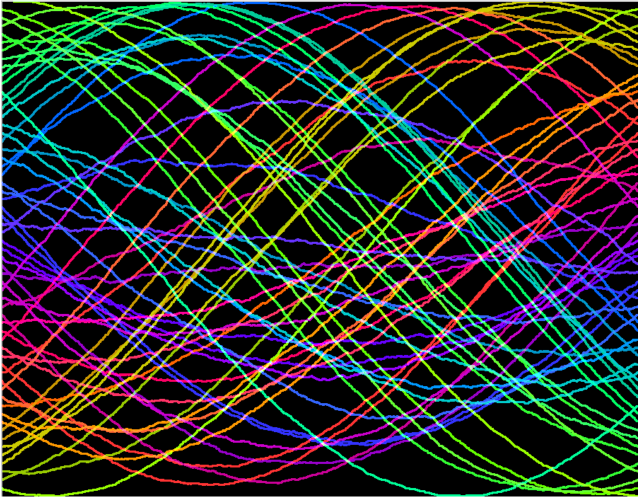
$$(0, \sigma_0(i)), \quad (1, \sigma_1(i)), \quad (2, \sigma_2(i)), \quad \ldots \quad (N, \sigma_N(i)).$$

and Φ_n, Ψ_N are usual rescaling maps $[0,n] \to [-1,1]$ and $[0,1] \to [0,N]$

T_i is "wire i" in diagram of w, rescaled to fit in rectangle $[0,1] \times [-1,1]$.

Simulated trajectories

Selected scaled trajectories from random 2000-element sorting network:



Sine trajectories conjecture

Conjecture (Angel, Holroyd, Romik, Virág 2006)

As $n \to \infty$, the scaled trajectories of random n-element sorting network almost always approach sine waves with the same frequency, but with random amplitude and phase.

Precisely, there are random variables $(A_i^n)_{i=1}^n$ and $(\Theta_i^n)_{i=1}^n$ such that if w is a random n-element sorting network, then for all $\epsilon > 0$

$$\mathrm{Prob}\left(\max_{i \in [n]} \| T_i(w) - A_i^n \sin\left(\pi x + \Theta_i^n\right) \|_\infty > \epsilon\right) \to 0 \quad \text{as } n \to \infty.$$

Sine trajectories conjecture (continued)

Theorem (Angel, Holroyd, Romik, Virág 2006)

As $n \to \infty$, scaled trajectories of random n-element sorting network are almost always Hölder$(\sqrt{8}, \frac{1}{2})$ continuous, in sense that

$$|T_i(x) - T_i(y)| \leq \sqrt{8}|x - y|^{1/2} \quad \text{for all } i \in [n] \text{ and all } x, y \in (0, 1)$$

with probability 1 as $n \to \infty$.

Scaled configurations

For n-element sorting network w, define scaled configuration

$$\mu_t = \mu_t(w) : \{\text{subsets of } \mathbb{R}^2\} \to [0,1]$$

at time $t \in [0,1]$ as sum of point measures

$$\mu_t = \frac{1}{n} \sum_{i=1}^{n} \delta\left(i, \sigma_{\lfloor tN \rfloor}(i)\right) \circ M_n$$

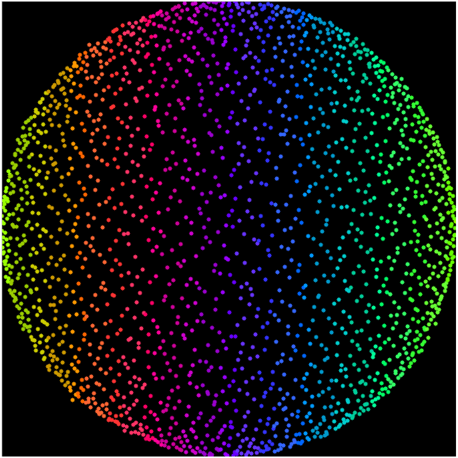
where $\sigma_k = \sigma_k(w)$ and M_n is usual rescaling map $[-1,1]^2 \to [0,n]^2$.

- Conceptually, we consider

$$\begin{pmatrix} \text{Scaled configuration} \\ \text{of } w \text{ at time } t = k/N \end{pmatrix} \Leftrightarrow \begin{pmatrix} \text{Permutation matrix of } \sigma_k(w) \\ \text{rescaled to fit in square } [-1,1]^2 \end{pmatrix}$$

Simulated configurations

Scaled configuration at time $t = N/2$ for 2000-element sorting network:



Archimedes measure

Let $\mathfrak{Arch}_t$ denote Archimedes measure with parameter $t \in [0,1]$:

- $\mathfrak{Arch}_{1/2}$ is probability measure with PDF $\frac{1}{2\pi\sqrt{1-x^2-y^2}}$.
- $\mathfrak{Arch}_t$ for general t defined by transforming $\mathfrak{Arch}_{1/2}$ by map

$$(x, y) \mapsto (x, x\cos(\pi t) + y\sin(\pi t)).$$

- This measure is supported in the ellipse

$$x^2 + y^2 - 2xy\cos(\pi t) \leq \sin^2(\pi t).$$

Archimedes configurations conjecture

Conjecture (Angel, Holroyd, Romik, Virág 2006)

As $n \to \infty$, the scaled configuration at time $t \in [0,1]$ of a random n-element sorting network is almost always supported in the ellipse

$$x^2 + y^2 - 2xy\cos(\pi t) \le \sin^2(\pi t).$$

Precisely, if w is a random n-element sorting network, then for all $t \in [0,1]$

$$\mu_t(w) \quad \Rightarrow \quad \mathfrak{Arch}_t \quad \text{as } n \to \infty$$

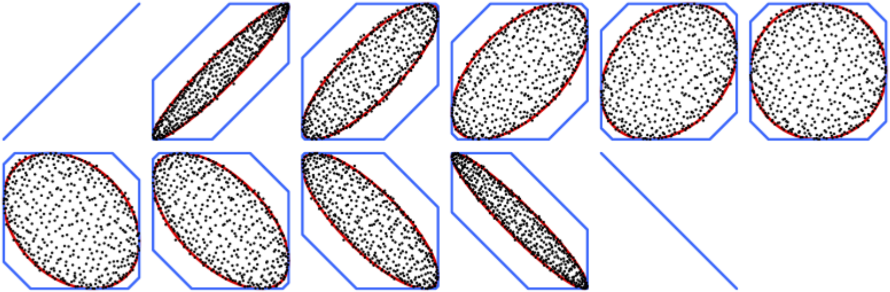
where $\Rightarrow$ denotes convergence of measures in vague topology.

Archimedes configurations conjecture (continued)

Theorem (Angel, Holroyd, Romik, Virág 2006)

As $n \to \infty$, scaled configurations of random n-element sorting network at time $t \in [0, 1]$ almost always supported in certain octagon $\mathcal{O}_t \subset [-1, 1]^2$.

Configurations of 500-element sorting network at times $0, \frac{N}{10}, \frac{2N}{10}, \ldots, N$ with octagon bounds and (conjectural) ellipse bounds:

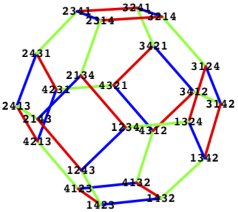


Permutahedron

Define the permutahedron as the natural embedding in Euclidean space of Cayley graph of symmetric group S_n with respect to generating set of simple transpositions $s_i = (i, i+1)$, sending

$$\sigma \in S_n \quad \mapsto \quad (\sigma(1), \sigma(2), \ldots, \sigma(n)) \in \mathbb{R}^n.$$

Permutahedron for $n = 4$:



Reasons to believe conjectures

- The permutahedron lies inside the $(n-2)$-sphere

$$\mathbb{S}_n = \underbrace{\left\{ z \in \mathbb{R}^n : \sum_{i=1}^{n} z_i = \sum_{i=1}^{n} i \right\}}_{\text{hyperplane}} \cap \underbrace{\left\{ z \in \mathbb{R}^n : \sum_{i=1}^{n} z_i^2 = \sum_{i=1}^{n} i^2 \right\}}_{(n-1)\text{-sphere}}.$$

- $1 \in S_n$ and reverse permutation $w_0 \in S_n$ are antipodal on $\mathbb{S}_n$.

- Sorting networks correspond to shortest paths in the permutahedron from 1 to w_0; therefore natural to suspect that such paths are typically closed to great circles in the sphere $\mathbb{S}_n$.

Reasons to believe conjectures (continued)

Conjecture (Angel, Holroyd, Romik, Virág 2006)

For any sequence $\{w_n\}$ of n-element sorting networks then there exists a sequence of great circles $C_n \subset \mathbb{S}_n$ with

$$\operatorname{dist}(w_n, C_n) \cdot \frac{1}{n} \to 0 \quad \text{as } n \to \infty.$$

This is a plausible geometric conjecture, overwhelmingly supported by numerical evidence, which gives reason to believe in earlier conjectures:

Theorem (Angel, Holroyd, Romik, Virág 2006)

Preceding conjecture implies both the sine trajectories conjecture and Archimedes configurations conjecture presented above.

Conjectures in this talk (from around 2006) remain open (in 2017).

References

Some references for this talk:

- O. Angel, A. E. Holroyd, D. Romik and B. Virág, Random Sorting Networks, *Adv. Math.*, 215(2) (2007), 839–868.
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- C. Greene, A. Nijenhuis, and H. S. Wilf, A probabilistic proof of a formula for the number of Young tableaux of a given shape, *Adv. Math.*, 31(1) (1979), 104–109.