HOMEWORK(MENG)

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1. Show that $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$ and $(\vec{a} \times \vec{b}) \times \vec{c} = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{b} \cdot \vec{c})\vec{a}$.

2. Show that $||\vec{a} \times \vec{b}||^2 = ||\vec{a}||^2||\vec{b}||^2 - (\vec{a} \cdot \vec{b})^2$.

3. Let $\vec{r}$ be a smooth function of t which satisfies the equation of motion $\vec{r}\,'' = -\frac{\vec{r}}{r^3}$. Let

$$\vec{L} = \vec{r} \times \vec{r}\,', \quad \vec{A} = \vec{L} \times \vec{r}\,' + \frac{\vec{r}}{r}, \quad E = \frac{1}{2}||\vec{r}\,'|^2 - \frac{1}{r}$$

Show that

$$\vec{L}' = 0, \quad \vec{A}' = 0, \quad E' = 0, \quad \vec{L} \cdot \vec{r} = 0, \quad \vec{L} \cdot \vec{A} = 0$$

and

$$r - \vec{A} \cdot \vec{r} = L^2, \quad A^2 = 2L^2E + 1.$$

Here, $A = ||\vec{A}||$, $L = ||\vec{L}||$, and $r = ||\vec{r}||$.

4. Let $\vec{L}$, $\vec{A}$ be some fixed vectors in $\mathbb{R}^3$, and $\mu = \vec{L} \cdot \vec{A}$. For vector variable $\vec{r} = x_1\vec{e}_1 + x_2\vec{e}_2 + x_3\vec{e}_3$, let us consider the system of equations

$$\vec{L} \cdot \vec{r} = \mu r, \quad r - \vec{A} \cdot \vec{x} = L^2 - \mu^2$$

where r is the length of $\vec{r}$, and L is the length of $\vec{L}$. Assume that $L^2 > \mu^2$.

(i) Show that this system of equations determines a conic.

(ii) Find the eccentricity of this conic in terms of $\vec{L}$ and $\vec{A}$.

5. On the Lorentzian vector space $\mathbb{R}^{1,3} := (\mathbb{R}^4, \cdot)$, the Lorentzian dot product $\cdot$ is given by formula

$$[x_0, x_1, x_2, x_3]^T \cdot [y_0, y_1, y_2, y_3]^T = x_0y_0 - x_1y_1 - x_2y_2 - x_3y_3.$$

Show that, if e is a time-like unit vector (i.e., $e \cdot e = 1$), then formula

$$\langle x, y\rangle := 2(x \cdot e)(y \cdot e) - x \cdot y$$

defines an inner product on $\mathbb{R}^4$. In particular, if $e = e_0 := [1, 0, 0, 0]^T$, we have

$$\langle [x_0, x_1, x_2, x_3]^T, [y_0, y_1, y_2, y_3]^T\rangle = x_0y_0 + x_1y_1 + x_2y_2 + x_3y_3.$$

6. Let l, a be some fixed 4-dimensional Lorentz vectors such that $l \cdot l = -1$, $l \cdot a = 0$, and $a_0 > 0$. Here a_0 denotes the temporal component of a.

(i) Show that the intersection of the plane

$$l \cdot x = 0, \quad a \cdot x = 1$$

with the future light cone

$$x \cdot x = 0, \quad x_0 > 0$$

is a conic.

(ii) Show that this conic is an eclipse, a parabola, or a branch of a hyperbola according as $a \cdot a$ is positive, zero and negative.

7. Let $\mathrm{H}_n(\mathbb{C})$ be the set of complex hermitian matrices of order n, $\mathbb{C}^n_*$ be the set of non-zero column matrices with n complex entries. Then $\mathrm{H}_n(\mathbb{C})$ is a real algebra where the multiplication (called the Jordan multiplication) is the symmetrized matrix multiplication:

$$(u, v) \mapsto u * v := \frac{1}{2}(uv + vu) \quad \text{for any } u, v \text{ in } \mathrm{H}_n(\mathbb{C})$$

For $u \in \mathrm{H}_n(\mathbb{C})$, we use L_u to denote the linear map $v \mapsto u * v$. Consider the map

$$q: \quad \mathbb{C}^n_* \to \mathrm{H}_n(\mathbb{C})$$

which maps $z \in \mathbb{C}^n_*$ to $\bar{z}^T z$. Here T stands for transpose and $\bar{z}$ is the complex conjugation of z.

(i) Show that the image of q, $\operatorname{Im} q$, is precisely the set of rank one, semi-positive hermitian matrices of order n. Let us denote this set by $\mathcal{C}_1$.

(ii) Let $\mathbb{C}P^k$ denote the set of 1-dimensional complex vector subspaces of the complex vector space $\mathbb{C}^{k+1}$. For matrix A, we use $\operatorname{tr} A$ to denote the trace of A and $\operatorname{Col} A$ to denote the column space of A. Show that the map

$$\mathcal{C}_1 \to (0, \infty) \times \mathbb{C}P^{n-1}$$

which maps $x \in \mathcal{C}_1$ to $(\operatorname{tr} x, \operatorname{Col} x)$ is a bijection.

(iii) Show that $\mathrm{H}_n(\mathbb{C})$ is a real vector space. (So it can be viewed as a real affine space with the same dimension.)

(iv) For any smooth map $\alpha\colon I \to \mathrm{H}_n(\mathbb{C})$ where I is an open interval containing 0, if the image of α is inside $\mathcal{C}_1$, we say that α is a smooth parametrized curve on $\mathcal{C}_1$, passing through point $\alpha(0)$. Show that, for any $x \in \mathcal{C}_1$, the image of L_x, $\operatorname{Im} L_x$, can be described this way: $u \in \operatorname{Im} L_x$ if and only if $u = \alpha'(0)$ for some smooth parametrized curve α on $\mathcal{C}_1$, passing through point x at time $t = 0$. (In case you know that $\mathcal{C}_1$ is a smooth manifold, this proves that the tangent space of $\mathcal{C}_1$ at point x is $\{x\} \times \operatorname{Im} L_x$.)