

# HOMEWORK(MENG)

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1. Let  $\vec{L}, \vec{A}$  be some fixed vectors in  $\mathbb{R}^3$ , and  $\mu = \vec{L} \cdot \vec{A}$ . For vector variable  $\vec{r} = x_1 \vec{e}_1 + x_2 \vec{e}_2 + x_3 \vec{e}_3$ , let us consider the system of equations

$$\vec{L} \cdot \vec{r} = \mu r, \quad r - \vec{A} \cdot \vec{r} = L^2 - \mu^2$$

where  $r$  is the length of  $\vec{r}$ , and  $L$  is the length of  $\vec{L}$ . Assume that  $L^2 > \mu^2$ .

- (i) Show that this system of equations determines a conic.  
 (ii) Find the eccentricity of this conic in terms of  $\vec{L}$  and  $\vec{A}$ .
2. On the Lorentzian vector space  $\mathbb{R}^{1,3} := (\mathbb{R}^4, \cdot)$ , the Lorentzian dot product  $\cdot$  is given by formula

$$[x_0, x_1, x_2, x_3]^T \cdot [y_0, y_1, y_2, y_3]^T = x_0 y_0 - x_1 y_1 - x_2 y_2 - x_3 y_3.$$

Show that, if  $e$  is a time-like unit vector (i.e.,  $e \cdot e = 1$ ), then formula

$$\langle x, y \rangle := 2(x \cdot e)(y \cdot e) - x \cdot y$$

defines an inner product on  $\mathbb{R}^4$ . In particular, if  $e = e_0 := [1, 0, 0, 0]^T$ , we have

$$\langle [x_0, x_1, x_2, x_3]^T, [y_0, y_1, y_2, y_3]^T \rangle = x_0 y_0 + x_1 y_1 + x_2 y_2 + x_3 y_3.$$

3. Let  $l, a$  be some fixed 4-dimensional Lorentz vectors such that  $l \cdot l = -1$ ,  $l \cdot a = 0$ , and  $a_0 > 0$ . Here  $a_0$  denotes the temporal component of  $a$ .

- (i) Show that the intersection of the plane

$$l \cdot x = 0, \quad a \cdot x = 1$$

with the future light cone

$$x \cdot x = 0, \quad x_0 > 0$$

is a conic.

- (ii) Show that this conic is an ellipse, a parabola, or a branch of a hyperbola according as  $a \cdot a$  is positive, zero and negative.

4. Let  $H_n(\mathbb{C})$  be the set of complex hermitian matrices of order  $n$ ,  $\mathbb{C}_*^n$  be the set of non-zero column matrices with  $n$  complex entries. Then  $H_n(\mathbb{C})$  is a real algebra where the multiplication (called the Jordan multiplication) is the symmetrized matrix multiplication:

$$(u, v) \mapsto u * v := \frac{1}{2}(uv + vu) \quad \text{for any } u, v \text{ in } H_n(\mathbb{C})$$

For  $u \in H_n(\mathbb{C})$ , we use  $L_u$  to denote the linear map  $v \mapsto u * v$ . Consider the map

$$q : \mathbb{C}_*^n \rightarrow H_n(\mathbb{C})$$

which maps  $z \in \mathbb{C}_*^n$  to  $\bar{z}^T z$ . Here  $T$  stands for transpose and  $\bar{z}$  is the complex conjugation of  $z$ .

(i) Show that the image of  $q$ ,  $\text{Im } q$ , is precisely the set of rank one, semi-positive hermitian matrices of order  $n$ . Let us denote this set by  $\mathcal{C}_1$ .

(ii) Let  $\mathbb{C}P^k$  denote the set of 1-dimensional complex vector subspaces of the complex vector space  $\mathbb{C}^{k+1}$ . For matrix  $A$ , we use  $\text{tr } A$  to denote the trace of  $A$  and  $\text{Col } A$  to denote the column space of  $A$ . Show that the map

$$\mathcal{C}_1 \rightarrow (0, \infty) \times \mathbb{C}P^{n-1}$$

which maps  $x \in \mathcal{C}_1$  to  $(\text{tr } x, \text{Col } x)$  is a bijection.

(iii) Show that  $H_n(\mathbb{C})$  is a real vector space. (So it can be viewed as a real affine space with the same dimension.)

(iv) For any smooth map  $\alpha: I \rightarrow H_n(\mathbb{C})$  where  $I$  is an open interval containing 0, if the image of  $\alpha$  is inside  $\mathcal{C}_1$ , we say that  $\alpha$  is a smooth parametrized curve on  $\mathcal{C}_1$ , passing through point  $\alpha(0)$ . Show that, for any  $x \in \mathcal{C}_1$ , the image of  $L_x$ ,  $\text{Im } L_x$ , can be described this way:  $u \in \text{Im } L_x$  if and only if  $u = \alpha'(0)$  for some smooth parametrized curve  $\alpha$  on  $\mathcal{C}_1$ , passing through point  $x$  at time  $t = 0$ . (In case you know that  $\mathcal{C}_1$  is a smooth manifold, this proves that the tangent space of  $\mathcal{C}_1$  at point  $x$  is  $\{x\} \times \text{Im } L_x$ .)