

2.6 Arrow

Arrow

Theory of voting (actually not equivalent to democracy)

1770: Jean-Charles de Borda and Marquis de Condorcet, two methods for electing members of French Academy of Sciences

1790: Apportionment of US congressman, methods by Alexander Hamilton, Thomas Jefferson, ...

1856: Carl Andrae and Thomas Hare, single transferable vote in Denmark

1870: Study of single-winner methods (revisit Borda and Condorcet) by Edward Nanson, Charles Dodgson (aka Lewis Carroll), ...

1899: Party-list proportional representation in Belgium election

1940: Game theory by John von Neumann, John Nash, ...

1951: **Impossibility Theorem** by Kenneth Arrow

Nobel Prize in Economics 1972

1973: Theorem by Allan Gibbard and Mark Satterthwaite,

“reasonable” voting system is always susceptible to tactical voting

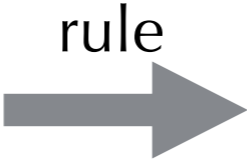


Arrow

5 people elect A and B

A	B	B	A	A
v ₁	v ₂	v ₃	v ₄	v ₅
B	A	A	B	B

election



A
v
B

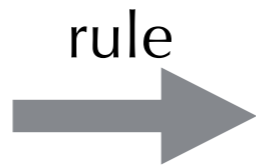
result

Arrow

5 people elect A and B

A	B	B	A	A
v ₁	v ₂	v ₃	v ₄	v ₅
B	A	A	B	B

election



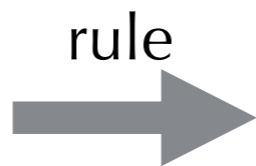
A
v
B

result

majority

B	B	B	B	B
v ₁	v ₂	v ₃	v ₄	v ₅
A	A	A	A	A

another election



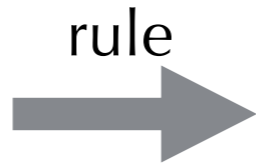
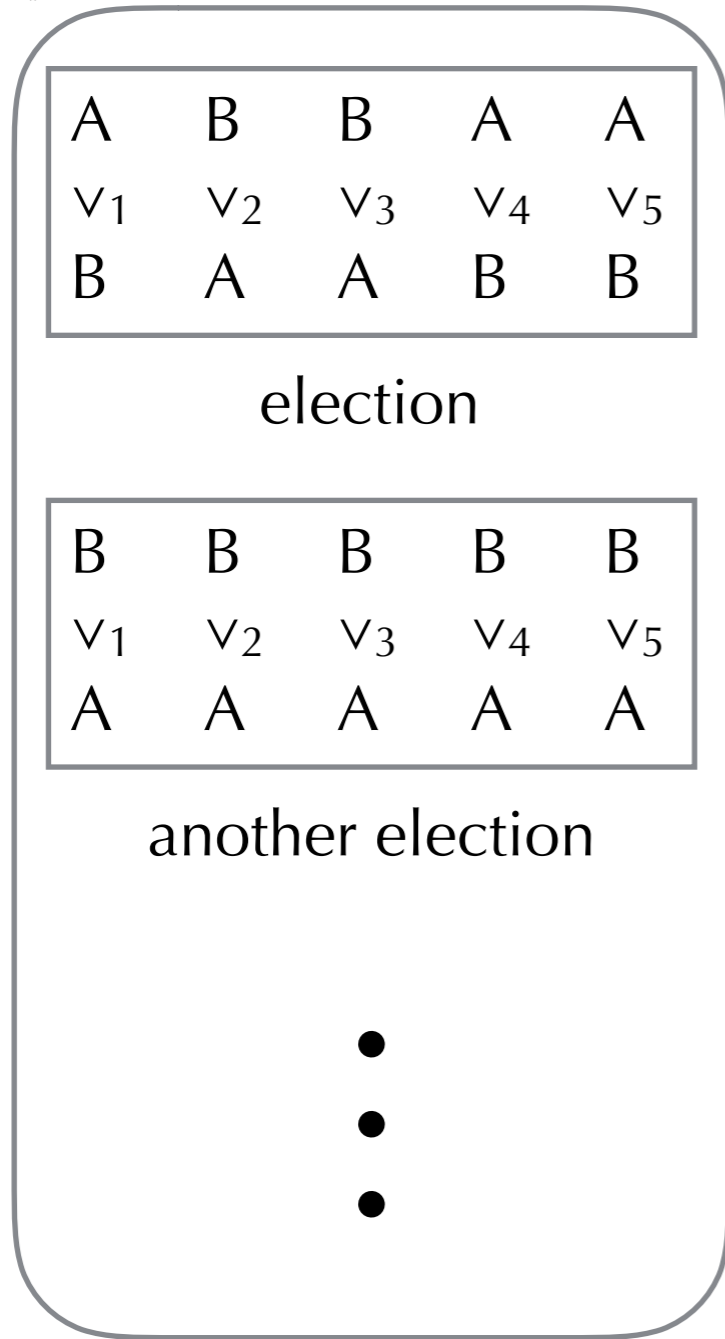
B
v
A

new result

unanimity

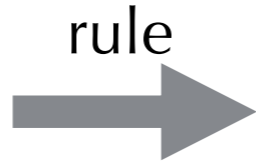
Arrow

5 people elect A and B



result

majority



new result

unanimity

Arrow

Definition An *election* of n people choosing m candidates consists of

- *vote*: each people has a ranking $x_1 > x_2 > \dots > x_m$ of all m candidates
- *rule*: vote (all the rankings) \Rightarrow result (a single ranking)

Arrow

Definition An *election* of n people choosing m candidates consists of

- *vote*: each people has a ranking $x_1 > x_2 > \dots > x_m$ of all m candidates
- *rule*: vote (all the rankings) \Rightarrow result (a single ranking)

Some reasonable requirements for a voting system

1. **Secrecy (S)**: Switching the rankings of two people does not change the result
2. **Monotonicity (M)**: If everybody moves x up, then the result moves x up
3. **Unanimity (U)**: If $x > y$ in all rankings, then $x > y$ in the result
4. **Fairness (F)**: If the result of an election has $x > y$, and everybody switches x and y in a second election, then the result of the second election has $x < y$
5. **Independence of Irrelevant Alternatives (I)**: If the result of an election has $x > y$, and everybody moves a third candidate z in a second election, then the result of the second election still has $x > y$

Arrow

Theorem For 2 candidates, the only rule satisfying S, F and M is the majority rule

Arrow

Theorem For 2 candidates, the only rule satisfying S, F and M is the majority rule

Let the 2 candidates be A and B.

Secrecy means the result depends only on the number a of $A > B$.

Denote $V(a) = \{ a[A > B] \text{ and } (n - a)[A < B] \}$. Switch A, B in $V(a)$ gives $V(n - a)$.

Denote the result of $V(a)$ by $R(a)$.

Monotonicity means that $R(a) = [A > B]$ and $b > a$ implies $R(b) = [A > B]$.

Let c be the smallest number with $R(c) = [A > B]$. Then $R(c - 1) = [A < B]$.

Switch A, B. By Fairness, we have $R(n - c) = [A < B]$ and $R(n - c + 1) = [A > B]$.

Apply M to $R(n - c) = [A < B]$ and $R(c) = [A > B]$, we have $n - c < c$.

Apply M to $R(c - 1) = [A < B]$ and $R(n - c + 1) = [A > B]$, we have $c - 1 < n - c + 1$.

So $2c < n < 2c + 2$. So n is odd and c is the smallest number $> n/2$.

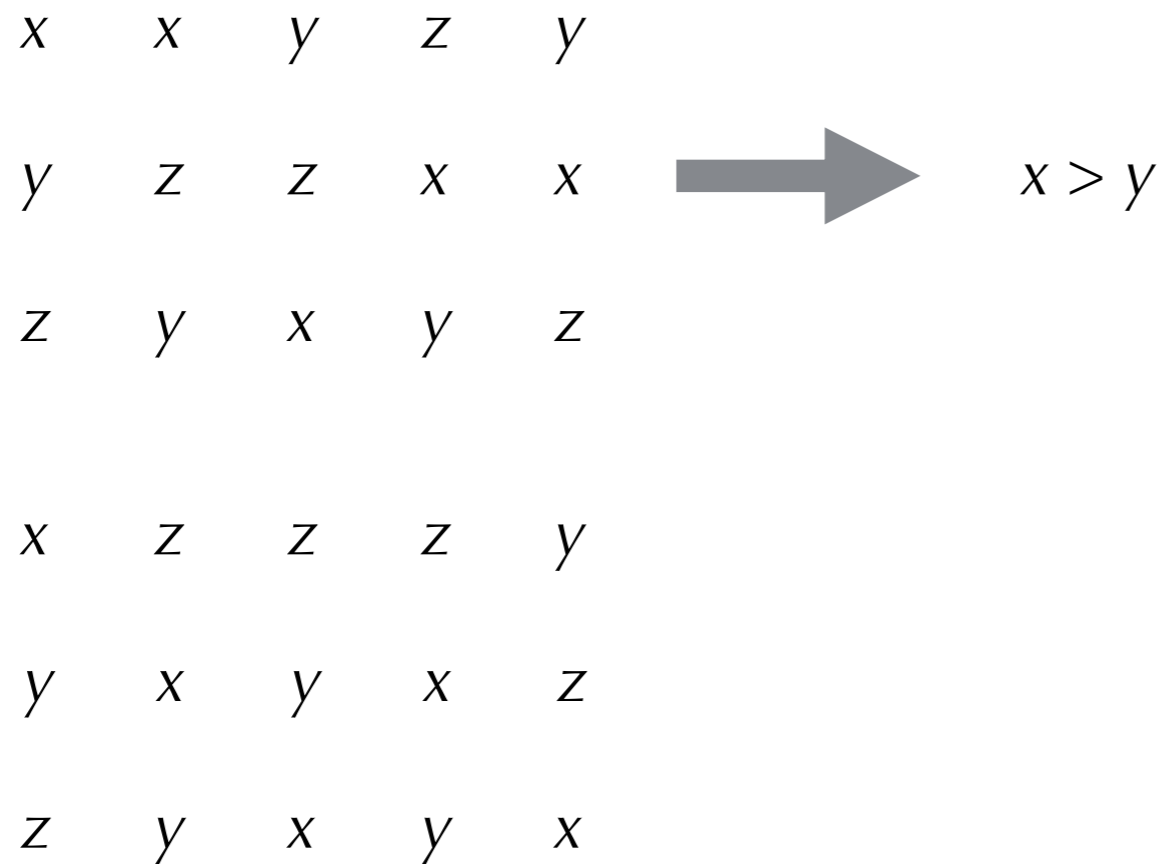
Arrow

Theorem For ≥ 3 candidates, the only election satisfying U and I is the dictatorship

Arrow

Theorem For ≥ 3 candidates, the only election satisfying U and I is the dictatorship

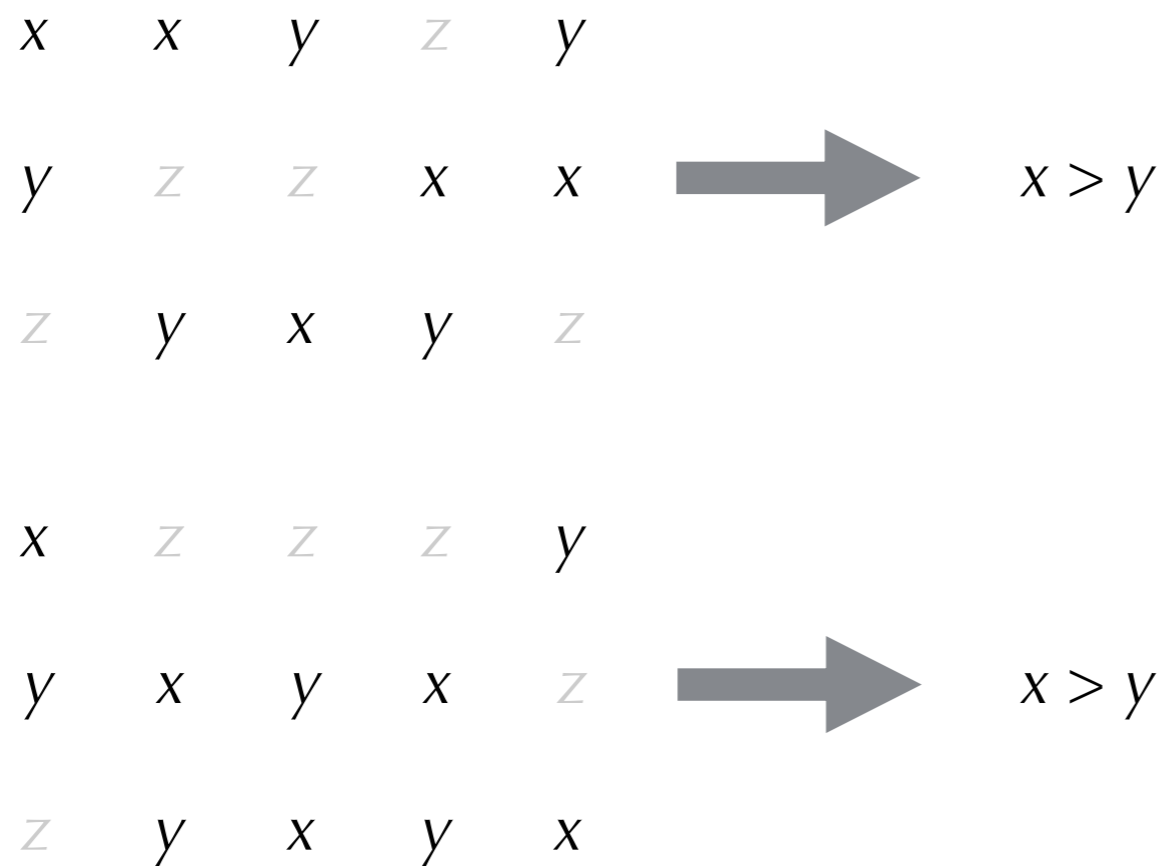
Independence of Irrelevant Alternatives (I): If the result of an election has $x > y$, and everybody moves a third candidate z in a second election, then the result of the second election still has $x > y$



Arrow

Theorem For ≥ 3 candidates, the only election satisfying U and I is the dictatorship

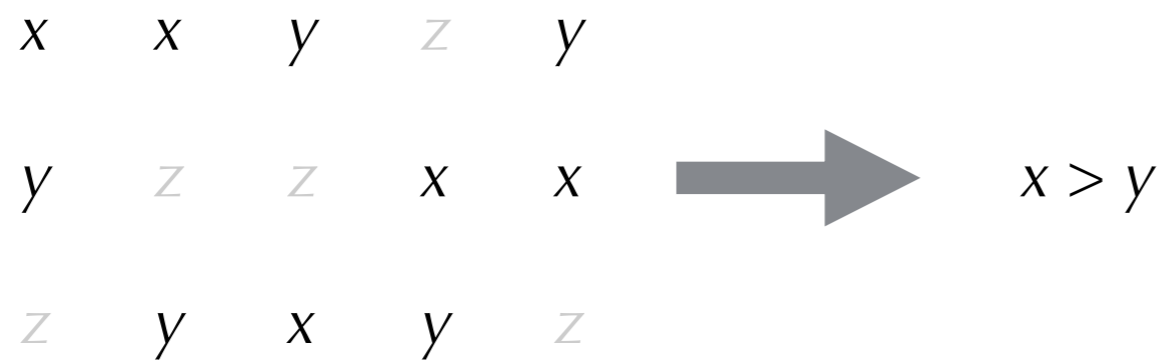
Independence of Irrelevant Alternatives (I): If the result of an election has $x > y$, and everybody moves a third candidate z in a second election, then the result of the second election still has $x > y$



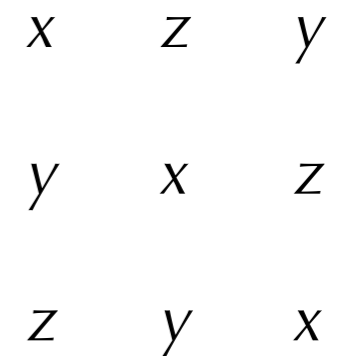
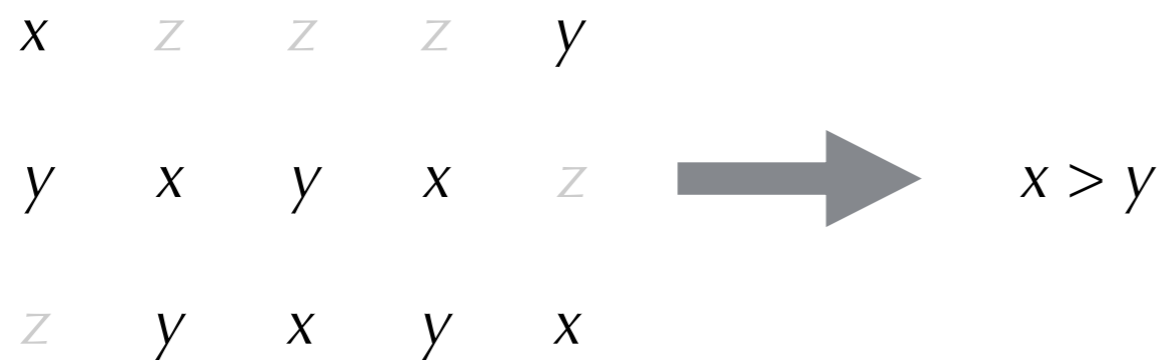
Arrow

Theorem For ≥ 3 candidates, the only election satisfying U and I is the dictatorship

Independence of Irrelevant Alternatives (I): If the result of an election has $x > y$, and everybody moves a third candidate z in a second election, then the result of the second election still has $x > y$



“I” means that the rule is a combination of pairwise rules
 If we also require S, F, M, this may lead to circular result



Arrow

Theorem For ≥ 3 candidates, the only election satisfying U and I is the dictatorship

Arrow

Theorem For ≥ 3 candidates, the only election satisfying U and I is the dictatorship

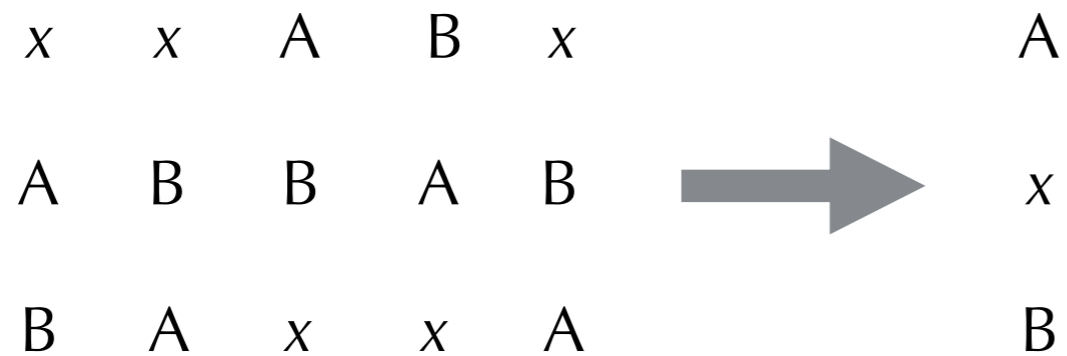
Lemma If x is always at top or bottom of every ranking in an election, then x is at top or bottom in the result

x	x	A	B	x
A	B	B	A	B
B	A	x	x	A

Arrow

Theorem For ≥ 3 candidates, the only election satisfying U and I is the dictatorship

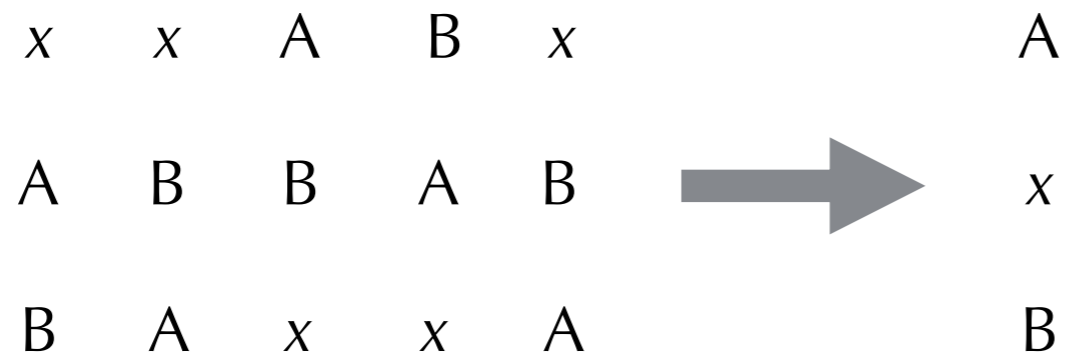
Lemma If x is always at top or bottom of every ranking in an election, then x is at top or bottom in the result



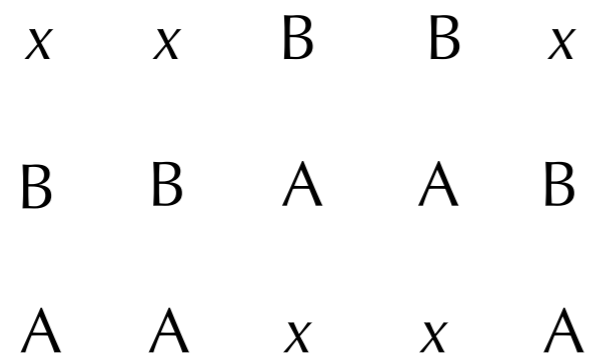
Arrow

Theorem For ≥ 3 candidates, the only election satisfying U and I is the dictatorship

Lemma If x is always at top or bottom of every ranking in an election, then x is at top or bottom in the result



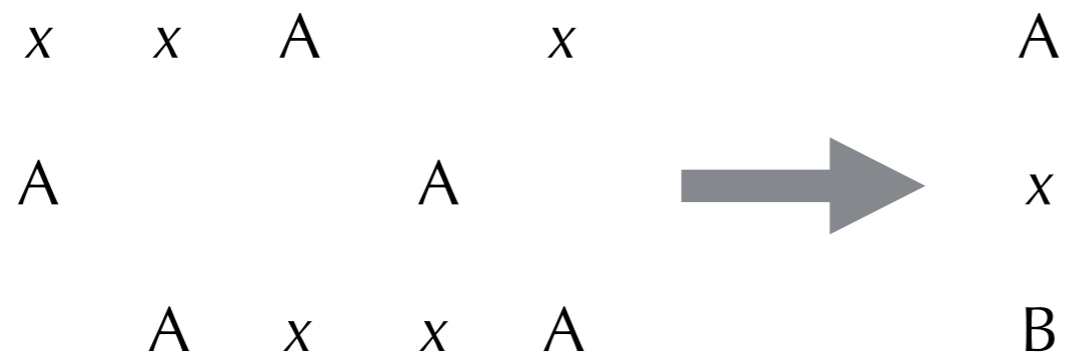
move B ahead of A



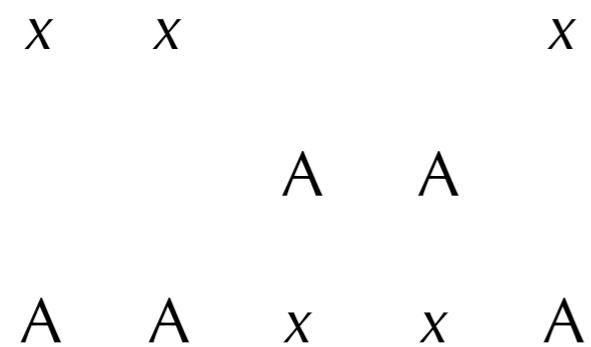
Arrow

Theorem For ≥ 3 candidates, the only election satisfying U and I is the dictatorship

Lemma If x is always at top or bottom of every ranking in an election, then x is at top or bottom in the result



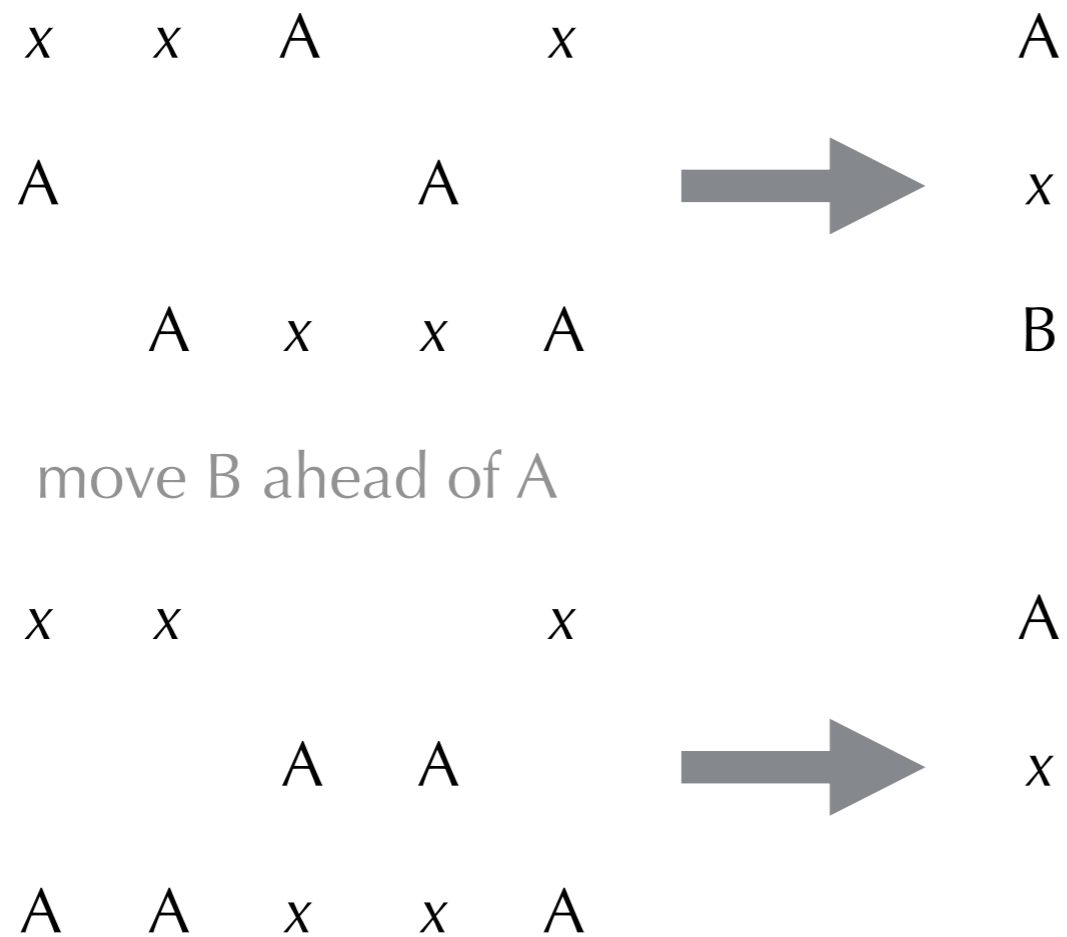
move B ahead of A



Arrow

Theorem For ≥ 3 candidates, the only election satisfying U and I is the dictatorship

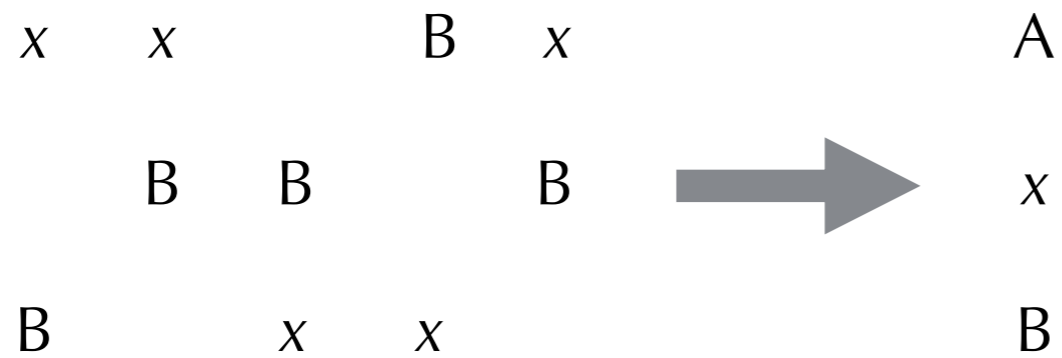
Lemma If x is always at top or bottom of every ranking in an election, then x is at top or bottom in the result



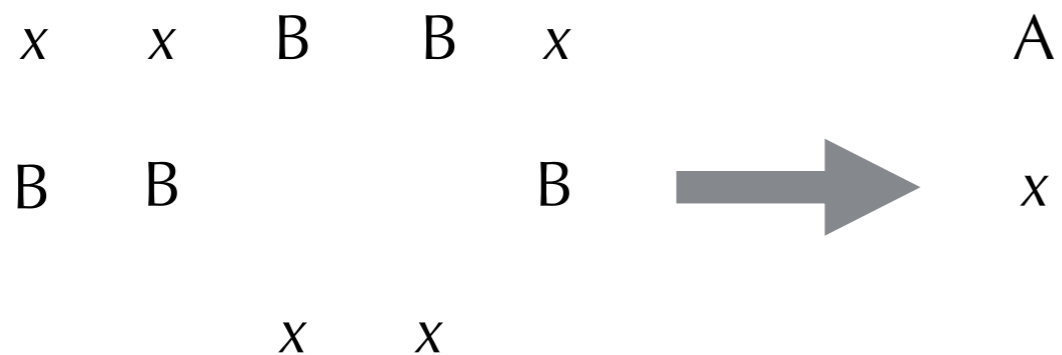
Arrow

Theorem For ≥ 3 candidates, the only election satisfying U and I is the dictatorship

Lemma If x is always at top or bottom of every ranking in an election, then x is at top or bottom in the result



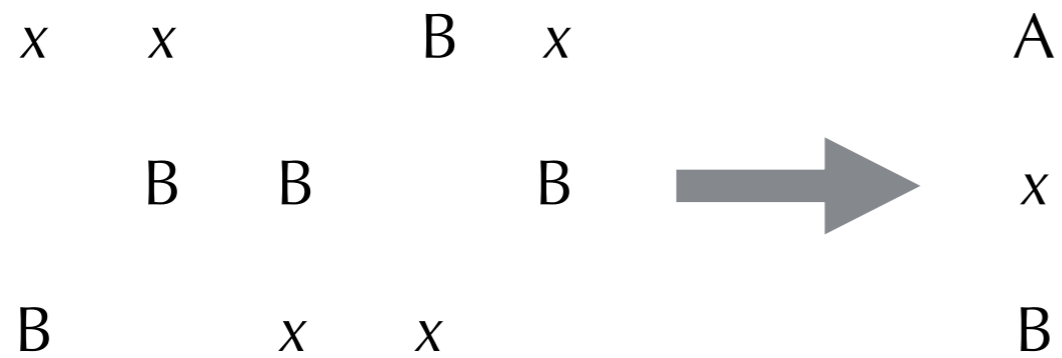
move B ahead of A



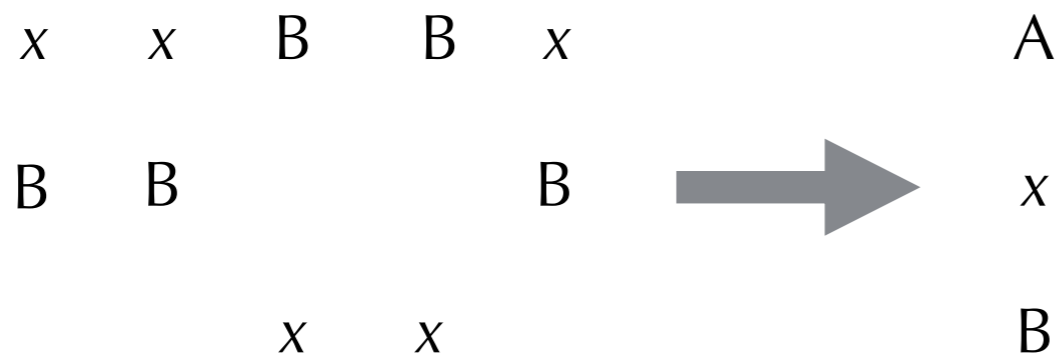
Arrow

Theorem For ≥ 3 candidates, the only election satisfying U and I is the dictatorship

Lemma If x is always at top or bottom of every ranking in an election, then x is at top or bottom in the result




move B ahead of A




Arrow

Theorem For ≥ 3 candidates, the only election satisfying U and I is the dictatorship

Lemma If x is always at top or bottom of every ranking in an election, then x is at top or bottom in the result

x	x	A	B	x		A
A	B	B	A	B		x
B	A	x	x	A		B

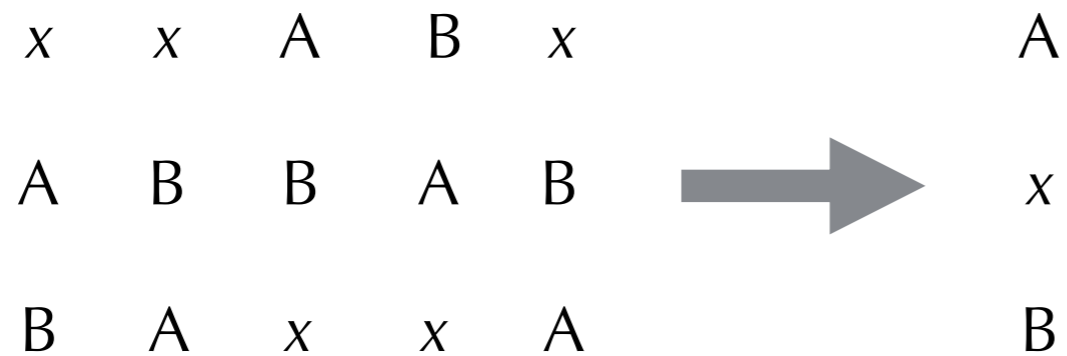
move B ahead of A

x	x	B	B	x		A
B	B	A	A	B		x
A	A	x	x	A		B

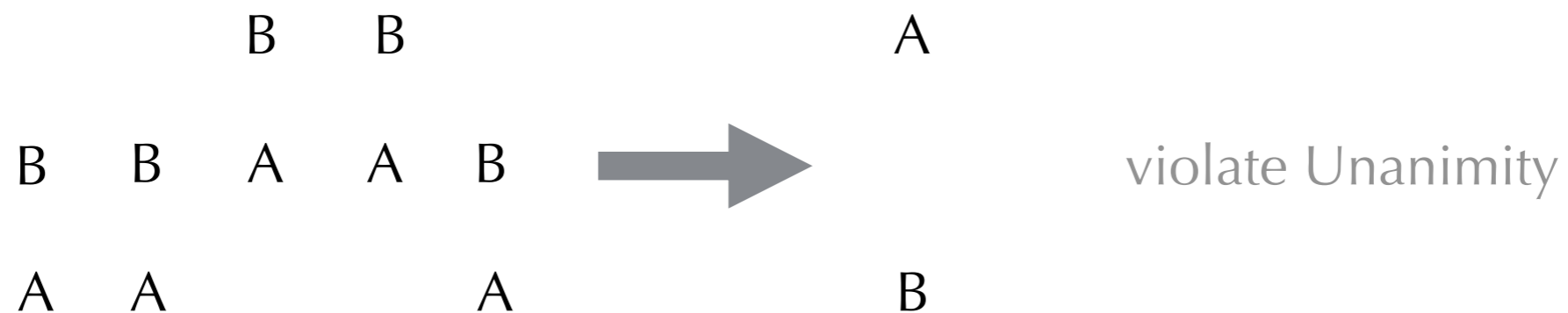
Arrow

Theorem For ≥ 3 candidates, the only election satisfying U and I is the dictatorship

Lemma If x is always at top or bottom of every ranking in an election, then x is at top or bottom in the result



move B ahead of A



Arrow

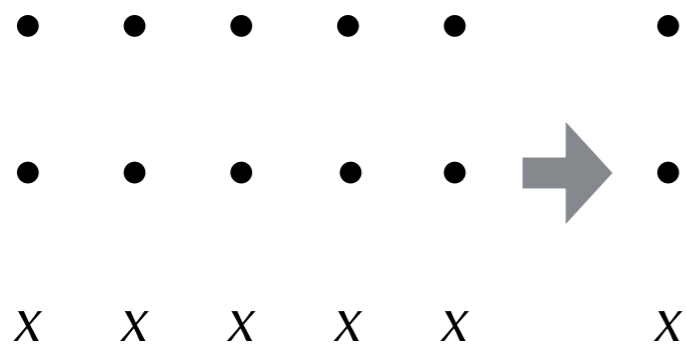
Theorem For ≥ 3 candidates, the only election satisfying U and I is the dictatorship

Lemma If x is always at top or bottom of every ranking in an election, then x is at top or bottom in the result

Arrow

Theorem For ≥ 3 candidates, the only election satisfying U and I is the dictatorship

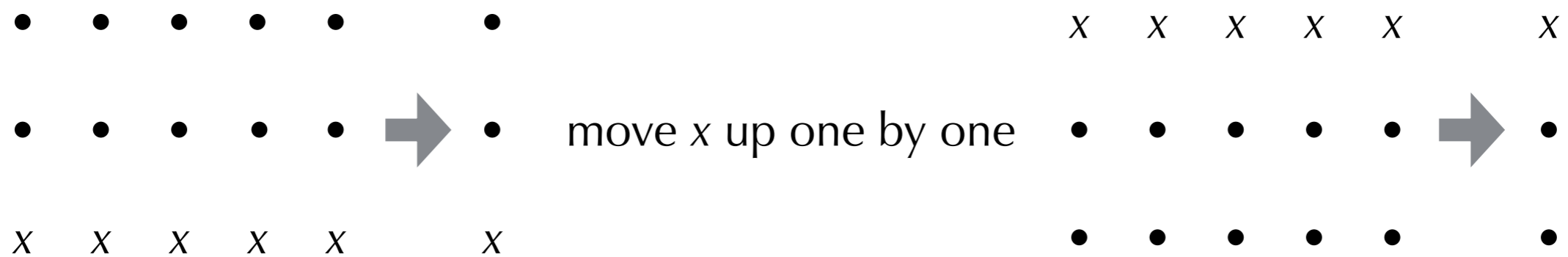
Lemma If x is always at top or bottom of every ranking in an election, then x is at top or bottom in the result



Arrow

Theorem For ≥ 3 candidates, the only election satisfying U and I is the dictatorship

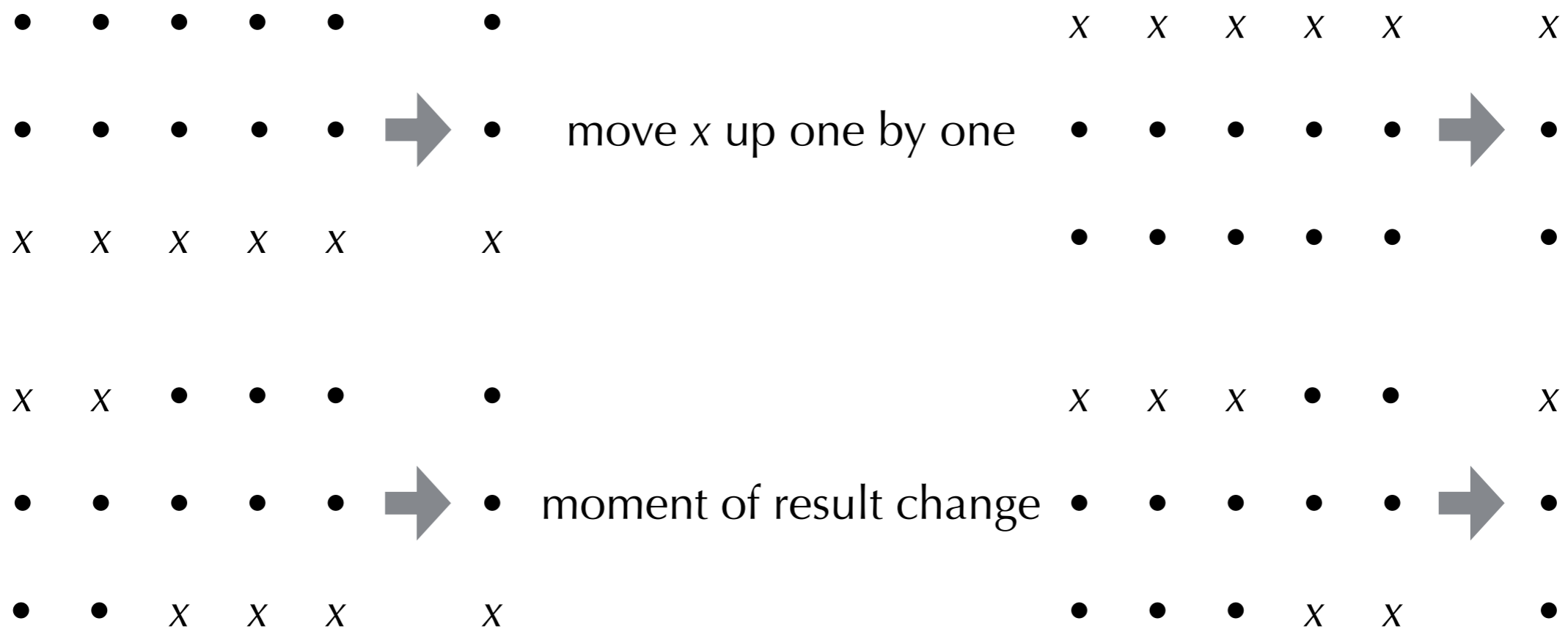
Lemma If x is always at top or bottom of every ranking in an election, then x is at top or bottom in the result



Arrow

Theorem For ≥ 3 candidates, the only election satisfying U and I is the dictatorship

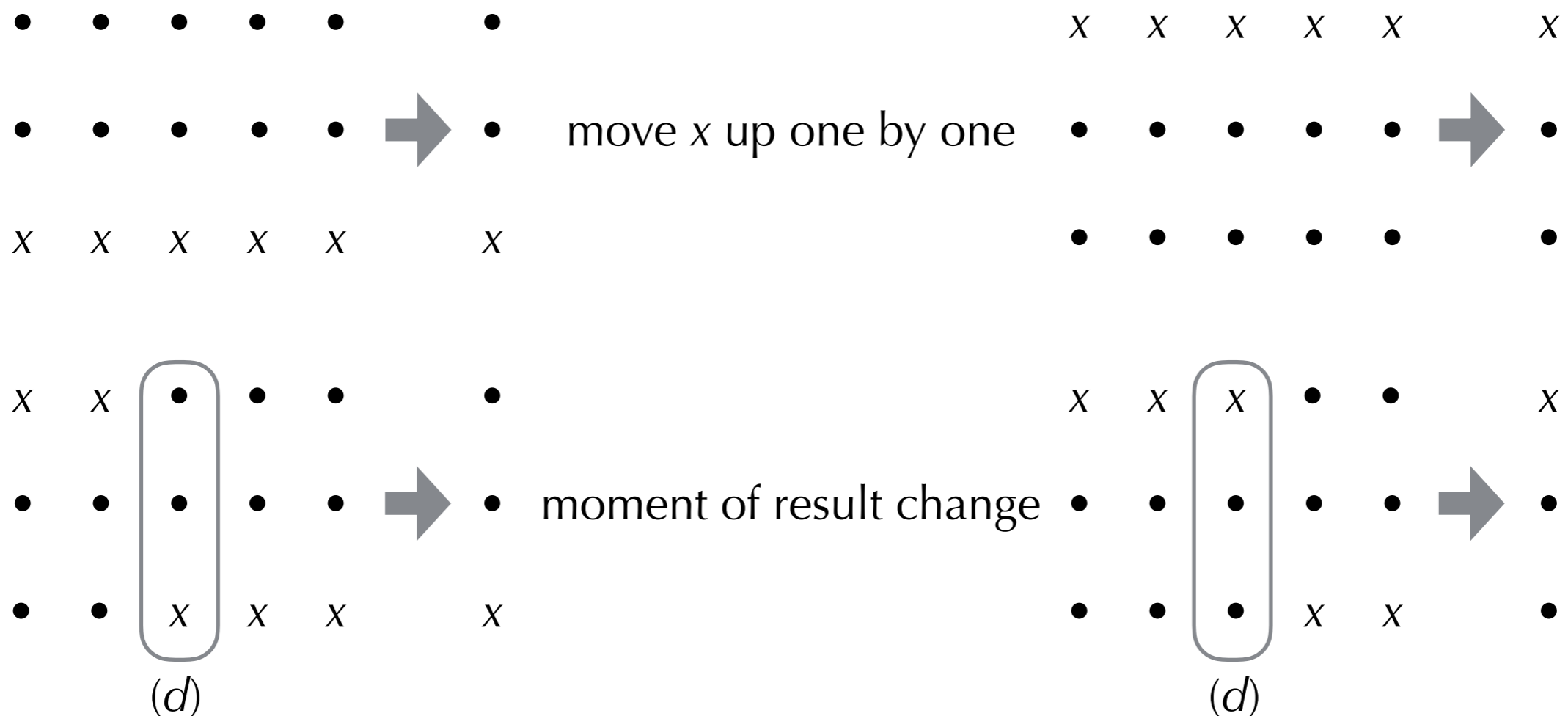
Lemma If x is always at top or bottom of every ranking in an election, then x is at top or bottom in the result



Arrow

Theorem For ≥ 3 candidates, the only election satisfying U and I is the dictatorship

Lemma If x is always at top or bottom of every ranking in an election, then x is at top or bottom in the result

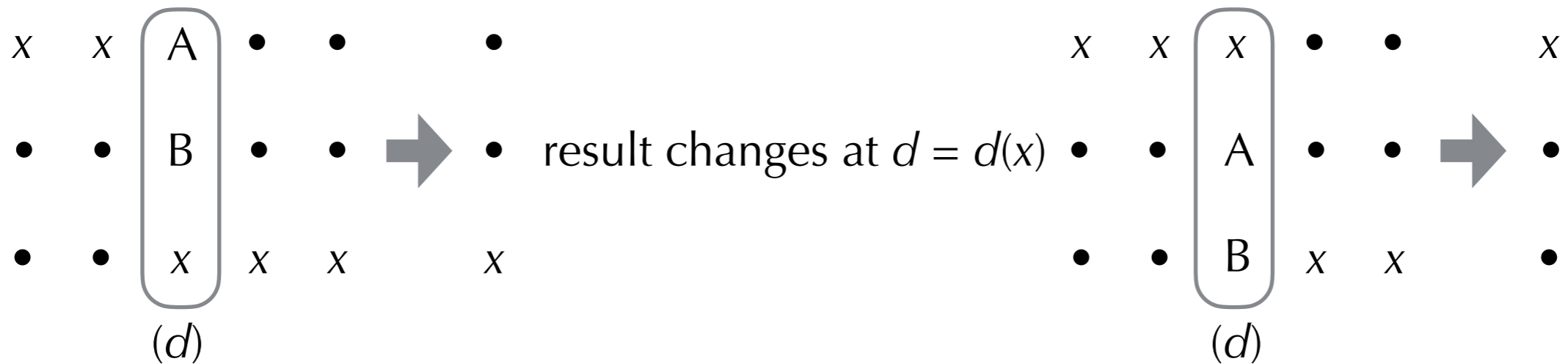


d is independent of all \bullet , $d = d(x)$

Arrow

Theorem For ≥ 3 candidates, the only election satisfying U and I is the dictatorship

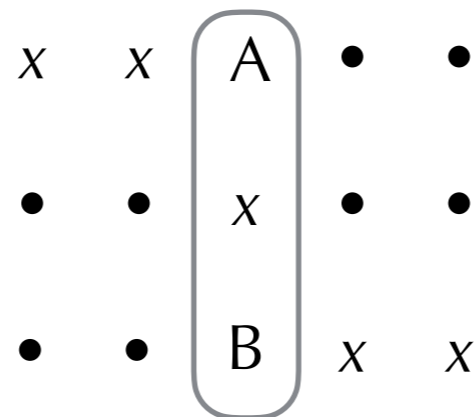
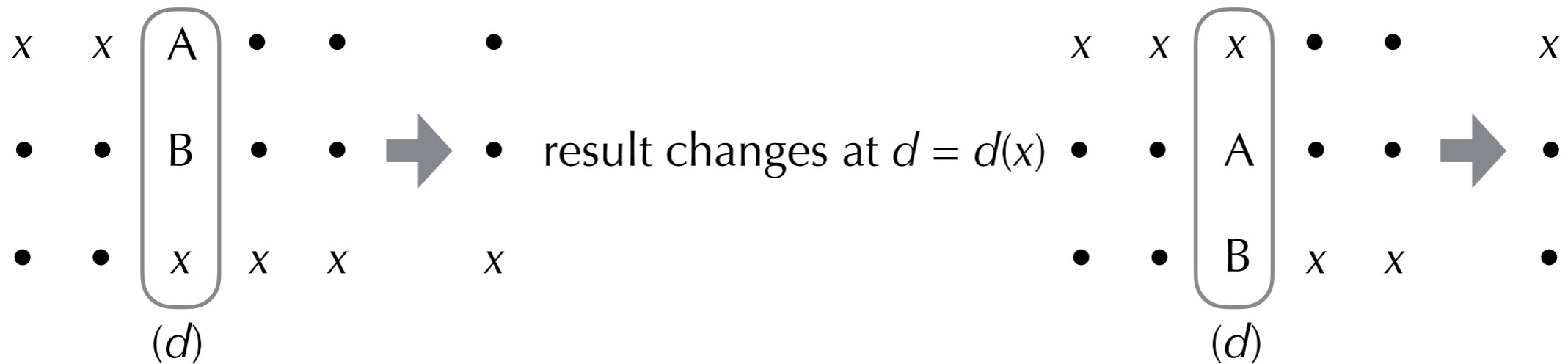
Lemma If x is always at top or bottom of every ranking in an election, then x is at top or bottom in the result



Arrow

Theorem For ≥ 3 candidates, the only election satisfying U and I is the dictatorship

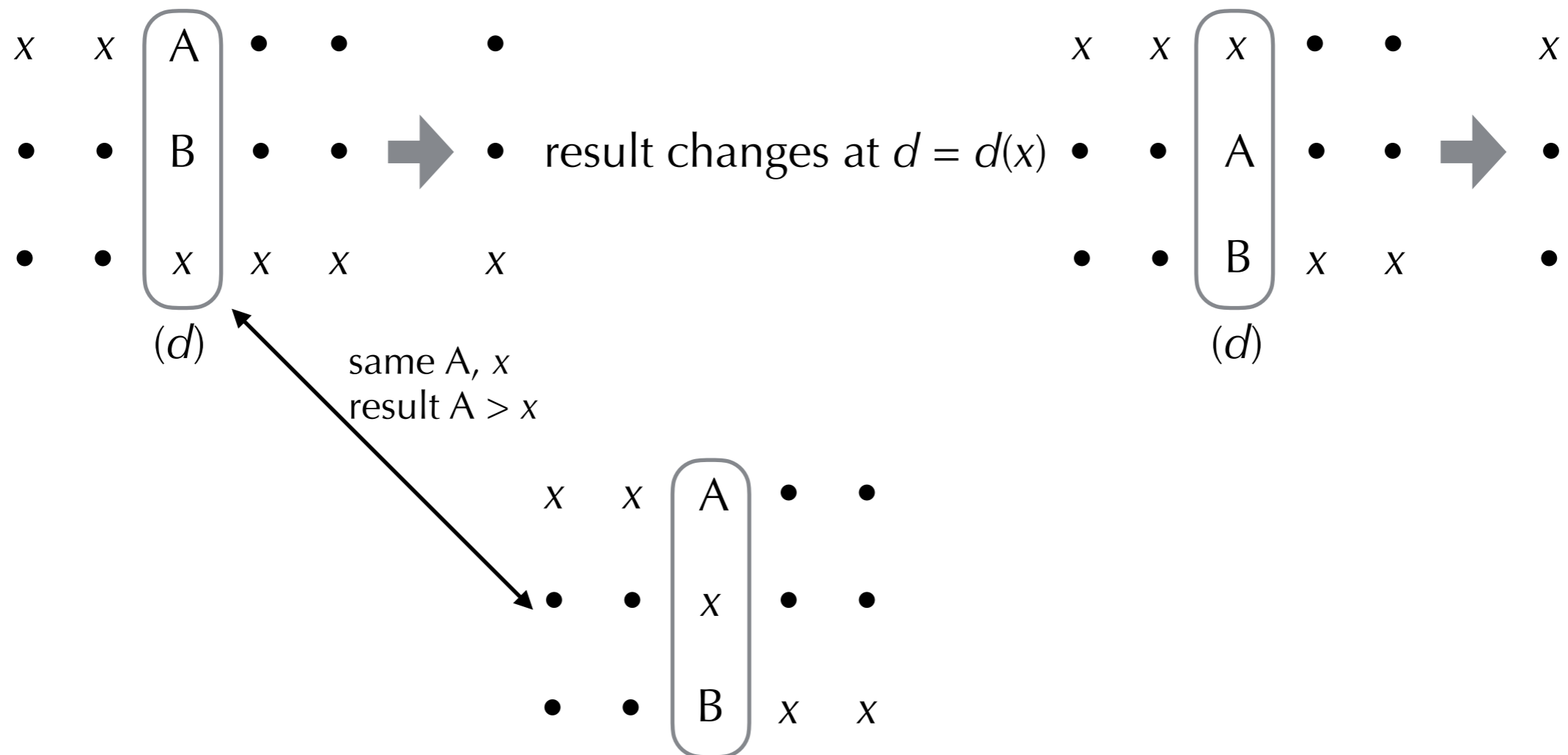
Lemma If x is always at top or bottom of every ranking in an election, then x is at top or bottom in the result



Arrow

Theorem For ≥ 3 candidates, the only election satisfying U and I is the dictatorship

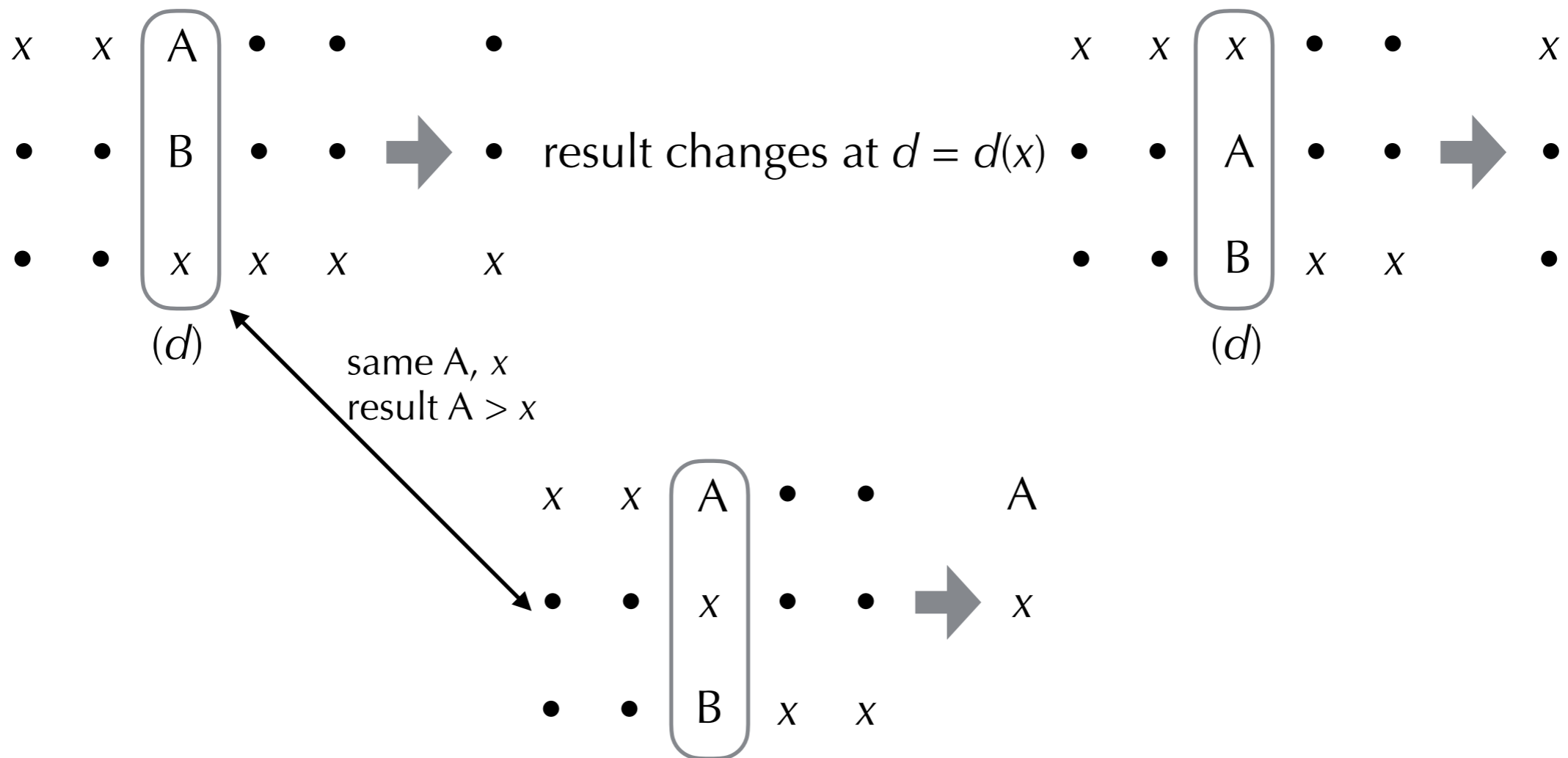
Lemma If x is always at top or bottom of every ranking in an election, then x is at top or bottom in the result



Arrow

Theorem For ≥ 3 candidates, the only election satisfying U and I is the dictatorship

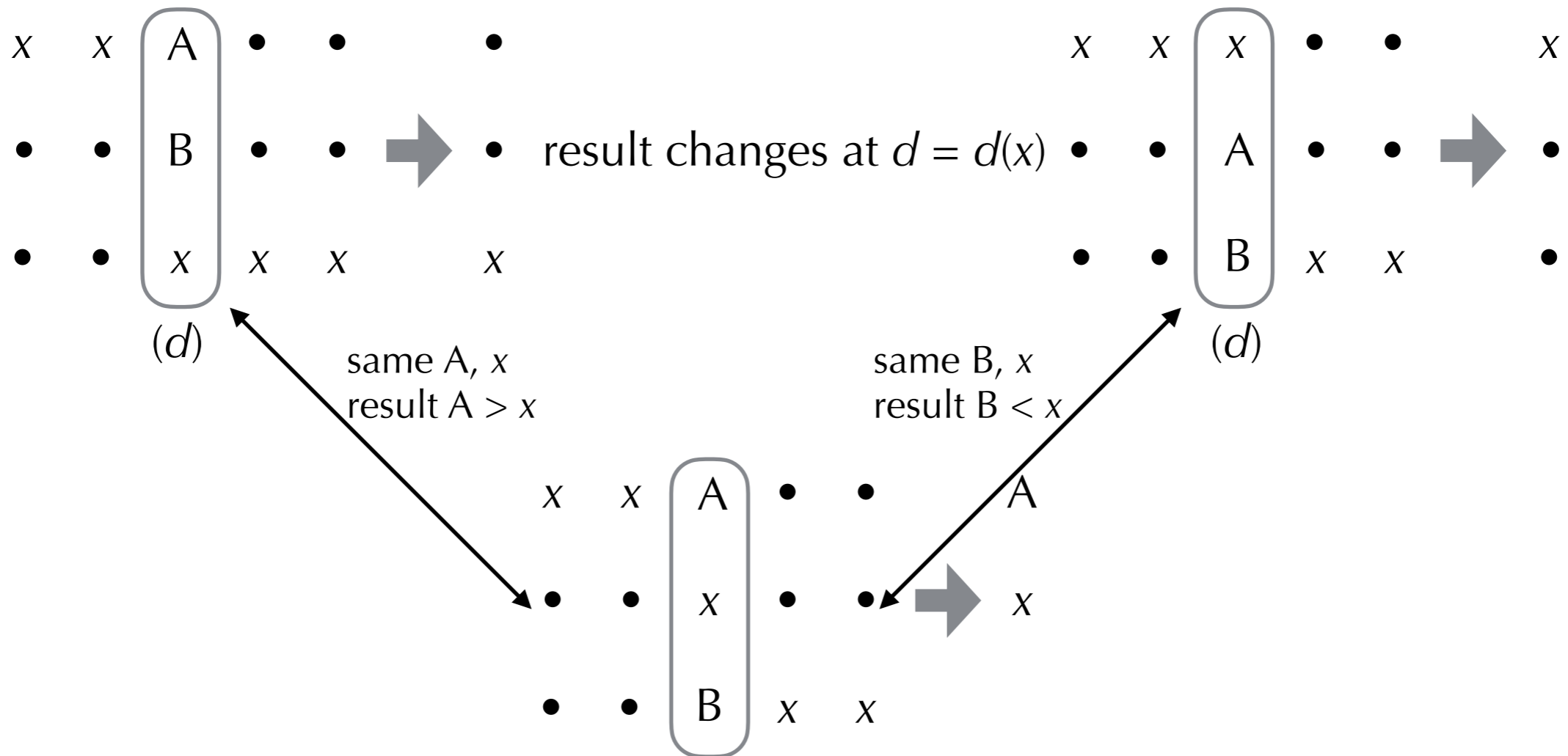
Lemma If x is always at top or bottom of every ranking in an election, then x is at top or bottom in the result



Arrow

Theorem For ≥ 3 candidates, the only election satisfying U and I is the dictatorship

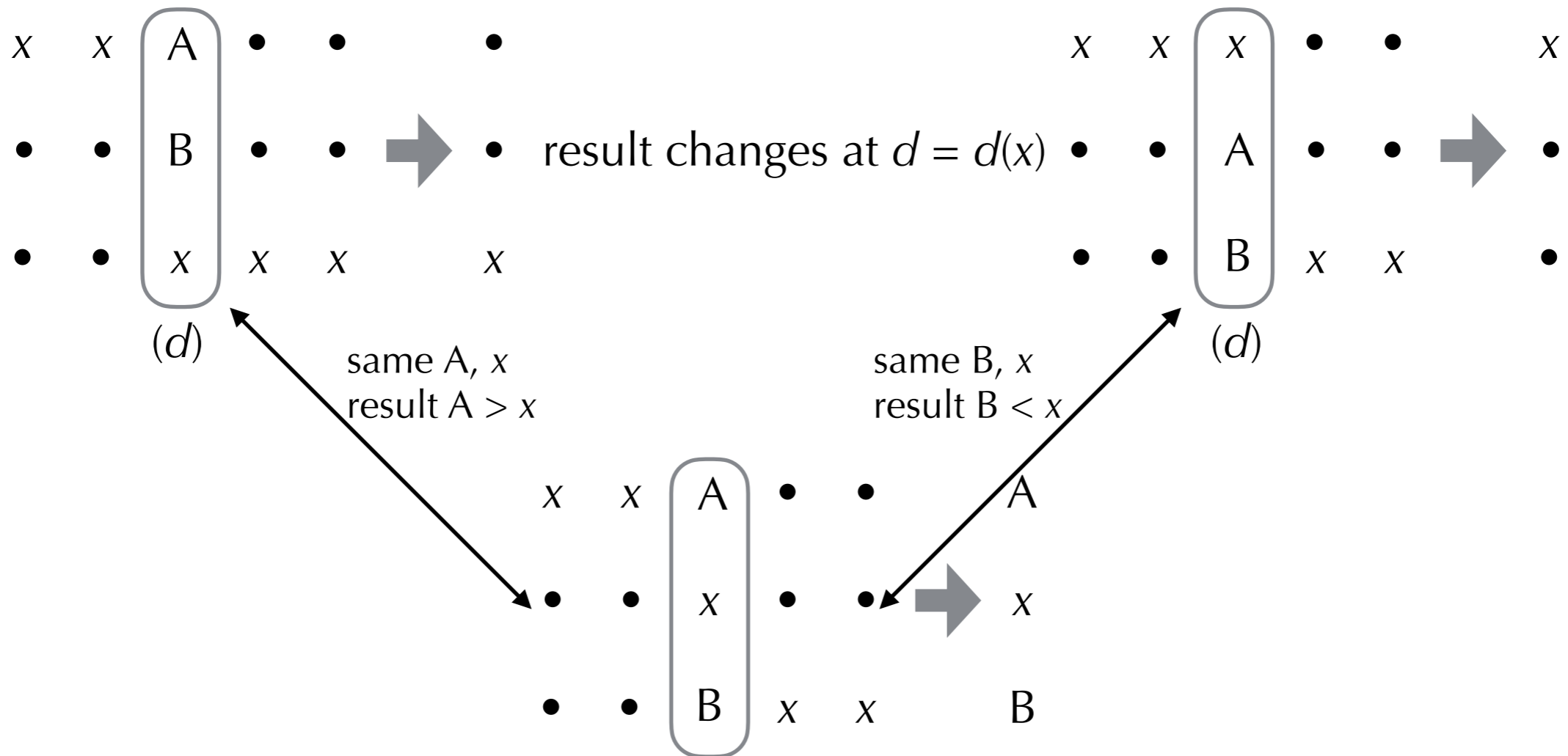
Lemma If x is always at top or bottom of every ranking in an election, then x is at top or bottom in the result



Arrow

Theorem For ≥ 3 candidates, the only election satisfying U and I is the dictatorship

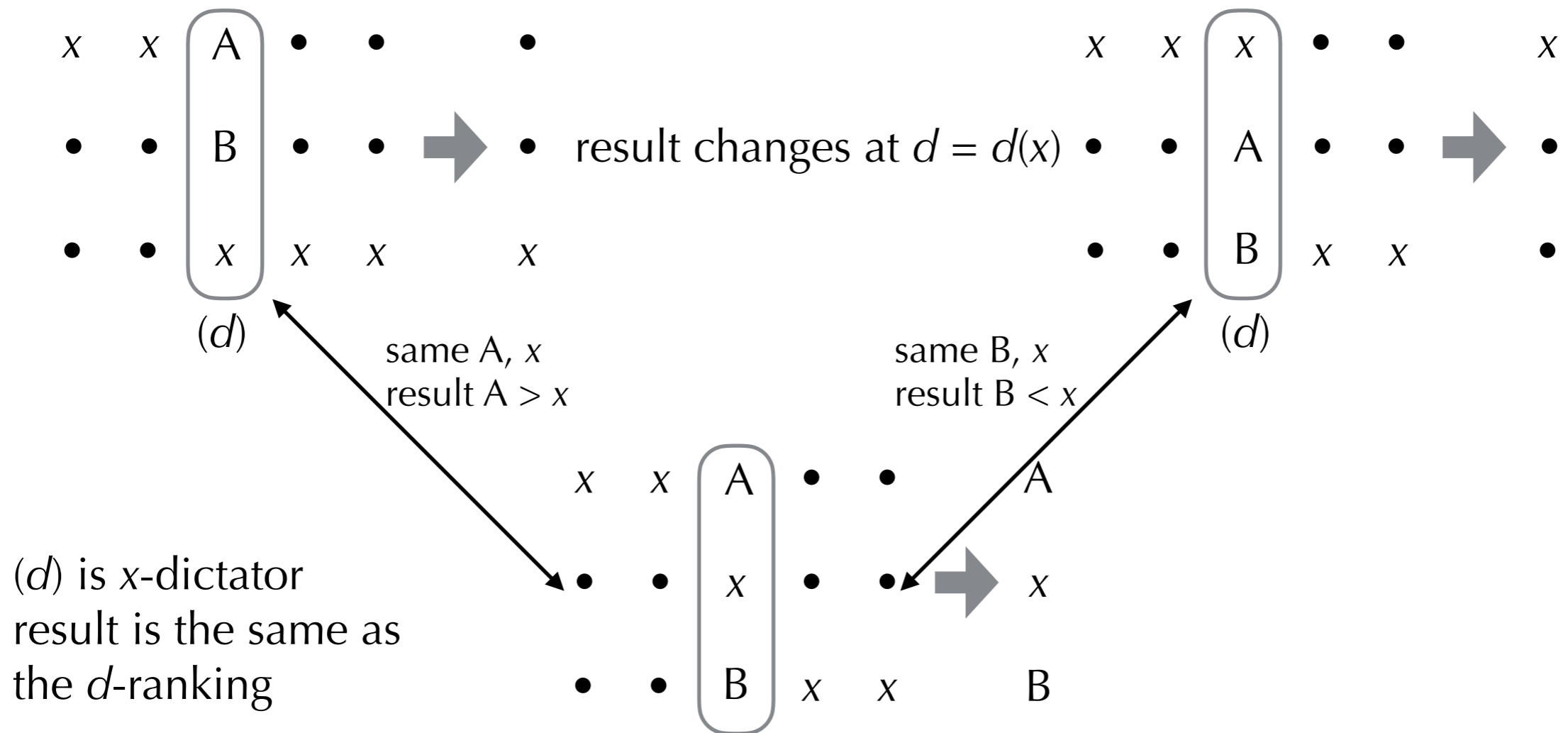
Lemma If x is always at top or bottom of every ranking in an election, then x is at top or bottom in the result



Arrow

Theorem For ≥ 3 candidates, the only election satisfying U and I is the dictatorship

Lemma If x is always at top or bottom of every ranking in an election, then x is at top or bottom in the result



Arrow

Theorem For ≥ 3 candidates, the only election satisfying U and I is the dictatorship

Lemma If x is always at top or bottom of every ranking in an election, then x is at top or bottom in the result

