Tilings of Sphere by Congruent Pentagons III

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Abstract

There are exactly eight edge-to-edge tilings of the sphere by congruent equilateral pentagons.

1 Introduction

The classification of edge-to-edge tilings of the sphere by congruent triangles was started in 1923 by Sommerville [5] and was finally completed in 2002 by Ueno and Agaoka [6]. The classification of similar pentagonal tilings was done for the case of minimal number of 12 tiles by Gao, Shi and Yan [1, 4], and for some cases of enough variety in edge lengths by Cheuk, Cheung and Yan [3]. The technique developed in these papers should be sufficient for dealing with all the other cases of enough variety in edge lengths. This paper develops a completely different method, for the opposite case of tilings by equilateral pentagons (i.e., no variety in edge lengths).

General discussions about spherical tilings can be found in the introduction of [3, 4]. Here we only mention that the tile in an edge-to-edge tiling of the sphere by congruent polygons must be triangle, quadrilateral or pentagon. We believe that pentagonal tilings are easier to study than the quadrilateral ones because 5 is an "extreme" among $3, 4, 5$.

The key idea for dealing with equilateral pentagons is the following. A general pentagon is determined by the free choice of 4 edge lengths and

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3 angles, yielding 7 degrees of freedom. The requirement that all 5 edges are equal imposes 4 equations, leaving $7 - 4 = 3$ degrees of freedom for equilateral pentagons. Therefore 3 more independent equations are enough to completely determine equilateral pentagons.

It is easy to show that most vertices in a spherical pentagonal tiling have degree 3. The 3 independent equations can often be obtained by studying the angle combinations at degree 3 vertices. By using the fact that the sum of all angles at any vertex is 2π, we may derive all such combinations (which we call *anglewise vertex combination at degree* 3 and denote by AVC_3) in Lemma 1 and Table 1. By further considering the possible angle combinations in the pentagonal tile, we get more restrictions in Section 3. Using such restrictions, we can get 3 independent equations (sometimes involving angle sums at degree 4 or 5 vertices) for all but one exceptional case. Therefore, with one exception, the equilateral pentagonal tiles can be completely determined. It is then not difficult to find the tilings.

Since the minimal case of spherical tilings by 12 congruent pentagons has benn classified [1, 4], we will assume the number of tiles $f > 12$ in this paper. By [7], we actually know that f is an even number ≥ 16. It turns out that we need to calculate 481 cases, including various angle arrangements in the pentagon. We use the MAPLE software to carry out all these calculations and find out that almost all cases either do not lead to equilateral pentagons, or lead to pentagons whose area is not 4π (the area of the unit sphere) divided by an even number ≥ 16 (the number of tiles). For the remaining limited number of cases, we find total of seven tilings. Together with the regular dodecahedron from [4], we get the following complete list.

Theorem. *There are eight tilings of the sphere by congruent equilateral pentagons. In the list below, the edge length is a, the angles are arranged as $[\alpha, \beta, \delta, \gamma, \epsilon]$ in the pentagon, and we always have $\alpha + \beta + \gamma = 2\pi$. Specifically, there are three pentagonal subdivisions (Figure 7):*

1. *$f = 12$, $a = 0.2322\pi$, $\alpha = \beta = \gamma = \delta = \epsilon = \frac{3}{2}\pi$. Regular dodecahedron, or subdivision of tetrahedron.*

2. *$f = 24$, $a = 0.1745\pi$, $\alpha = 0.8010\pi$, $\beta = 0.5113\pi$, $\gamma = 0.6875\pi$, $\delta = \frac{2}{3}\pi$, $\epsilon = \frac{1}{2}\pi$. Subdivision of (cube, octahedron). See Case 4.2c in Section 5.3.*

3. *$f = 60$, $a = 0.1186\pi$, $\alpha = 0.9059\pi$, $\beta = 0.4093\pi$, $\gamma = 0.6847\pi$, $\delta = \frac{2}{3}\pi$,*

$\epsilon = \frac{2}{5}\pi$, $f = 60$. Subdivision of (dodecahedron, icosahedron). See Case 4.2d in Section 5.3.

There are four earth map tilings (Figure 16):

4. *$f = 16$, $a = 0.2155\pi$, $\alpha = 0.4536\pi$, $\beta = 0.8823\pi$, $\gamma = 0.6639\pi$, $\delta = \frac{1}{2}\pi$, $\epsilon = \frac{3}{4}\pi$. See Case 1.5b in Section 6.5.*

5. *$f = 20$, $a = 0.2168\pi$, $\alpha = 0.3095\pi$, $\beta = 1.0615\pi$, $\gamma = 0.6288\pi$, $\delta = \frac{2}{5}\pi$, $\epsilon = \frac{4}{5}\pi$. See Case 2.6b in Section 6.5.*

6. *$f = 24$, $a = 0.2501\pi$, $\alpha = 0.1440\pi$, $\beta = \frac{4}{3}\pi$, $\gamma = 0.5226\pi$, $\delta = \frac{1}{3}\pi$, $\epsilon = \frac{5}{6}\pi$. See the first solution in Section 6.2.*

7. *$f = 24$, $a = 0.2614\pi$, $\alpha = 0.1192\pi$, $\beta = 1.3807\pi$, $\gamma = \frac{1}{2}\pi$, $\delta = \frac{1}{3}\pi$, $\epsilon = \frac{5}{6}\pi$. See the second solution in Section 6.2.*

And there is one special tiling (Figure 21):

8. *$f = 20$, $a = 0.2168\pi$, $\alpha = 0.3095\pi$, $\beta = 1.0615\pi$, $\gamma = 0.6288\pi$, $\delta = \frac{2}{5}\pi$, $\epsilon = \frac{4}{5}\pi$. See Case 1.4b in Section 6.5.*

Moreover, the fifth and eighth tilings have the same pentagonal tile.

Throughout this paper, decimal values are effective digits. For example, $a = 0.2322\pi$ means $a \in [0.2322\pi, 0.2323\pi]$. We provide enough digits so that the approximate values are enough for rigorous conclusions. Moreover, we will provide the exact values of all data in the first, sixth and seventh tilings. For example, we have $a = \arccos \frac{\sqrt{5}}{3}$ for the regular decahedron and $a = \arccos \sqrt{-3 + 2\sqrt{3}}$ for the seventh tiling.

2 Angle Combinations at Degree 3 Vertices

We study all the possible angle combinations at degree 3 vertices. We assume limited number (up to five) of distinct angles at degree 3 vertices. The only fact used in this section is that the angle sum at each vertex is 2π (called the *angle sum equation at the vertex*). The only criterion we use is that the angle sum equations do not force distinct angles to become the same. So the conclusion actually applies to any edge-to-edge tiling of any surface. As a matter of fact, we do not even need a tiling because we are only concerned with angles at vertices.

We say (an angle combination at) a vertex is of $\alpha\beta\gamma$-*type* if it is $\alpha'\beta'\gamma'$ for some distinct angles α', β', γ'. A degree 3 vertex can also be of $\alpha\beta^2$-type or α^3-type. There are altogether three types of degree 3 vertices.

Case (1)**.** There is only one angle α at degree 3 vertices.

The only possible degree 3 vertex is α^3. We denote this by

$$\text{AVC}_3 = \{\alpha^3\}.$$

Here AVC stands for *anglewise vertex combination*. The subscript 3 indicates the combinations at degree 3 vertices.

Case (2)**.** There are two distinct angles α and β at degree 3 vertices.

The degree 3 vertices must be $\alpha^3, \alpha^2\beta, \alpha\beta^2$ or β^3. If any two appear simultaneously, then the angle sum equations imply $\alpha = \beta$. For example, if both $\alpha^2\beta$ and $\alpha\beta^2$ are vertices, then we have $2\alpha + \beta = 2\pi = \alpha + 2\beta$, which implies $\alpha = \beta$. Therefore only one of the four combinations can appear. In order for both α, β to appear, therefore, either $\alpha^2\beta$ or $\alpha\beta^2$ is the only vertex. Up to the symmetry of exchanging α and β, we get

$$\text{AVC}_3 = \{\alpha\beta^2\}.$$

Case (3)**.** There are three distinct angles α, β and γ at degree 3 vertices.

If $\alpha\beta\gamma$ is a vertex, then all three angles already appear and we get $\{\alpha\beta\gamma\} \subset \text{AVC}_3$. Now the problem is whether the AVC can admit additional degree 3 vertices. The additional vertex cannot be of the $\alpha\beta^2$-type, because the corresponding angle sum equations will force some angles to become equal. On the other hand, it is possible for one of $\alpha^3, \beta^3, \gamma^3$ to appear while still keeping α, β, γ distinct. But simultaneous appearance of two of $\alpha^3, \beta^3, \gamma^3$ will force the corresponding angles to become equal. Up to symmetry, therefore, we get

$$\{\alpha\beta\gamma\} \subset \text{AVC}_3 \subset \{\alpha\beta\gamma, \alpha^3\}.$$

We denote this by writing

$$\text{AVC}_3 = \{\alpha\beta\gamma \mid \alpha^3\}.$$

Here $\alpha\beta\gamma$ is the *necessary* part to make sure all three angles appear at degree 3 vertices, and α^3 is the *optional part* that can be added without causing distinct angles to become equal. The necessary part is the lower bound for the AVC_3 and the necessary plus optional part is the upper bound for the AVC_3.

Next we assume that there are no $\alpha\beta\gamma$-type vertices, and there are $\alpha\beta^2$-type vertices. Up to symmetry, we may assume that $\alpha\beta^2$ is a vertex. Then γ must appear as $\alpha^2\gamma$, $\beta\gamma^2$ or γ^3, without forcing some angles to become equal. Since $\{\alpha\beta^2, \alpha^2\gamma\}$ can be transformed to $\{\alpha\beta^2, \beta\gamma^2\}$ via $\alpha \to \beta \to \gamma \to \alpha$, up to symmetry, we get two possible necessary parts

$$\{\alpha\beta^2, \alpha^2\gamma\}, \quad \{\alpha\beta^2, \gamma^3\}.$$

It can be easily verified that neither allow optional vertices.

Finally we assume that there are no $\alpha\beta\gamma$-type and $\alpha\beta^2$-type vertices. In other words, only α^3, β^3 and γ^3 can appear. There is no way for all three angles to appear in this way without forcing them to become equal. So we get no AVC_3.

Case (4)**.** There are four distinct angles α, β, γ and δ at degree 3 vertices.

If $\alpha\beta\gamma$ is a vertex, then up to symmetry, the angle δ must appear as $\alpha\delta^2$, $\alpha^2\delta$ or δ^3. This gives three possible necessary parts

$$\{\alpha\beta\gamma, \alpha\delta^2\}, \quad \{\alpha\beta\gamma, \alpha^2\delta\}, \quad \{\alpha\beta\gamma, \delta^3\}.$$

The appearance of $\alpha\beta\gamma$ excludes any other optional vertices of $\alpha\beta\gamma$-type. The necessary part $\{\alpha\beta\gamma, \alpha\delta^2\}$ only allows $\beta^2\delta$, $\gamma^2\delta$, β^3, γ^3 to be optional vertices, and the simultaneous appearance of any two from the four forces some angles to become equal. The necessary part $\{\alpha\beta\gamma, \alpha^2\delta\}$ only allows $\beta\delta^2$, $\gamma\delta^2$, β^3, γ^3 to be optional vertices, and the simultaneous appearance of any two forces some angles to become equal. The necessary part $\{\alpha\beta\gamma, \delta^3\}$ does not allow optional vertices. Therefore up to symmetry, we get five possible AVC_3s

$$\{\alpha\beta\gamma, \alpha\delta^2 \mid \beta^2\delta\}, \quad \{\alpha\beta\gamma, \alpha\delta^2 \mid \beta^3\},$$
$$\{\alpha\beta\gamma, \alpha^2\delta \mid \beta\delta^2\}, \quad \{\alpha\beta\gamma, \alpha^2\delta \mid \beta^3\}, \quad \{\alpha\beta\gamma, \delta^3\}.$$

Next we assume that there are no $\alpha\beta\gamma$-type vertices, and $\alpha\beta^2$ is a vertex. Then γ must appear as $\alpha^2\gamma$, $\beta\gamma^2$, $\gamma\delta^2$, $\gamma^2\delta$ or γ^3. Up to symmetry, we may drop $\beta\gamma^2$ and $\gamma^2\delta$. Moreover, for the combinations $\{\alpha\beta^2, \alpha^2\gamma\}$ and $\{\alpha\beta^2, \gamma^3\}$, we need to further consider the way δ appears. Up to symmetry, this leads to six possible necessary parts

$$\{\alpha\beta^2, \alpha^2\gamma, \beta\delta^2\}, \quad \{\alpha\beta^2, \alpha^2\gamma, \gamma^2\delta\}, \quad \{\alpha\beta^2, \alpha^2\gamma, \delta^3\},$$

$$\{\alpha\beta^2, \gamma\delta^2\}, \quad \{\alpha\beta^2, \gamma^3, \alpha^2\delta\}, \quad \{\alpha\beta^2, \gamma^3, \beta\delta^2\}.$$

The first is a special case of the fourth via the transformation $\alpha \to \beta \to \gamma \to \alpha$. The second is also a special case of the fourth. The fifth becomes the third via $\gamma \leftrightarrow \delta$. The sixth becomes the third via $\alpha \to \gamma \to \delta \to \beta \to \alpha$. This reduces the necessary parts under consideration to the third and fourth only.

The third does not allow optional vertices. Under the assumption of no $\alpha\beta\gamma$-type vertices, the fourth only allows $\alpha^2\delta$ and $\beta\gamma^2$ to be optional vertices, and the two optional vertices cannot appear simultaneously. Up to symmetry, we get two possible AVC$_3$s

$$\{\alpha\beta^2, \alpha^2\gamma, \delta^3\}, \quad \{\alpha\beta^2, \gamma\delta^2 \mid \alpha^2\delta\}.$$

Finally, it is easy to see that we cannot have all vertices to be of α^3-type.

Case (5)**.** There are five distinct angles α, β, γ, δ, ϵ at degree 3 vertices.

If all vertices are of α^3-type, then all the angles must be equal. Therefore either there are $\alpha\beta\gamma$-type vertices, or there are $\alpha\beta^2$-type vertices. Moreover, given five distinct angles, there can be at most two $\alpha\beta\gamma$-type vertices. This leads to three subcases.

Case (5.1)**.** There are two $\alpha\beta\gamma$-type vertices.

Up to symmetry, we may assume that $\alpha\beta\gamma$ and $\alpha\delta\epsilon$ are all the $\alpha\beta\gamma$-type vertices. We get one necessary part $\{\alpha\beta\gamma, \alpha\delta\epsilon\}$ and need to find optional vertices, which are of either $\alpha\beta^2$-type or α^3-type. It is easy to see that the only possible optional vertex involving α is α^3.

We look for optional vertices of $\alpha\beta^2$-type. Since such a vertex cannot involve α, up to symmetry (preserving the collection $\{\alpha\beta\gamma, \alpha\delta\epsilon\}$), such an optional vertex is $\beta\delta^2$. Next we ask whether $\{\alpha\beta\gamma, \alpha\delta\epsilon, \beta\delta^2\}$ allows any further optional vertices of $\alpha\beta^2$-type. The answer is $\beta^2\epsilon$, $\gamma\epsilon^2$, $\gamma^2\delta$ or $\gamma^2\epsilon$. Up to symmetry, we get $\{\alpha\beta\gamma, \alpha\delta\epsilon, \beta\delta^2, \beta^2\epsilon\}$, $\{\alpha\beta\gamma, \alpha\delta\epsilon, \beta\delta^2, \gamma\epsilon^2\}$ or $\{\alpha\beta\gamma, \alpha\delta\epsilon, \beta\delta^2, \gamma^2\epsilon\}$. Then we ask again whether any of the three allows further optional vertices. The answer is that only the second one allows α^3, and $\{\alpha\beta\gamma, \alpha\delta\epsilon, \beta\delta^2, \gamma\epsilon^2, \alpha^3\}$ does not allow any more optional vertices. So altogether we get three possible AVC$_3$s

$$\{\alpha\beta\gamma, \alpha\delta\epsilon \mid \beta\delta^2, \beta^2\epsilon\}, \quad \{\alpha\beta\gamma, \alpha\delta\epsilon \mid \beta\delta^2, \gamma\epsilon^2, \alpha^3\}, \quad \{\alpha\beta\gamma, \alpha\delta\epsilon \mid \beta\delta^2, \gamma^2\epsilon\}.$$

Having exhausted optional vertices of $\alpha\beta^2$-type for $\{\alpha\beta\gamma, \alpha\delta\epsilon, \beta\delta^2\}$, we still need to consider optional α^3-type vertices, which can only be α^3, γ^3 or ϵ^3. Since α^3 is already included in the second AVC$_3$ above, we get two more possible AVC$_3$s

$$\{\alpha\beta\gamma, \alpha\delta\epsilon \mid \beta\delta^2, \gamma^3\}, \quad \{\alpha\beta\gamma, \alpha\delta\epsilon \mid \beta\delta^2, \epsilon^3\}.$$

Finally, we need to consider the case there are no $\alpha\beta^2$-type vertices, so the only optional vertices are of α^3-type. Up to symmetry, we get the AVC$_3$s $\{\alpha\beta\gamma, \alpha\delta\epsilon \mid \alpha^3\}$ and $\{\alpha\beta\gamma, \alpha\delta\epsilon \mid \beta^3\}$. The first is included in one of the five AVC$_3$s above, and the second is also included via $\beta \leftrightarrow \gamma$.

Case (5.2)**.** There is only one $\alpha\beta\gamma$-type vertex.

Up to symmetry, we may assume that $\alpha\beta\gamma$ is the only $\alpha\beta\gamma$-type vertex.

Since δ^3 and ϵ^3 cannot appear simultaneously, one of δ and ϵ must appear in an $\alpha\beta^2$-type vertex. Up to symmetry, we may assume that one of $\alpha\delta^2$, $\alpha^2\delta$, $\delta\epsilon^2$ is a vertex. The first two cases really mean that either δ or ϵ is combined with α, β or γ to form a vertex. So in the third case, we may additionally assume that δ and ϵ are never combined with α, β or γ to form a vertex, which means that $\delta\epsilon^2$ is the only degree 3 vertex involving δ and ϵ.

If $\alpha\beta\gamma$ and $\alpha\delta^2$ are vertices, then under the assumption of no more $\alpha\beta\gamma$-type vertices and up to the symmetry of exchanging β and γ, the angle ϵ must appear as $\alpha^2\epsilon$, $\beta\epsilon^2$, $\beta^2\epsilon$, $\delta\epsilon^2$ or ϵ^3. Similarly, if $\alpha\beta\gamma$ and $\alpha^2\delta$ are vertices, then ϵ must appear as $\alpha\epsilon^2$, $\beta\epsilon^2$, $\beta^2\epsilon$, $\delta^2\epsilon$ or ϵ^3. So we get total of ten combinations in which all angles appear. Up to symmetry, the ten combinations are reduced to eight possible necessary parts, which we divide into four groups

$$\begin{aligned}
&\{\alpha\beta\gamma, \alpha\delta^2, \alpha^2\epsilon\}, \\
&\{\alpha\beta\gamma, \alpha\delta^2, \delta\epsilon^2\}, \quad \{\alpha\beta\gamma, \alpha^2\delta, \delta^2\epsilon\}, \\
&\{\alpha\beta\gamma, \alpha\delta^2, \beta\epsilon^2\}, \quad \{\alpha\beta\gamma, \alpha\delta^2, \beta^2\epsilon\}, \quad \{\alpha\beta\gamma, \alpha^2\delta, \beta^2\epsilon\}, \\
&\{\alpha\beta\gamma, \alpha\delta^2, \epsilon^3\}, \quad \{\alpha\beta\gamma, \alpha^2\delta, \epsilon^3\}.
\end{aligned}$$

Under the assumption of no more $\alpha\beta\gamma$-type vertices, the necessary part $\{\alpha\beta\gamma, \alpha\delta^2, \alpha^2\epsilon\}$ only allows $\beta\epsilon^2$, $\beta^2\delta$, $\gamma\epsilon^2$, $\gamma^2\delta$, β^3, γ^3 to be optional vertices. Up to the symmetry $\beta \leftrightarrow \gamma$ of the necessary part, the list may be reduced to $\beta\epsilon^2$, $\beta^2\delta$, β^3. Then we find that the three combinations do not allow any of the original six to be further optional vertices. Hence we get three possible AVC_3s

$$\{\alpha\beta\gamma, \alpha\delta^2, \alpha^2\epsilon \mid \beta\epsilon^2\}, \quad \{\alpha\beta\gamma, \alpha\delta^2, \alpha^2\epsilon \mid \beta^2\delta\}, \quad \{\alpha\beta\gamma, \alpha\delta^2, \alpha^2\epsilon \mid \beta^3\}.$$

The necessary part $\{\alpha\beta\gamma, \alpha\delta^2, \delta\epsilon^2\}$ only allows $\beta^2\epsilon$, $\gamma^2\epsilon$, β^3, γ^3 to be optional vertices. Up to symmetry, the list may be reduced to $\beta^2\epsilon$, β^3. The two combinations do not allow any further optional vertices, and we get two possible AVC_3s

$$\{\alpha\beta\gamma, \alpha\delta^2, \delta\epsilon^2 \mid \beta^2\epsilon\}, \quad \{\alpha\beta\gamma, \alpha\delta^2, \delta\epsilon^2 \mid \beta^3\}.$$

Applying similar argument to $\{\alpha\beta\gamma, \alpha^2\delta, \delta^2\epsilon\}$ gives two more possible AVC_3s

$$\{\alpha\beta\gamma, \alpha^2\delta, \delta^2\epsilon \mid \beta^2\epsilon\}, \quad \{\alpha\beta\gamma, \alpha^2\delta, \delta^2\epsilon \mid \beta^3\}.$$

The necessary part $\{\alpha\beta\gamma, \alpha\delta^2, \beta\epsilon^2\}$ only allows $\alpha^2\epsilon$, $\beta^2\delta$, $\gamma^2\delta$, $\gamma^2\epsilon$, γ^3 to be optional vertices. Up to symmetry, the list may be reduced to $\alpha^2\epsilon$, $\gamma^2\delta$, γ^3. The three combinations do not allow any further optional vertices, and we get three possible AVC_3s

$$\{\alpha\beta\gamma, \alpha\delta^2, \beta\epsilon^2 \mid \alpha^2\epsilon\}, \quad \{\alpha\beta\gamma, \alpha\delta^2, \beta\epsilon^2 \mid \gamma^2\delta\}, \quad \{\alpha\beta\gamma, \alpha\delta^2, \beta\epsilon^2 \mid \gamma^3\}.$$

The necessary part $\{\alpha\beta\gamma, \alpha\delta^2, \beta^2\epsilon\}$ only allows $\gamma\epsilon^2$, $\gamma^2\delta$, γ^3 to be optional vertices. The three combinations do not allow any further optional vertices, and we get three possible AVC_3s

$$\{\alpha\beta\gamma, \alpha\delta^2, \beta^2\epsilon \mid \gamma\epsilon^2\}, \quad \{\alpha\beta\gamma, \alpha\delta^2, \beta^2\epsilon \mid \gamma^2\delta\}, \quad \{\alpha\beta\gamma, \alpha\delta^2, \beta^2\epsilon \mid \gamma^3\}.$$

The necessary part $\{\alpha\beta\gamma, \alpha^2\delta, \beta^2\epsilon\}$ only allows $\alpha\epsilon^2$, $\beta\delta^2$, $\gamma\delta^2$, $\gamma\epsilon^2$, γ^3 to be optional vertices. Up to symmetry, the list may be reduced to $\alpha\epsilon^2$, $\gamma\delta^2$, γ^3. The three combinations do not allow any further optional vertices, and we get three possible AVCs

$$\{\alpha\beta\gamma, \alpha^2\delta, \beta^2\epsilon \mid \alpha\epsilon^2\}, \quad \{\alpha\beta\gamma, \alpha^2\delta, \beta^2\epsilon \mid \gamma\delta^2\}, \quad \{\alpha\beta\gamma, \alpha^2\delta, \beta^2\epsilon \mid \gamma^3\}.$$

The necessary part $\{\alpha\beta\gamma, \alpha\delta^2, \epsilon^3\}$ only allows $\beta^2\delta$, $\gamma^2\delta$ to be optional vertices. Up to symmetry, we only need to consider $\beta^2\delta$. The combination does not allow any further optional vertices, and we get one possible AVC_3

$$\{\alpha\beta\gamma, \alpha\delta^2, \epsilon^3 \mid \beta^2\delta\}.$$

Applying similar argument to $\{\alpha\beta\gamma, \alpha^2\delta, \epsilon^3\}$ gives another possible AVC_3

$$\{\alpha\beta\gamma, \alpha^2\delta, \epsilon^3 \mid \beta\delta^2\}.$$

It remains to consider the case $\delta\epsilon^2$ is the only $\alpha\beta^2$-type vertex involving δ and ϵ. The optional vertices allowed by the necessary part $\{\alpha\beta\gamma, \delta\epsilon^2\}$ can only involve α, β, γ. By the discussion about the AVC_3 of three distinct angles with the necessary part $\{\alpha\beta\gamma\}$ in **Case** (3), we get one possible AVC_3

$$\{\alpha\beta\gamma, \delta\epsilon^2 \mid \alpha^3\}.$$

Case (5.3)**.** There are no $\alpha\beta\gamma$-type vertices.

Since there can be at most one α^3-type vertex, we may assume that $\alpha^3, \beta^3, \gamma^3, \delta^3$ are not vertices. Up to the symmetry of exchanging $\alpha, \beta, \gamma, \delta$ (and under the assumption of no $\alpha\beta\gamma$-type vertices), we may assume that $\alpha\beta^2$ and $\gamma\delta^2$ are vertices. Moreover, up to the symmetry of $\alpha \leftrightarrow \gamma$ and $\beta \leftrightarrow \delta$, we may further assume that ϵ appears as $\alpha^2\epsilon$, $\beta\epsilon^2$ or ϵ^3. This gives three possible necessary parts

$$\{\alpha\beta^2, \gamma\delta^2, \alpha^2\epsilon\}, \quad \{\alpha\beta^2, \gamma\delta^2, \beta\epsilon^2\}, \quad \{\alpha\beta^2, \gamma\delta^2, \epsilon^3\}.$$

Since the second becomes the first via $\alpha \to \epsilon \to \beta \to \alpha$, we only need to consider the first and the third.

Since there are no $\alpha\beta\gamma$-type vertices, the necessary part $\{\alpha\beta^2, \gamma\delta^2, \alpha^2\epsilon\}$ only allows $\beta\gamma^2$, $\delta\epsilon^2$ to be optional vertices, and the two cannot appear simultaneously. Therefore we get two possible AVC_3s

$$\{\alpha\beta^2, \gamma\delta^2, \alpha^2\epsilon \mid \beta\gamma^2\}, \quad \{\alpha\beta^2, \gamma\delta^2, \alpha^2\epsilon \mid \delta\epsilon^2\},$$

The necessary part $\{\alpha\beta^2, \gamma\delta^2, \epsilon^3\}$ only allows $\alpha^2\delta$, $\beta\gamma^2$ to be optional vertices, and the two cannot appear simultaneously. Up to symmetry, we get one possible AVC_3

$$\{\alpha\beta^2, \gamma\delta^2, \epsilon^3 \mid \alpha^2\delta\}.$$

The following is the summary of our discussion.

<table>
<tr><th colspan="3">Necessary</th><th>Optional</th></tr>
<tr><td>α^3</td><td colspan="2"></td><td></td></tr>
<tr><td>$\alpha\beta^2$</td><td colspan="2"></td><td></td></tr>
<tr><td colspan="2">$\alpha\beta\gamma$</td><td rowspan="3"></td><td>α^3</td></tr>
<tr><td rowspan="2">$\alpha\beta^2$</td><td>$\alpha^2\gamma$</td><td rowspan="2"></td></tr>
<tr><td>γ^3</td></tr>
<tr><td colspan="2" rowspan="5">$\alpha\beta\gamma$</td><td rowspan="2">$\alpha\delta^2$</td><td>$\beta^2\delta$</td></tr>
<tr><td>β^3</td></tr>
<tr><td rowspan="2">$\alpha^2\delta$</td><td>$\beta\delta^2$</td></tr>
<tr><td>β^3</td></tr>
<tr><td>δ^3</td><td></td></tr>
<tr><td rowspan="2">$\alpha\beta^2$</td><td colspan="2">$\gamma\delta^2$</td><td>$\alpha^2\delta$</td></tr>
<tr><td colspan="2">$\alpha^2\gamma, \delta^3$</td><td></td></tr>
<tr><td rowspan="5">$\alpha\beta\gamma$</td><td colspan="2" rowspan="5">$\alpha\delta\epsilon$</td><td>$\beta\delta^2, \beta^2\epsilon$</td></tr>
<tr><td>$\beta\delta^2, \gamma\epsilon^2, \alpha^3$</td></tr>
<tr><td>$\beta\delta^2, \gamma^2\epsilon$</td></tr>
<tr><td>$\beta\delta^2, \gamma^3$</td></tr>
<tr><td>$\beta\delta^2, \epsilon^3$</td></tr>
</table>

<table>
<tr><th colspan="3">Necessary</th><th>Optional</th></tr>
<tr><td rowspan="19">$\alpha\beta\gamma$</td><td rowspan="12">$\alpha\delta^2$</td><td rowspan="3">$\alpha^2\epsilon$</td><td>$\beta\epsilon^2$</td></tr>
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<tr><td>β^3</td></tr>
<tr><td rowspan="3">$\beta\epsilon^2$</td><td>$\alpha^2\epsilon$</td></tr>
<tr><td>$\gamma^2\delta$</td></tr>
<tr><td>γ^3</td></tr>
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<tr><td rowspan="6">$\alpha^2\delta$</td><td rowspan="3">$\beta^2\epsilon$</td><td>$\alpha\epsilon^2$</td></tr>
<tr><td>$\gamma\delta^2$</td></tr>
<tr><td>γ^3</td></tr>
<tr><td rowspan="2">$\delta^2\epsilon$</td><td>$\beta^2\epsilon$</td></tr>
<tr><td>β^3</td></tr>
<tr><td>ϵ^3</td><td>$\beta\delta^2$</td></tr>
<tr><td colspan="2">$\delta\epsilon^2$</td><td>α^3</td></tr>
<tr><td colspan="2" rowspan="3">$\alpha\beta^2, \gamma\delta^2$</td><td rowspan="2">$\alpha^2\epsilon$</td><td>$\beta\gamma^2$</td></tr>
<tr><td>$\delta\epsilon^2$</td></tr>
<tr><td>ϵ^3</td><td>$\alpha^2\delta$</td></tr>
</table>

Table 1: Anglewise vertex combinations at degree 3 vertices.

Lemma 1. *If an edge-to-edge tiling of a surface has at most five distinct angles at degree* 3 *vertices, then after suitable relabeling of the distinct angles, the anglewise vertex combination at degree* 3 *vertices is in Table 1.*

We note that the argument leading to the table follows a specific sequence of cases, and the discussion of a case often assumes the exclusion of the earlier cases. Following a different sequence of cases may lead to a different table, and not excluding earlier cases may introduce many overlapings between various cases.

3 Angle Combinations in the Pentagon

Consider an edge-to-edge tiling of the sphere by pentagons, such that all vertices have degree ≥ 3. Let v, e, f be the numbers of vertices, edges and tiles. Let v_k be the number of degree k vertices. Then we have

$$v-e+f=2, \quad v=v_3+v_4+v_5+\cdots, \quad 5f=2e=3v_3+4v_4+5v_5+\cdots. \tag{3.1}$$

The equalities easily implies (see [4, page 750], for example) the following *vertex counting equation* (the two are equivalent by the third equality in (3.1))

$$v_3=20+\sum_{k\geq 4}(3k-10)v_k, \quad \frac{f}{2}-6=\sum_{k\geq 4}(k-3)v_k. \tag{3.2}$$

The first equality shows that most vertices have degree 3. We call vertices of degree >3 *high degree vertices*. The second equality implies that f is even and $f\geq 12$. Since spherical pentagonal tilings are completely understood for $f=12$ by [1, 4], and we cannot have $f=14$ by [7], we will assume that f is even and $f\geq 16$ throughout this paper.

Now we further assume that all pentagonal tiles have the same angle combination. For example, by the combination $\alpha^2\beta^2\gamma$, we mean that α, β, γ are distinct angles, and the angles in each tile are two α, two β and one γ. This implies that the total number of times α, β, γ appear in the tiling are respectively $2f, 2f, f$.

Lemma 2. *In a spherical tiling by f pentagons with the same angle combination, the sum of five angles in the pentagon is $3\pi+\frac{4\pi}{f}$.*

The equality in the lemma is the *angle sum equation for the pentagon.* If all tiles are (geometrically) congruent, then the area of each tile is $\frac{4\pi}{f}$, and the equality follows from the fact that the area of a spherical pentagon (with great arc edges) is the sum of five angles minus 3π. So the lemma says that the angle sum equation still holds under much weaker assumption.

We also note that, if all angles in the tiling are the same, then the existence of degree 3 vertices imply that the angle must be $\frac{2\pi}{3}$. So the lemma implies $f=12$. Since we are only concerned with $f\geq 16$, the case is dismissed.

Proof. Since the sum of angles at each vertex is 2π, the total sum of all angles is $2\pi v$. Since all tiles have the same angle combination, the sum Σ of five angles is the same for all the tiles, and the total sum of all angles is also $f\Sigma$.

Therefore we have $2\pi v = f\Sigma$. It is also easy to derive $3f = 2v - 4$ from (3.1). Then we get

$$\Sigma = 2\pi \frac{v}{f} = 3\pi + \frac{4\pi}{f}. \qquad \square$$

Next we study how the appearance of angles at degree 3 vertices affect their appearance in the pentagonal tile.

Lemma 3. *If an angle appears at every degree* 3 *vertex in a spherical tiling by pentagons with the same angle combination, then the angle appears at least twice in the pentagon.*

Proof. If an angle θ appears only once in the pentagon, then the total number of times θ appears in the whole tiling is f, and the total number of non-θ angles is $4f$. If we also know that θ appears at every degree 3 vertex, then $f \geq v_3$, and non-θ angles appear $\leq 2v_3$ times at degree 3 vertices. Moreover, non-θ angles appear $\leq \sum_{k\geq 4} kv_k$ times at high degree vertices. Therefore

$$4v_3 \leq 4f \leq 2v_3 + \sum_{k\geq 4} kv_k.$$

This implies

$$v_3 \leq \sum_{k\geq 4} \frac{1}{2} kv_k.$$

Since $k \geq 4$ implies $\frac{1}{2}k \leq 3k - 10$, we get a contradiction to the first equality in (3.2). $\square$

Lemma 4. *If an angle appears at least twice at every degree* 3 *vertex in a spherical tiling by pentagons with the same angle combination, then the angle appears at least three times in the pentagon.*

Proof. If an angle θ appear twice in the pentagon, then the total number of times θ appears in the whole tiling is $2f$, and the total number of non-θ angles is $3f$. If we also know that θ appears at least twice at every degree 3 vertex, then $2f \geq 2v_3$, and non-θ angles appear $\leq v_3$ times at degree 3 vertices. Moreover, non-θ angles appear $\leq \sum_{k\geq 4} kv_k$ times at high degree vertices. Therefore

$$3v_3 \leq 3f \leq v_3 + \sum_{k\geq 4} kv_k.$$

This leads to the same contradiction as in the proof of Lemma 3.

If an angle appears at least twice at every degree 3 vertex, then by Lemma 3, it cannot appear only once in the pentagon. By the proof above, it cannot appear only twice in the pentagon. So it appears at least three times in the pentagon. □

The proof of Lemma 4 can be easily modified to get the following.

Lemma 5. *If two angles together appear at least twice at every degree* 3 *vertex in a spherical tiling by pentagons with the same angle combination, then the two angles together appear at least three times in the pentagon.*

In contrast, the following is about angles not appearing at degree 3 vertices.

Lemma 6. *Suppose an angle* θ *does not appear at degree* 3 *vertices in a spherical tiling by pentagons with the same angle combination.*

1. *There can be at most one such angle* θ.
2. *The angle* θ *appears only once in the pentagon.*
3. $2v_4 + v_5 \geq 12$.
4. *One of* $\alpha\theta^3$, θ^4, θ^5 *is a vertex, where* $\alpha \neq \theta$.

The first statement implies that the angle α in the fourth statement must appear at a degree 3 vertex.

Proof. Suppose two angles θ_1 and θ_2 do not appear at degree 3 vertices. Then the total number of times these two angles appear is at least $2f$, and is at most the total number $\sum_{k\geq 4} kv_k$ of angles at high degree vertices. Therefore we have $2f \leq \sum_{k\geq 4} kv_k$. On the other hand, by the vertex counting equation (3.2), we have

$$2f - \sum_{k\geq 4} kv_k = 24 + \sum_{k\geq 4} 3(k-4)v_k > 0.$$

The contradiction proves the first statement.

The argument above also applies to the case $\theta_1 = \theta_2$, which means the same angle appearing at least twice in the pentagon. This proves the second statement.

The first two statements imply that θ appears exactly f times. Since this should be no more than the total number $\sum_{k\geq 4} kv_k$ of angles at high degree vertices, by the vertex counting equation (3.2), we get

$$f - \sum_{k\geq 4} kv_k = 12 - 2v_4 - v_5 + \sum_{k\geq 6}(k-6)v_k \leq 0.$$

This implies the third statement.

For the last statement, we assume that $\alpha\theta^3, \theta^4, \theta^5$ are not vertices. This means that θ appears at most twice at any degree 4 vertex, and at most four times at any degree 5 vertex. Since θ also does not appear at degree 3 vertices, the total number of times θ appears is $\leq 2v_4 + 4v_5 + \sum_{k\geq 6} kv_k$. However, the number of times θ appears should also be f. Then we get the following contradiction

$$f = 12 + 2\sum_{k\geq 4}(k-3)v_k \leq 2v_4 + 4v_5 + \sum_{k\geq 6} kv_k. \qquad \square$$

The constraints obtained so far may be used to eliminate certain angle combinations. For example, if a spherical tiling by pentagons with the same angle combination has $\text{AVC}_3 = \{\alpha\beta^2\}$, then by Lemmas 3 and 4, the pentagon must have the angle combination $\alpha^2\beta^3$.

So far we have not used any edge information. In fact, the results so far do not even need to assume the edges are great arcs. Next we use the stronger assumption that all the tiles are congruent equilateral pentagons (with great arc edges). Recall the following result from [3, Lemma 7].

Lemma 7. *Suppose two pairs of edges in the spherical pentagon in Figure 1 are equal. Then $\beta > \epsilon$ if and only if $\gamma < \delta$.*

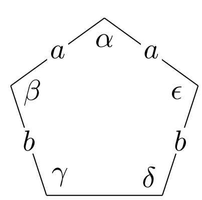

Figure 1: Spherical pentagon with two pairs of equal edges.

For an equilateral pentagon, the lemma easily implies that the number of distinct angles in the pentagon must be odd. In particular, this eliminates the possibility that $\mathrm{AVC}_3 = \{\alpha\beta^2\}$. We have also dismissed the case that all angles in the pentagon are the same. By Lemma 1, therefore, we only need to consider the following cases:

- Three distinct angles, all appearing at degree 3 vertices.
- Five distinct angles, four appearing at degree 3 vertices.
- Five distinct angles, all appearing at degree 3 vertices.

4 Three Distinct Angles

Consider the spherical equilateral pentagon in Figure 2, with edge length a and five (not necessarily distinct) angles $\alpha, \beta, \gamma, \delta, \epsilon$. By [1] and [3, Section 3], we may calculate the great arc x connecting β and ϵ vertices, from the triangle above x as well as the quadrilateral below x

$$\begin{aligned}\cos x &= \cos^2 a + \sin\alpha \sin^2 a,\\ \cos x &= (1-\cos\gamma)(1-\cos\delta)\cos^3 a - \sin\gamma\sin\delta\cos^2 a\\ &\quad + (\cos\gamma + \cos\delta - \cos\gamma\cos\delta)\cos a + \sin\gamma\sin\delta.\end{aligned}$$

Equating the two formulae for $\cos x$ and dividing $1-\cos a$, we get a quadratic equation for $\cos a$

$$L\cos^2 a + M\cos a + N = 0,$$

where the coefficients depend only on α, γ, δ,

$$\begin{aligned}L &= (1-\cos\gamma)(1-\cos\delta),\\ M &= \cos\alpha + \cos(\gamma+\delta) - \cos\gamma - \cos\delta,\\ N &= \cos\alpha - \sin\gamma\sin\delta.\end{aligned}$$

Let x_i be the arcs connecting respectively (β, ϵ), (α, γ), (β, δ), (γ, ϵ), (α, δ). By the two ways of calculating each x_i, we get five quadratic equations

$$L_i\cos^2 a + M_i\cos a + N_i = 0, \quad i = 1, 2, 3, 4, 5$$

that share a common root $\cos a$. The sharing of a root among two quadratic equations can be detected by (and is equivalent to) the vanishing of the resultant

$$R_{ij} = (L_iN_j - L_jN_i)^2 - (L_iM_j - L_jM_i)(M_iN_j - M_jN_i).$$

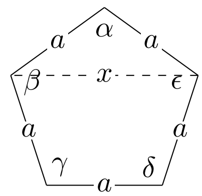

Figure 2: Spherical equilateral pentagon.

The sharing of a root among five quadratic equations can be detected by (but may not be equivalent to) the vanishing of four resultants relating all five equations together.

As pointed out in the introduction, the equilateral pentagon has three degrees of freedom. If we have three independent relations among the five angles, then we may express five angles in terms of two, and the resultants become functions of two free variables. We may find the pentagon by looking for the common zero of four resultant functions of two variables.

Consider the simplest case that the tiling has three distinct angles. Then the angle combinations at degree 3 vertices are given by the three angle part of Table 1. Moreover, by Lemma 7, we know the angle combination in the pentagon must be $\alpha^2\beta^2\gamma$, $\alpha^2\beta\gamma^2$, or $\alpha\beta^2\gamma^2$.

For $\text{AVC}_3 = \{\alpha\beta\gamma \mid \alpha^3\}$, by Lemma 3, α appears at least twice in the pentagon. Up to the symmetry of exchanging β and γ, therefore, we may assume that the pentagon has the angle combination $\alpha^2\beta^2\gamma$. Then $\alpha\beta\gamma$ cannot be the only degree 3 vertex, because otherwise γ appears at every degree 3 vertex, and by Lemma 3 must appear twice in the pentagon. We denote the case by $\{\alpha^2\beta^2\gamma\colon \alpha\beta\gamma, \alpha^3\}$, where α^3 becomes necessary because it must appear.

For $\text{AVC}_3 = \{\alpha\beta^2, \alpha^2\gamma\}$, by Lemma 3, α again appears at least twice in the pentagon. Therefore the pentagon has the angle combination $\alpha^2\beta^2\gamma$ or $\alpha^2\beta\gamma^2$. If we exchange β and γ in the second case, then we get two possible cases $\{\alpha^2\beta^2\gamma\colon \alpha\beta^2, \gamma^3\}$ and $\{\alpha^2\beta^2\gamma\colon \alpha\gamma^2, \beta^3\}$.

For $\text{AVC}_3 = \{\alpha\beta^2, \gamma^3\}$, the pentagon cannot have the angle combination $\alpha^2\beta^2\gamma$ because otherwise the angle sums at the two degree 3 vertices would imply that the sum of five angles in the pentagon (see Lemma 2) is

$$\alpha + 2\beta + 2\gamma = (\alpha + 2\beta) + \frac{2}{3}3\gamma = 2\pi + \frac{4\pi}{3},$$

which further implies that $f = 12$. We conclude that the pentagon must have either $\alpha^2\beta^2\gamma$ or $\alpha^2\beta\gamma^2$ as the angle combination. If we exchange β and γ in the second case, then we get two possible cases $\{\alpha^2\beta^2\gamma\colon \alpha\beta^2, \gamma^3\}$ and $\{\alpha^2\beta^2\gamma\colon \alpha\gamma^2, \beta^3\}$.

In summary, we get the following complete list of all cases with three distinct angles.

3.1 $\{\alpha^2\beta^2\gamma\colon \alpha\beta\gamma, \alpha^3\}$.

3.2a $\{\alpha^2\beta^2\gamma\colon \alpha\beta^2, \alpha^2\gamma\}$.

3.2b $\{\alpha^2\beta^2\gamma\colon \alpha^2\beta, \alpha\gamma^2\}$.

3.3a $\{\alpha^2\beta^2\gamma\colon \alpha\beta^2, \gamma^3\}$.

3.3b $\{\alpha^2\beta^2\gamma\colon \alpha\gamma^2, \beta^3\}$.

Note that for the angle combination $\alpha^2\beta^2\gamma$, there are two possible angle arrangements in the pentagon (see Figure 3)

$$[\alpha, \alpha, \beta, \gamma, \beta],\ [\alpha, \beta, \beta, \alpha, \gamma].$$

Taking into account of the two arrangements, we get total of 10 cases.

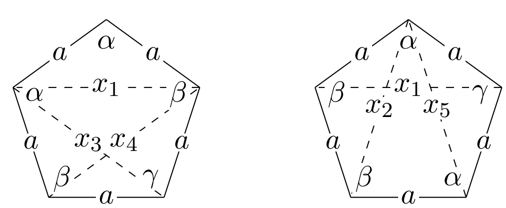

Figure 3: Two arrangements and three quadratic equations for $\cos a$.

Next we use spherical trigonometry to show that none of the three angle cases lead to tilings. Note that by the symmetry of the pentagon, we have $x_1 = x_2$, $x_3 = x_5$ for the first arrangement in Figure 3, and $x_1 = x_4$, $x_2 = x_3$ for the second arrangement. Hence we only have three quadratic equations for $\cos a$. Moreover, the two vertices in AVC_3 enable us to express all three angles in terms of one angle. Therefore we only look for the common zero of two resultant functions of a single angle variable.

For example, in Case 3.1, the angle sum equations $\alpha + \beta + \gamma = 3\alpha = 2\pi$ imply that

$$\alpha = \frac{2}{3}\pi,\ \gamma = \frac{4}{3}\pi - \beta,\ 2\alpha + 2\beta + \gamma - 3\pi = \beta - \frac{1}{3}\pi = \frac{4}{f}\pi.$$

The condition $f \geq 16$ means $\frac{1}{3}\pi < \beta \leq \frac{7}{12}\pi$. For the first arrangement, Figure **??** gives the graph of the resultants R_{13} (in red) and R_{14} (in blue) on the interval $[0.3\pi, 0.6\pi]$ containing $[\frac{1}{3}\pi, \frac{7}{12}\pi]$. In Figure **??**, we find that the common zero of the two resultants is approximately $\beta = \frac{1}{3}\pi$. The exact value can be further confirmed by symbolic calculation. Since this implies $f = \infty$, the solution is dismissed.

?????? more words about approximate value

In Figure **??**, we omit π in the coordinates values. So 0.6 for β really means $\beta = 0.6\pi$. We will adopt the same convention in Figures **??** and **??**.

We carry out the similar calculation for all the three angle cases and two arrangements for each case. We find no pentagon suitable for tiling.

5 Four Distinct Angles at Degree 3 Vertices

By the remark after Lemma 7, after the discussion about three distinct angles in the pentagon in Section 4, we may assume that the pentagon has five distinct angles $\alpha, \beta, \gamma, \delta, \epsilon$. The remark also splits the consideration into two cases. The first is that only four angles appear at degree 3 vertices. The second is that all five angles appear at degree 3 vertices.

This section discusses the first case, which corresponds to the four angle part of Table 1, with ϵ appearing only at high degree vertices. Up to the symmetry of flipping, there are generally twelve ways of arranging the angles in the pentagon

$$[\alpha,\beta,\gamma,\delta,\epsilon], [\alpha,\beta,\gamma,\epsilon,\delta], [\alpha,\beta,\delta,\gamma,\epsilon], [\alpha,\beta,\delta,\epsilon,\gamma],$$
$$[\alpha,\beta,\epsilon,\gamma,\delta], [\alpha,\beta,\epsilon,\delta,\gamma], [\alpha,\gamma,\beta,\delta,\epsilon], [\alpha,\gamma,\beta,\epsilon,\delta],$$
$$[\alpha,\gamma,\delta,\beta,\epsilon], [\alpha,\gamma,\epsilon,\beta,\delta], [\alpha,\delta,\beta,\gamma,\epsilon], [\alpha,\delta,\gamma,\beta,\epsilon].$$

Of course, further symmetries in some AVC_3s may reduce the number of arrangements we need to consider.

5.1 Calculation of Pentagon

For $\text{AVC}_3 = \{\alpha\beta\gamma, \alpha\delta^2 \mid \beta^2\delta \text{ or } \beta^3\}$, since α appears only once in the pentagon, by Lemma 3, $\alpha\beta\gamma$ and $\alpha\delta^2$ cannot be the only degree 3 vertices. Therefore one of the two optional vertices necessarily appear, and we get two AVC_3s

$$\{\alpha\beta\gamma, \alpha\delta^2, \beta^2\delta\}, \quad \{\alpha\beta\gamma, \alpha\delta^2, \beta^3\}.$$

Similar argument for $\text{AVC}_3 = \{\alpha\beta\gamma, \alpha^2\delta \mid \cdots\}$ also gives two AVC_3s

$$\{\alpha\beta\gamma, \alpha^2\delta, \beta\delta^2\}, \quad \{\alpha\beta\gamma, \alpha^2\delta, \beta^3\}.$$

Since $\alpha \leftrightarrow \beta$ exchanges $\{\alpha\beta\gamma, \alpha\delta^2, \beta^2\delta\}$ and $\{\alpha\beta\gamma, \alpha^2\delta, \beta\delta^2\}$, the four AVC_3s may be reduced to three.

For $\text{AVC}_3 = \{\alpha\beta^2, \gamma\delta^2 \mid \alpha^2\delta\}$, we may apply Lemma 5 to conclude that, if $\alpha^2\delta$ does not appear, then β and δ together appear at least three times in the pentagon, contradicting to five distinct angles in the pentagon. Therefore $\alpha^2\delta$ necessarily appear and we get $\text{AVC}_3 = \{\alpha\beta^2, \gamma\delta^2, \alpha^2\delta\}$.

For $\text{AVC}_3 = \{\alpha\beta\gamma, \delta^3\}$, we use Lemma 6 to conclude that one of $\alpha\epsilon^3$, $\beta\epsilon^3$, $\gamma\epsilon^3$, $\delta\epsilon^3$, ϵ^4, ϵ^5 must be a vertex. Up to the symmetry of exchanging α, β, γ, we may assume that one of $\alpha\epsilon^3$, $\delta\epsilon^3$, ϵ^4, ϵ^5 is a vertex.

Finally, $\text{AVC}_3 = \{\alpha\beta^2, \alpha^2\gamma, \delta^3\}$ already has three vertices.

In summary, we get the following complete list of all cases with four distinct angles at degree 3 vertices.

4.1a $\{\alpha\beta\gamma\delta\epsilon\colon \alpha\beta\gamma, \alpha\delta^2, \beta^2\delta\}$, 12 arrangements.

4.1b $\{\alpha\beta\gamma\delta\epsilon\colon \alpha\beta\gamma, \alpha\delta^2, \beta^3\}$, 12 arrangements.

4.1c $\{\alpha\beta\gamma\delta\epsilon\colon \alpha\beta\gamma, \alpha^2\delta, \beta^3\}$, 12 arrangements.

4.2a $\{\alpha\beta\gamma\delta\epsilon\colon \alpha\beta\gamma, \delta^3, \alpha\epsilon^3\}$, 6 arrangements by β, γ exchange.

4.2b $\{\alpha\beta\gamma\delta\epsilon\colon \alpha\beta\gamma, \delta^3, \delta\epsilon^3\}$, 2 arrangements by α, β, γ exchange.

4.2c $\{\alpha\beta\gamma\delta\epsilon\colon \alpha\beta\gamma, \delta^3, \epsilon^4\}$, 2 arrangements by α, β, γ exchange.

4.2d $\{\alpha\beta\gamma\delta\epsilon\colon \alpha\beta\gamma, \delta^3, \epsilon^5\}$, 2 arrangements by α, β, γ exchange.

4.3 $\{\alpha\beta\gamma\delta\epsilon\colon \alpha\beta^2, \gamma\delta^2, \alpha^2\delta\}$, 12 arrangements.

4.4 $\{\alpha\beta\gamma\delta\epsilon\colon \alpha\beta^2, \alpha^2\gamma, \delta^3\}$, 12 arrangements.

The reductions of arrangements by further symmetries are also indicated. All 12 arrangements need to be considered if there is not further indication. Taking into account of the various arrangements, we get total of 72 cases.

Next we calculate the pentagon for each case. We illustrate the process by the example of the first arrangement of Case 4.2a

$$\{[\alpha, \beta, \gamma, \delta, \epsilon]\colon \alpha\beta\gamma, \delta^3, \alpha\epsilon^3\}.$$

From the angle sum equations $\alpha+\beta+\gamma = 3\delta = \alpha+3\epsilon = 2\pi$ at the three vertices, we can express all five angles in terms of two angles

$$\gamma = 2\pi - \alpha - \beta,\ \delta = \frac{2}{3}\pi,\ \epsilon = \frac{2}{3}\pi - \frac{1}{3}\alpha.$$

We determine the *domain* for the two angles α, β by the following conditions

1. All angles are positive.

2. Lemma 7 is satisfied in five ways.

3. The number of tiles f as calculated from the angle sum equation for the pentagon (see Lemma 2) satisfies $16 \le f < \infty$.

The first two conditions gives the union of three polygonal regions, listed below in terms of their vertices and appearing in Figure 4 with solid boundaries

$$\begin{aligned}
&\Omega_1\colon (0,2\pi), (\tfrac{1}{2}\pi, \pi), (\tfrac{1}{2}\pi, \tfrac{3}{2}\pi);\\
&\Omega_2\colon (\tfrac{2}{3}\pi, \tfrac{2}{3}\pi), (\pi, \tfrac{1}{3}\pi), (2\pi, 0);\\
&\Omega_3\colon (\tfrac{4}{5}\pi, \tfrac{2}{5}\pi), (\pi, 0), (\tfrac{4}{3}\pi, 0), (\pi, \tfrac{1}{3}\pi).
\end{aligned}$$

The third condition means

$$0 < \alpha+\beta+\gamma+\delta+\epsilon-3\pi = \frac{1}{3}(\pi-\alpha) = \frac{4}{f}\pi \le \frac{1}{4}\pi,$$

which is the strip between parallel dashed lines $f = 16$ and $f = \infty$ in Figure 4. The domain is then the intersection of $\Omega_1 \cup \Omega_2 \cup \Omega_3$ and the strip.

The resultants are functions of five angles. Substituting the formulae for γ, δ, ϵ, the resultants become functions of α, β. The zeros of resultants are plotted (using MAPLE) in Figure **??**. Within the domain, there are two intersections (or solutions) of the zeros of all ten resultants. To see the solutions more clearly, we zoom in, and get the two intersections on the right of Figure **??**.

The first solution is approximately $(\alpha,\beta) = (0.2584\pi, 1.5603\pi)$. Substituting the values into the angle sum equation for the pentagon in Lemma 2, we find $f = 16.18$. Since this is not an even number ≥ 16, the solution can be dismissed for the purpose of spherical pentagonal tiling.

The second solution appears to be $(\alpha,\beta) = (\pi, \frac{1}{3}\pi)$. We may use symbolic calculation to confirm that the exact value is indeed a common zero of all

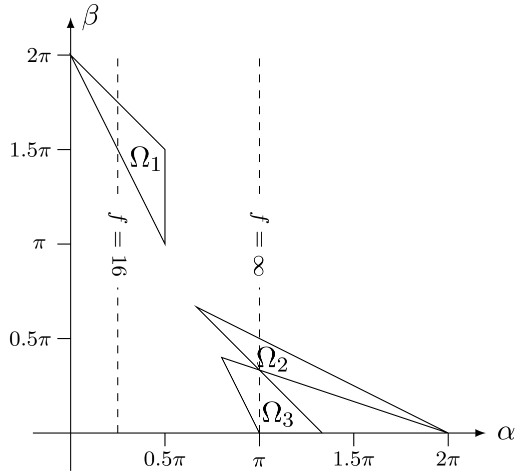

Figure 4: Domain for Case 4.2a, first arrangement.

ten resultants. Since this solution gives $f = \infty$, the solution can also be dismissed.

We need to be concerned with the reliability of numerical calculation. We give here the outline of how to achieve the reliability. The full detail and the calculation for all the cases are given in [2].

We may improve the visualisation of solutions by plotting the regions for

$$\max R = \max_{1\le i<j\le 5} |R_{ij}| > \epsilon.$$

Figure **??** gives the regions for the choices $\epsilon = 0.5$ and 0.05. The red region is $\max R > 0.5$, and the red plus the blue region is $\max R > 0.05$. The (boundary of) polygonal regions $\Omega_1, \Omega_2, \Omega_3$ are indicated in green, and the lines corresponding to $f = 16$ and ∞ are indicated in yellow. Moreover, the lines corresponding to some other values of f are also indicated.

The picture suggests that we should have $\max R > 0.5$ in the red region. Specifically, we can set up some grid and then numerically verify $\max R > 0.5$ at all the grid points in the red part of the domain. The step size σ of the grid can be determined as follows. Since the resultants are polynomials of $\cos\alpha, \sin\alpha, \cos\beta, \sin\beta$, we can easily estimate the bound M for the partial derivatives of all the resultants on the domain. If we know $\max R > 0.5$ at some grid point, then we know $\max R \ge 0.5 - \sigma M$ on the square of side length σ centred at the grid point. This means that, if we choose the step size $\sigma < \frac{0.5}{M}$, then the verification of $\max R > 0.5$ at all the grid points implies $\max R > 0$ in the red part of the domain. This rigorously shows that the red part of the domain does not contain any solution.

It turns out that the numerical calculation with $\max R > 0.5$ is not enough. So we carry out the similar numerical calculation with $\max R > 0.05$ for the blue part of the domain. This means verifying $\max R > 0.05$ at all the grid points in a grid in the blue part with step size $< \frac{0.05}{M}$. Then we are sure that the solutions lie in the two white regions in the middle and right of Figure **??**. Since the first white region (in the middle of Figure **??**) lies in $16 < f < 18$, there is no valid solutions in the first region. However, the same reason cannot be applied to the second white region (on the right of Figure **??**), because of the possibility of another solution very close to the (symbolically verified) special solution $(\alpha, \beta) = (\pi, \frac{1}{3}\pi)$ with f being a huge even number. Instead, we notice the non-singular Jacobian at the special solution

$$\left.\frac{\partial(R_{12}, R_{45})}{\partial(\alpha, \beta)}\right|_{(\pi, \frac{1}{3}\pi)} = \frac{\sqrt{3}}{4} \begin{pmatrix} 2 & -6 \\ -1 & -1 \end{pmatrix}.$$

For a suitable small radius r, we may estimate the bound of the second order derivatives of R_{12} and R_{45} on the small square of side length $2r$ centred at the special solution. The estimation on the second order derivatives and the invertibility of the Jacobian enables us to find a small r, such that the special solution is the only place in the small square where the map (R_{12}, R_{45}) vanishes. For our case, we find that $r = 0.01$ is enough.

The argument for the first permutation of Case 4.2a is typical. In [2], we also show further tricks to reduce the amount of calculation. For example, on the left of Figure **??**, for the part of the domain in the first region Ω_1, we only need to verify $\max R > 0.5$ at those grid points along the five yellow lines corresponding to even f between 16 and 24. We also note the special Cases 4.2b, 4.2c, 4.2d, where f is fixed at $36, 24, 60$. In fact, we get valid solutions only in the three special cases.

5.2 Case 4.2b

Up to the permutation of α, β, γ, we only need to consider the first and third arrangements. We have

$$\gamma = 2\pi - \alpha - \beta,\ \delta = \frac{2}{3}\pi,\ \epsilon = \frac{4}{9}\pi,\ f = 36.$$

By the calculation in [2], there are two solutions for the first arrangement

$$(\alpha, \beta) = (0.29539\pi, 1.62453\pi),\ (0.8757\pi, 0.4299\pi),$$

and one solution for the third arrangement

$$(\alpha, \beta) = (0.855\pi, 0.455\pi).$$

The values are approximate, and are accurate up to the last digit. For example, the first solution is accuracy up to $\pm 0.00001\pi$. In fact, we show in [2] that the first solution lies in the box $[0.29539\pi, 0.29540\pi] \times [1.62453\pi, 1.62454\pi]$.

For the solution we have $\gamma = 2\pi - \alpha - \beta = 0.08008\pi$ accurate up to $\pm 0.00002\pi$ (i.e., $\gamma \in [0.08006\pi, 0.08010\pi]$). Then we try to find all the possible angle combinations $\alpha^i\beta^j\gamma^k\delta^l\epsilon^m$ at vertices by solving

$$\alpha i + \beta j + \gamma k + \delta l + \epsilon m = 2\pi.$$

Since we only have approximate values for α, β, γ, we cannot solve the exact equation. Still, since all five terms on the left are positive, the approximate values of the five angles imply

$$i \le 6,\ j \le 1,\ k \le 25,\ l \le 3,\ m \le 4.$$

Therefore any solution to the exact equation also satisfies

$$\left| 0.29539i + 1.62453j + 0.08008k + \frac{2}{3}l + \frac{4}{9}m - 2 \right| < 0.0006,$$

where the right side comes from

$$(6+1) \cdot 0.00001 + 25 \cdot 0.00002 < 0.0006.$$

We substitute all combinations of indices i, j, k, l, m within the bounds to the inequality above and find exactly three combinations $\alpha\beta\gamma$, δ^3, $\delta\epsilon^3$ satisfying the inequality.

Applying the same argument to the other two solutions, we also get exactly three combinations $\alpha\beta\gamma$, δ^3, $\delta\epsilon^3$. We need 5 digit approximation for the first solution because γ is very small, which means k can be as big as 25. The bounds for the second and third solutions are much smaller, and 4 digit and 3 digit approximations are sufficient.

The problem is now reduced to finding the tiling for

$$\text{AVC} = \{\alpha\beta\gamma\delta\epsilon\colon \alpha\beta\gamma, \delta^3, \delta\epsilon^3\},$$

the *full anglewise vertex combination* consisting of all the angle combinations at all vertices. We will denote by P_i the i-th tile, and indicate the tile as circled i. We also denote by θ_i the angle θ in P_i, and denote by $V_{\theta,i}$ the vertex where the angle θ_i is located. The notations are unambiguous because the five angles in the pentagon are distinct.

Consider four tiles P_1, P_2, P_3, P_4 around a vertex $\delta\epsilon^3$. For the first arrangement, up to symmetry, we may assume that the angles of P_1 are arranged as on the left of Figure 5. By the AVC, we know $V_{\delta,1} = \delta^3$ or $\delta\epsilon^3$. Since P_2 has only one ϵ, we get $V_{\delta,1} = \delta^3$ and a tile P_5 outside P_1, P_2. We also know the location of δ_5. The locations of δ_2, ϵ_2 determine all the angles of P_2. Then one of the question marks is the angle ϵ_5 adjacent to δ_5. This implies the existence of a vertex $\gamma\epsilon\cdots$, contradicting to the AVC.

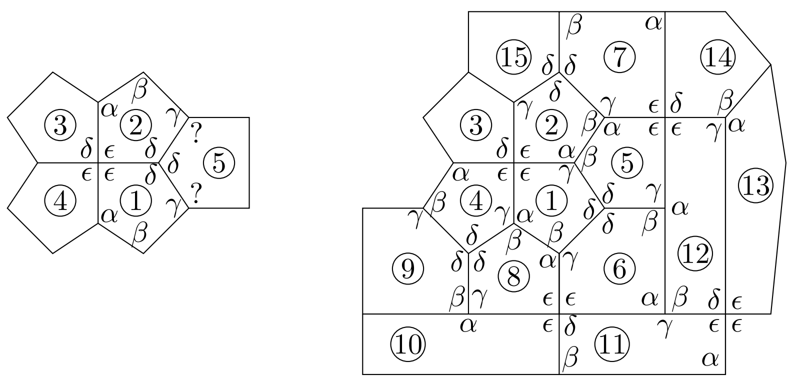

Figure 5: Tiling for $\{\alpha\beta\gamma\delta\epsilon\colon \alpha\beta\gamma, \delta^3, \delta\epsilon^3\}$.

For the third arrangement, up to symmetry, we may assume that the angle of P_1 are arranged as on the right of Figure 5. By the AVC (we will henceforth omit mentioning AVC), we have $V_{\gamma,1} = \alpha\beta\gamma$. Since β_2, ϵ_2 are not adjacent, we get (a tile) P_5 outside P_1, P_2 and (the locations of) α_2, β_5. Then α_2, ϵ_2 determine (all the angles of) P_2. By $V_{\delta,1} = \delta^3$ or $\delta\epsilon^3$ and the non-adjacency of β_5, ϵ_5, we have $V_{\delta,2} = \delta^3$, and get P_6, δ_5, δ_6. Then β_5, δ_5 determine P_5. By $V_{\beta,2} = V_{\alpha,5} = \alpha\beta\gamma$, we get P_7, γ_7. By $V_{\beta,1} = \alpha\beta\gamma$ and the fact that δ_6 is adjacent only to β_6, γ_6, we get γ_6, P_8, α_8. Then γ_6, δ_6 determine P_6. By $V_{\alpha,1} = \alpha\beta\gamma$ and the non-adjacency of α_8, γ_8, we get β_8, γ_4.

Then γ_4, ϵ_4 determine P_4 and α_8, β_8 determine P_8. By $V_{\delta,4} = V_{\delta,8} = \delta^3$, we get P_9, δ_9. By $V_{\gamma,8} = \alpha\beta\gamma$ and the fact that δ_9 is adjacent only to β_9, γ_9, we get $\beta_9, P_{10}, \alpha_{10}$. By $V_{\epsilon,6} = V_{\epsilon,8} = \delta\epsilon^3$ and the non-adjacency of α_{10}, δ_{10}, we get $\epsilon_{10}, P_{11}, \delta_{11}$. By $V_{\gamma,5} = V_{\beta,6} = \alpha\beta\gamma$, we get P_{12}, α_{12}. By $V_{\alpha,6} = \alpha\beta\gamma$ and the non-adjacency of α_{12}, γ_{12}, we get β_{12}, γ_{11}. Then γ_{11}, δ_{11} determine P_{11} and α_{12}, β_{12} determine P_{12}. By $V_{\epsilon,11} = V_{\delta,12} = \delta\epsilon^3$, we get P_{13}, ϵ_{13}. By $V_{\gamma,12} = \alpha\beta\gamma$ and the non-adjacency of $\beta_{13}, \epsilon_{13}$, we get $\alpha_{13}, P_{14}, \beta_{14}$. By $V_{\epsilon,5} = V_{\epsilon,12} = \delta\epsilon^3$ and the non-adjacency of $\beta_{14}, \epsilon_{14}$, we get δ_{14}, ϵ_7. Then γ_7, ϵ_7 determine P_7. By $V_{\delta,2} = V_{\delta,7} = \delta^3$, we get P_{15}, δ_{15}. Since $V_{\gamma,2} = \alpha\beta\gamma$ has degree 3, P_{15} is also glued to P_3 as indicated. Now the angle α at $V_{\gamma,2} = \alpha\beta\gamma$ must be either α_3 or α_{15}, so that α and δ must be adjacent in either P_3 or P_{15}, a contradiction.

We conclude that Case 4.2b does not admit tiling.

5.3 Cases 4.2c and 4.2d

Again we only need to consider the first and third arrangements. For Case 4.2c, we have

$$\gamma = 2\pi - \alpha - \beta,\ \delta = \frac{2}{3}\pi,\ \epsilon = \frac{1}{2}\pi,\ f = 24,$$

and by the calculation in [2], three solutions

$$(\alpha, \beta) = (0.27849\pi, 1.59984\pi),\ (0.820\pi, 0.484\pi),\ (0.801\pi, 0.511\pi).$$

For Case 4.2d, we have

$$\gamma = 2\pi - \alpha - \beta,\ \delta = \frac{2}{3}\pi,\ \epsilon = \frac{2}{5}\pi,\ f = 60,$$

and also three solutions

$$(\alpha, \beta) = (0.31031\pi, 1.64260\pi),\ (0.9229\pi, 0.3890\pi),\ (0.9059\pi, 0.4093\pi).$$

In both cases, the first and second solutions are for the first arrangement $[\alpha, \beta, \gamma, \delta, \epsilon]$, and the third solution is for the third arrangement $[\alpha, \beta, \delta, \gamma, \epsilon]$.

For each of the six solutions, we may calculate the full AVC similar to Case 4.2b. We get the full AVC $\{\alpha\beta\gamma\delta\epsilon\colon \alpha\beta\gamma, \delta^3, \epsilon^4\}$ for Case 4.2c, and the full AVC $\{\alpha\beta\gamma\delta\epsilon\colon \alpha\beta\gamma, \delta^3, \epsilon^5\}$ for Case 4.2d.

For Case 4.2c, we consider four tiles P_1, P_2, P_3, P_4 around a vertex ϵ^4. For the first arrangement depicted on the left of Figure 6, the same argument for

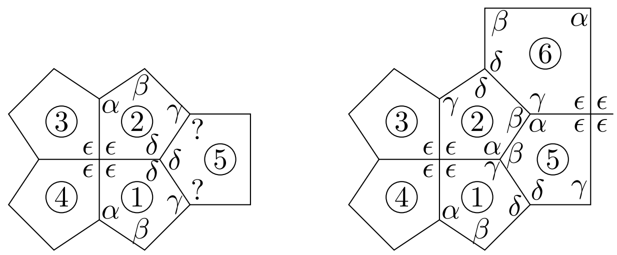

Figure 6: Tiling for $\{\alpha\beta\gamma\delta\epsilon\colon \alpha\beta\gamma, \delta^3, \epsilon^4\}$.

the first arrangement of the Case 4.2b leads to the same contradiction that $\gamma\epsilon\cdots$ is a vertex.

For the third arrangement, up to symmetry, we may assume that the angles of P_1 are arranged as on the right of Figure 6. By $V_{\gamma,1} = \alpha\beta\gamma$ and the non-adjacency of β_2, ϵ_2, we get α_2, P_5, β_5. Then α_2, ϵ_2 determine P_2. By $V_{\delta,1} = \delta^3$, we get δ_5. Then β_5, δ_5 determine P_5. By $V_{\beta,2} = V_{\alpha,5} = \alpha\beta\gamma$, we get P_6, γ_6. By $V_{\epsilon,5} = \epsilon^4$, we get ϵ_6. Then γ_6, ϵ_6 determine P_6.

Note that the starting point for deriving the right of Figure 6 is the angle arrangement of a tile P_1 at an ϵ^4 vertex. Now we know the angle arrangements of P_2 and P_6 at their respective ϵ^4 vertices. So we can repeat the same argument by starting with P_2 and P_6 in place of P_1 and get more tiles and their angle arrangements. More repetitions of the same argument give the whole tiling. The tiling consists of $f = 24$ tiles and is the left of Figure 7, with dot vertices indicating δ^3 and circle vertices indicating ϵ^4.

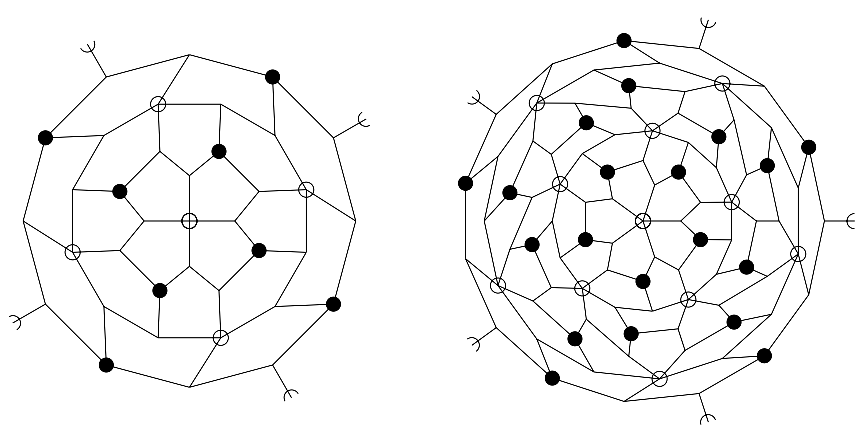

Figure 7: Tilings for Cases 4.2c and 4.2d.

The argument for Case 4.2d is completely similar. The key point is that the previous argument only concerns two adjacent tiles P_1, P_2 around a degree 4 vertex ϵ^4. Such argument is clearly still valid for two adjacent tiles around a degree 5 vertex ϵ^5 in the AVC for Case 4.2d. The only difference is more complicated combinatorial structure of the tiling we get at the end. The tiling consists of $f = 60$ tiles and is the right of Figure 7, with dot vertices indicating δ^3 and circle vertices indicating ϵ^5.

The two tilings in Figure 7 are the *pentagonal subdivisions* of platonic solids. For any tiling of the sphere (or any compact oriented surface), we add two vertices to each edge and add one vertex at the center of each tile. For each tile, the orientation of the surface can be used to label the two new vertices on each boundary edge as the first and second. Then we connect the center vertex to the second new vertex of each boundary edge. The subdivision divides an n-gon tile into n pentagons. See Figure 8.

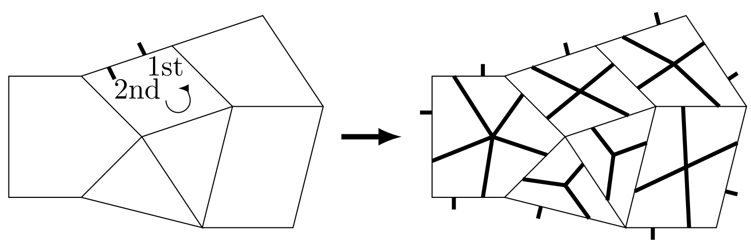

Figure 8: Pentagonal subdivision.

If we start with a regular tiling, then by triple dividing each edge in such a way that the first and third sub-edges have equal length a (see the left of Figure 9), we get a tiling by geometrically congruent pentagons (see the middle of Figure 9). Moreover, by only requiring $\alpha + \beta + \gamma = 2\pi$ instead of $\alpha + \gamma = \beta = 2\pi$ (i.e., no longer keeping the original edges straight), we get a tiling by geometrically congruent pentagons given on the right of Figure 9. In the formulae for δ and ϵ, the original regular tiling is made up of n-gons and each vertex has degree m. By the same argument as in [4], the pentagonal tile allows free continuous choice of two variables, which can be two angles from α, β, γ (a, b, c are determined by these two angles).

The pentagonal subdivision of the regular tetrahedron is the deformed dodecahedron tiling in [1, 4]. The pentagonal subdivision of the regular cube on the left of Figure 10 is a tiling of the sphere by 24 geometrically congruent pentagons. The pentagonal subdivision of the regular octahedron

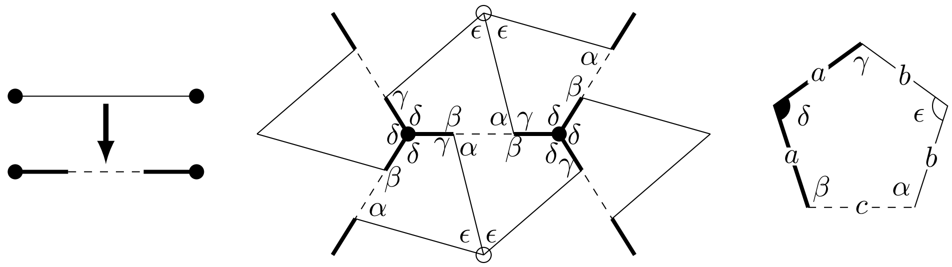

Figure 9: Tiling by congruent pentagons, $\alpha+\beta+\gamma=2\pi$, $\delta=\frac{2}{m}\pi$, $\epsilon=\frac{2}{n}\pi$.

on the right of Figure 10 is the same tiling, with the duality between the cube and the octahedron exchanging a and b and the corresponding angles. Similarly, the pentagonal subdivisions of the regular dodecahedron and the regular icosahedron are the same tiling of the sphere by 60 geometrically congruent pentagons. Hence we get three families of tilings of the sphere by $f=12,24,60$ congruent pentagons allowing free continuous choice of two variables. The two tilings in Figure 7 are the special equilateral cases.

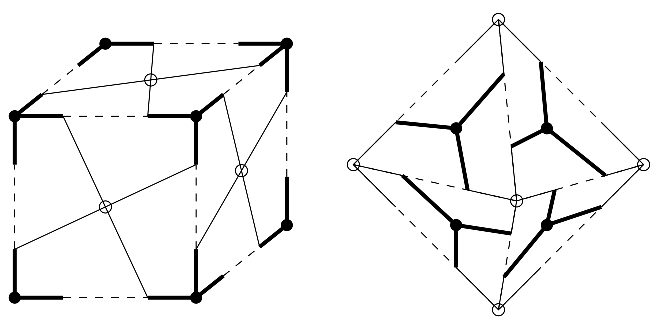

Figure 10: Pentagonal subdivisions of cube and octahedron.

It remains to verify that the third solutions (the only solutions for the third arrangement) of Cases 4.2c and 4.2d can be realised by actual spherical pentagons. By solving the quadratic equations of $\cos a$, we can find the approximate values of edge length a. For example, we find the approximate value $a=0.17\pi$ for the third solution of Case 4.2c. This means that we expect the exact value $a\in[a_-,a_+]=[0.17\pi,0.18\pi]$. We will see that $[a_-,a_+]\subset(0,\frac{1}{2}\pi)$ for two solutions here as well as all the later solutions.

Figure 11 shows the possible pentagons in the third arrangement, with A, B, C, D, E being respectively the vertices where the angles α, β, γ, δ, ϵ

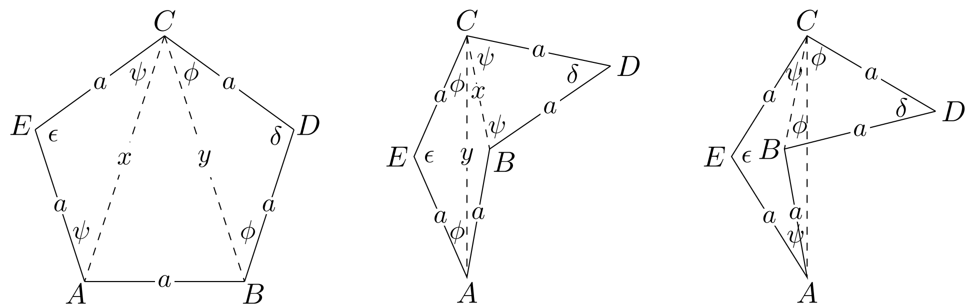

Figure 11: Possible shapes of the pentagon in third arrangement.

are located. We consider the pentagon as obtained by glueing the isosceles triangles $\triangle ACE$, $\triangle BCD$ and the middle triangle $\triangle ABC$ together. This is indeed the case for the left and middle situations, where $\triangle ABC$ lies inside the pentagon. Our subsequent discussion will also be based on this assumption. In the right situation, $\triangle ABC$ is not inside the pentagon, and we will explain why this and the similar situation do not happen for our solutions.

The known precise values of δ and ϵ determine $\triangle ACE$ and $\triangle BCD$ as functions of a. In particular,

$$\begin{aligned}
x &= BC = \arccos(\cos^2 a + \sin^2 a \cos\delta), \\
y &= AC = \arccos(\cos^2 a + \sin^2 a \cos\epsilon), \\
\phi &= \angle CBD = \arctan\left(\sec a \cot\frac{\delta}{2}\right), \\
\psi &= \angle CAE = \arctan\left(\sec a \cot\frac{\epsilon}{2}\right).
\end{aligned}$$

For the range $[a_-, a_+]$ of a, we find the corresponding ranges $[x_-, x_+]$, $[y_-, y_+]$, $[\phi_-, \phi_+]$, $[\psi_-, \psi_+]$ for x, y, ϕ, ψ. Then we verify that the ranges satisfy

$$a + x + y < 2\pi, \quad a < x + y, \quad x < a + y, \quad y < a + x.$$

This implies the existence of $\triangle ABC$ and therefore the existence of the pentagon with the given precise values of δ, ϵ and a range $[a_-, a_+]$ for a.

We further verify that the ranges of ϕ and ψ and the initial approximate values of α, β satisfy

$$\alpha > \psi, \quad \beta > \phi, \quad \gamma = 2\pi - \alpha - \beta > \phi + \psi.$$

This implies that the shape of the pentagon must be the left and middle of Figure 11, so that

$$\alpha = \psi + \angle CAB, \quad \beta = \phi + \angle CBA, \quad \gamma = \phi + \psi + \angle ACB.$$

To verify the original definition of Cases 4.2c and 4.2d, it remains to show that the equality $\alpha + \beta + \gamma = 2\pi$ can be achieved for some $a \in [a_-, a_+]$. For this purpose, we have the angles of $\triangle ABC$

$$\angle CAB = \arccos\left(\frac{\cos x - \cos a \cos y}{\sin a \sin y}\right),$$
$$\angle CBA = \arccos\left(\frac{\cos y - \cos a \cos x}{\sin a \sin x}\right),$$
$$\angle ACB = \arccos\left(\frac{\cos a - \cos x \cos y}{\sin x \sin y}\right).$$

Using the expressions of x, y, ϕ, ψ as functions of a, the following can be expressed as a function of a

$$f(a) = \alpha + \beta + \gamma - 2\pi = \angle CAB + \angle CBA + \angle ACB + 2\phi + 2\psi - 2\pi.$$

By showing that $f(a_-)$ and $f(a_+)$ have opposite signs and then applying the intermediate value theorem, we conclude that $\alpha+\beta+\gamma = 2\pi$ can be achieved.

For the third solution of Case 4.2c, we have approximately $a = 0.17\pi$. By

$$f(0.17\pi) = -0.028, \quad f(0.18\pi) = 0.035,$$

and the intermediate value theorem, there is $a \in [0.17\pi, 0.18\pi]$ satisfying $f(a) = 0$. We may further apply the intermediate value theorem to $f(a) = 0$ to get more and more digits for a. In fact, we get

$$a = 0.1745273185424745966984738102 6\pi$$

because for this value of a, we have

$$f(a) = -3.8 \times 10^{-29} < 0, \quad f(a + 10^{-29}\pi) = 2.6 \times 10^{-29} > 0.$$

Using this value of a, we get

$$\alpha = 0.80106832905992046260731242296 9\pi,$$
$$\beta = 0.51139177170631338496460382209\pi,$$
$$\gamma = 0.68753989923376615242808375493\pi,$$

and

$$\phi = 0.189\pi, \quad \psi = 0.275\pi, \quad x = 0.298\pi, \quad y = 0.240\pi.$$

We verify that the inequalities between a, x, y and between $\alpha, \beta, \phi, \psi$ are satisfied, so that the pentagon indeed exists and is shaped like the left of Figure 11. The numerical data for the pentagon is depicted on the left of Figure 12.

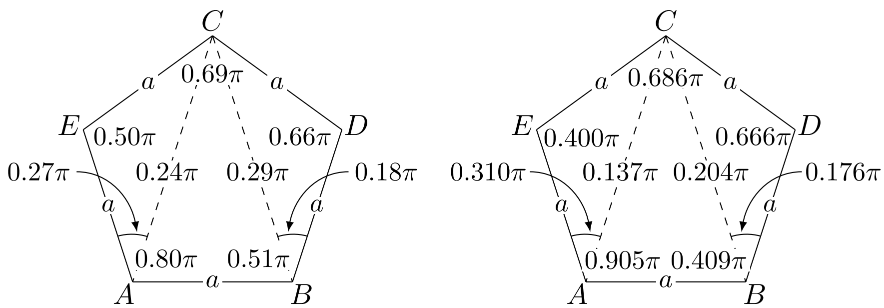

Figure 12: Pentagon for $\{[\alpha, \beta, \delta, \gamma, \epsilon] \colon \alpha\beta\gamma, \delta^3, \epsilon^4 \text{ or } \epsilon^5\}$.

Similarly, for the third solution of Case 4.2d, we have

$$\begin{aligned}
a &= 0.1186473348655018935829311189\,86\pi, \\
\alpha &= 0.905942593574543832769182439026\pi, \\
\beta &= 0.409303454898146180685546402290\pi, \\
\gamma &= 0.684753951527309986545271158\,68\pi,
\end{aligned}$$

and

$$\phi = 0.176\pi, \quad \psi = 0.310\pi, \quad x = 0.204\pi, \quad y = 0.137\pi.$$

This also implies the existence of the pentagon, which is also shaped like the left of Figure 11, with the numerical data depicted on the right of Figure 12.

6 Five Distinct Angles at Degree 3 Vertices

The remark after Lemma 7 splits the search for equilateral pentagonal tiling into three cases. We studied the first and second cases in Sections 4 and 5. In this section, we study the third case, that the pentagon has five distinct angles $\alpha, \beta, \gamma, \delta, \epsilon$, and all five angles appear at degree 3 vertices. This means that we need to consider the five angle part of Table 1.

6.1 Calculation of Pentagon

By Lemma 3, no angle can appear at all the degree 3 vertices. For the first $\text{AVC}_3 = \{\alpha\beta\gamma, \alpha\delta\epsilon \mid \cdots\}$ in the five angle part of Table 1, therefore, some optional vertex not involving α must appear. Up to the symmetry of symbols, we may assume that either $\gamma\epsilon^2$ or γ^3 appears. This gives $\text{AVC}_3 = \{\alpha\beta\gamma, \alpha\delta\epsilon, \gamma\epsilon^2 \mid \cdots\}$ or $\text{AVC}_3 = \{\alpha\beta\gamma, \alpha\delta\epsilon, \gamma^3 \mid \cdots\}$, both have three degree 3 vertices in the necessary part.

For the other AVC_3s in the five angle part of Table 1, we always get three necessary vertices with the only exception of $\text{AVC}_3 = \{\alpha\beta\gamma, \delta\epsilon^2 \mid \alpha^3\}$. If we include the case that α^3 also appears for this particular AVC_3, then we get the following complete list of all cases with five distinct angles at degree 3 vertices and with three degree 3 vertices.

5.1a $\{\alpha\beta\gamma\delta\epsilon\colon \alpha\beta\gamma, \alpha\delta\epsilon, \gamma\epsilon^2\}$, 12 arrangements.

5.1b $\{\alpha\beta\gamma\delta\epsilon\colon \alpha\beta\gamma, \alpha\delta\epsilon, \gamma^3\}$, 6 arrangements by δ, ϵ exchange.

5.2 $\{\alpha\beta\gamma\delta\epsilon\colon \alpha\beta\gamma, \alpha\delta^2, \alpha^2\epsilon\}$, 6 arrangements by β, γ exchange.

5.3 $\{\alpha\beta\gamma\delta\epsilon\colon \alpha\beta\gamma, \alpha\delta^2, \beta\epsilon^2\}$, 8 arrangements by $(\alpha, \delta), (\beta, \epsilon)$ exchange.

5.4 $\{\alpha\beta\gamma\delta\epsilon\colon \alpha\beta\gamma, \alpha\delta^2, \beta^2\epsilon\}$, 12 arrangements.

5.5 $\{\alpha\beta\gamma\delta\epsilon\colon \alpha\beta\gamma, \alpha\delta^2, \delta\epsilon^2\}$, 6 arrangements by β, γ exchange.

5.6 $\{\alpha\beta\gamma\delta\epsilon\colon \alpha\beta\gamma, \alpha\delta^2, \epsilon^3\}$, 6 arrangements by β, γ exchange.

5.7 $\{\alpha\beta\gamma\delta\epsilon\colon \alpha\beta\gamma, \alpha^2\delta, \beta^2\epsilon\}$, 8 arrangements by $(\alpha, \delta), (\beta, \epsilon)$ exchange.

5.8 $\{\alpha\beta\gamma\delta\epsilon\colon \alpha\beta\gamma, \alpha^2\delta, \delta^2\epsilon\}$, 12 arrangements.

5.9 $\{\alpha\beta\gamma\delta\epsilon\colon \alpha\beta\gamma, \alpha^2\delta, \epsilon^3\}$, 6 arrangements by β, γ exchange.

5.10 $\{\alpha\beta\gamma\delta\epsilon\colon \alpha\beta\gamma, \delta\epsilon^2, \alpha^3\}$, 6 arrangements by β, γ exchange.

5.11 $\{\alpha\beta\gamma\delta\epsilon\colon \alpha\beta^2, \gamma\delta^2, \alpha^2\epsilon\}$, 12 arrangements.

5.12 $\{\alpha\beta\gamma\delta\epsilon\colon \alpha\beta^2, \gamma\delta^2, \epsilon^3\}$, 8 arrangements by $(\alpha, \beta), (\gamma, \delta)$ exchange.

Taking into account of various arrangements, we get total of 108 cases.

After the list above, it only remains to consider $\text{AVC}_3 = \{\alpha\beta\gamma, \delta\epsilon^2\}$, which means that $\alpha\beta\gamma$ and $\delta\epsilon^2$ are the only degree 3 vertices. If there are degree 4 vertices, then we consider all the possible combinations at a degree 4 vertex and get the following complete list. Here the angle combinations at the degree 4 vertex are ordered by the types (to borrow a terminology from degree 3 vertices) $\alpha\beta\gamma\delta$, $\alpha\beta\gamma^2$, $\alpha^2\beta^2$, $\alpha\beta^3$, α^4. We also note that $\{\alpha\beta\gamma, \delta\epsilon^2, \epsilon^4\}$ is dismissed because it implies $f = 8$.

1.1 $\{\alpha\beta\gamma\delta\epsilon\colon \alpha\beta\gamma, \delta\epsilon^2, \alpha\beta\delta\epsilon\}$, 6 arrangements by α, β exchange.

1.2a $\{\alpha\beta\gamma\delta\epsilon\colon \alpha\beta\gamma, \delta\epsilon^2, \alpha\beta^2\delta\}$, 12 arrangements.

1.2b $\{\alpha\beta\gamma\delta\epsilon\colon \alpha\beta\gamma, \delta\epsilon^2, \alpha\beta^2\epsilon\}$, 12 arrangements.

1.2c $\{\alpha\beta\gamma\delta\epsilon\colon \alpha\beta\gamma, \delta\epsilon^2, \alpha\beta\delta^2\}$, 6 arrangements by α, β exchange.

1.2d $\{\alpha\beta\gamma\delta\epsilon\colon \alpha\beta\gamma, \delta\epsilon^2, \alpha\beta\epsilon^2\}$, 6 arrangements by α, β exchange.

1.2e $\{\alpha\beta\gamma\delta\epsilon\colon \alpha\beta\gamma, \delta\epsilon^2, \alpha\delta^2\epsilon\}$, 6 arrangements by β, γ exchange.

1.3a $\{\alpha\beta\gamma\delta\epsilon\colon \alpha\beta\gamma, \delta\epsilon^2, \alpha^2\beta^2\}$, 6 arrangements by α, β exchange.

1.3b $\{\alpha\beta\gamma\delta\epsilon\colon \alpha\beta\gamma, \delta\epsilon^2, \alpha^2\delta^2\}$, 6 arrangements by β, γ exchange.

1.3c $\{\alpha\beta\gamma\delta\epsilon\colon \alpha\beta\gamma, \delta\epsilon^2, \alpha^2\epsilon^2\}$, 6 arrangements by β, γ exchange.

1.4a $\{\alpha\beta\gamma\delta\epsilon\colon \alpha\beta\gamma, \delta\epsilon^2, \alpha\beta^3\}$, 12 arrangements.

1.4b $\{\alpha\beta\gamma\delta\epsilon\colon \alpha\beta\gamma, \delta\epsilon^2, \alpha\delta^3\}$, 6 arrangements by β, γ exchange.

1.4c $\{\alpha\beta\gamma\delta\epsilon\colon \alpha\beta\gamma, \delta\epsilon^2, \alpha\epsilon^3\}$, 6 arrangements by β, γ exchange.

1.4d $\{\alpha\beta\gamma\delta\epsilon\colon \alpha\beta\gamma, \delta\epsilon^2, \alpha^3\delta\}$, 6 arrangements by β, γ exchange.

1.4e $\{\alpha\beta\gamma\delta\epsilon\colon \alpha\beta\gamma, \delta\epsilon^2, \alpha^3\epsilon\}$, 6 arrangements by β, γ exchange.

1.4f $\{\alpha\beta\gamma\delta\epsilon\colon \alpha\beta\gamma, \delta\epsilon^2, \delta^3\epsilon\}$, 2 arrangements by α, β, γ exchange.

1.5a $\{\alpha\beta\gamma\delta\epsilon\colon \alpha\beta\gamma, \delta\epsilon^2, \alpha^4\}$, 6 arrangements by β, γ exchange.

1.5b $\{\alpha\beta\gamma\delta\epsilon\colon \alpha\beta\gamma, \delta\epsilon^2, \delta^4\}$, 2 arrangements by α, β, γ exchange.

Taking into account of various arrangements, we get total of 112 cases.

Similarly, if there are degree 5 vertices, then we consider all the possible combinations at a degree 5 vertex and get the following complete list. Here the angle combinations at the degree 5 vertex are ordered by the types $\alpha\beta\gamma\delta^2$, $\alpha\beta^2\gamma^2$, $\alpha\beta^3$, $\alpha\beta^4$, α^5. We also note that $\{\alpha\beta\gamma, \delta\epsilon^2, \epsilon^5\}$ is dismissed because it implies $f = \frac{20}{3}$.

2.1a $\{\alpha\beta\gamma\delta\epsilon\colon \alpha\beta\gamma, \delta\epsilon^2, \alpha\beta^2\delta\epsilon\}$, 12 arrangements.

2.1b $\{\alpha\beta\gamma\delta\epsilon\colon \alpha\beta\gamma, \delta\epsilon^2, \alpha\beta\delta^2\epsilon\}$, 6 arrangements by α, β exchange.

2.2a $\{\alpha\beta\gamma\delta\epsilon\colon \alpha\beta\gamma, \delta\epsilon^2, \alpha^2\beta^2\delta\}$, 6 arrangements by α, β exchange.

2.2b $\{\alpha\beta\gamma\delta\epsilon\colon \alpha\beta\gamma, \delta\epsilon^2, \alpha^2\beta^2\epsilon\}$, 6 arrangements by α, β exchange.

2.2c $\{\alpha\beta\gamma\delta\epsilon\colon \alpha\beta\gamma, \delta\epsilon^2, \alpha^2\delta^2\epsilon\}$, 6 arrangements by β, γ exchange.

2.2d $\{\alpha\beta\gamma\delta\epsilon\colon \alpha\beta\gamma, \delta\epsilon^2, \alpha\beta^2\delta^2\}$, 12 arrangements.

2.2e $\{\alpha\beta\gamma\delta\epsilon\colon \alpha\beta\gamma, \delta\epsilon^2, \alpha\beta^2\epsilon^2\}$, 12 arrangements.

2.3a $\{\alpha\beta\gamma\delta\epsilon\colon \alpha\beta\gamma, \delta\epsilon^2, \alpha^3\delta\epsilon\}$, 6 arrangements by β, γ exchange.

2.3b $\{\alpha\beta\gamma\delta\epsilon\colon \alpha\beta\gamma, \delta\epsilon^2, \alpha\beta^3\delta\}$, 12 arrangements.

2.3c $\{\alpha\beta\gamma\delta\epsilon\colon \alpha\beta\gamma, \delta\epsilon^2, \alpha\beta^3\epsilon\}$, 12 arrangements.

2.3d $\{\alpha\beta\gamma\delta\epsilon\colon \alpha\beta\gamma, \delta\epsilon^2, \alpha\delta^3\epsilon\}$, 6 arrangements by β, γ exchange.

2.4a $\{\alpha\beta\gamma\delta\epsilon\colon \alpha\beta\gamma, \delta\epsilon^2, \alpha^2\beta^3\}$, 12 arrangements.

2.4b $\{\alpha\beta\gamma\delta\epsilon\colon \alpha\beta\gamma, \delta\epsilon^2, \alpha^2\delta^3\}$, 6 arrangements by β, γ exchange.

2.4c $\{\alpha\beta\gamma\delta\epsilon\colon \alpha\beta\gamma, \delta\epsilon^2, \alpha^2\epsilon^3\}$, 6 arrangements by β, γ exchange.

2.4d $\{\alpha\beta\gamma\delta\epsilon\colon \alpha\beta\gamma, \delta\epsilon^2, \alpha^3\delta^2\}$, 6 arrangements by β, γ exchange.

2.4e $\{\alpha\beta\gamma\delta\epsilon\colon \alpha\beta\gamma, \delta\epsilon^2, \alpha^3\epsilon^2\}$, 6 arrangements by β, γ exchange.

2.5a $\{\alpha\beta\gamma\delta\epsilon\colon \alpha\beta\gamma, \delta\epsilon^2, \alpha\beta^4\}$, 12 arrangements.

2.5b $\{\alpha\beta\gamma\delta\epsilon\colon \alpha\beta\gamma, \delta\epsilon^2, \alpha\delta^4\}$, 6 arrangements by β, γ exchange.

2.5c $\{\alpha\beta\gamma\delta\epsilon\colon \alpha\beta\gamma, \delta\epsilon^2, \alpha\epsilon^4\}$, 6 arrangements by β, γ exchange.

2.5d $\{\alpha\beta\gamma\delta\epsilon\colon \alpha\beta\gamma, \delta\epsilon^2, \alpha^4\delta\}$, 6 arrangements by β, γ exchange.

2.5e $\{\alpha\beta\gamma\delta\epsilon\colon \alpha\beta\gamma, \delta\epsilon^2, \alpha^4\epsilon\}$, 6 arrangements by β, γ exchange.

2.5f $\{\alpha\beta\gamma\delta\epsilon\colon \alpha\beta\gamma, \delta\epsilon^2, \delta^4\epsilon\}$, 2 arrangements by α, β, γ exchange.

2.6a $\{\alpha\beta\gamma\delta\epsilon\colon \alpha\beta\gamma, \delta\epsilon^2, \alpha^5\}$, 6 arrangements by β, γ exchange.

2.6b $\{\alpha\beta\gamma\delta\epsilon\colon \alpha\beta\gamma, \delta\epsilon^2, \delta^5\}$, 2 arrangements by α, β, γ exchange.

Taking into account of various arrangements, we get total of 178 cases.

Finally, we need to consider the only remaining exceptional case that $\alpha\beta\gamma$ and $\delta\epsilon^2$ are the only degree 3 vertices, and there are no vertices of degree 4 or 5. In Section 6.2, we will show that the only possibility is that δ^6 is a vertex, and δ, ϵ are not adjacent in the pentagon.

For each case, we carry out the calculation similar to what is outlined for Case 4.2a in Section 5.1. See [2] for the details. We find solutions only in Cases 1.2e, 1.4f, 1.5a, 1.5b, 2.4b, 2.5f, 2.6b, 5.5, and the exceptional case.

6.2 Exceptional Case $\{\alpha\beta\gamma\delta\epsilon\colon \alpha\beta\gamma, \delta\epsilon^2\}$, $v_4 = v_5 = 0$

We first study the exceptional case. By [3, Proposition 1], any pentagonal spherical tiling must have a tile with four vertices having degree 3, and the fifth vertex having degree 3, 4, or 5. By $v_4 = v_5 = 0$, we have a tile with all vertices having degree 3. We call such a tile 3^5-*tile*.

The neighborhood of a 3^5-tile is given in Figure 13. We use the notations P_i, θ_i, $V_{\theta,i}$ introduced in Section 5.2. We will also denote by $A_{i,jk}$ the angle of P_i at the vertex shared by P_i, P_j, P_k.

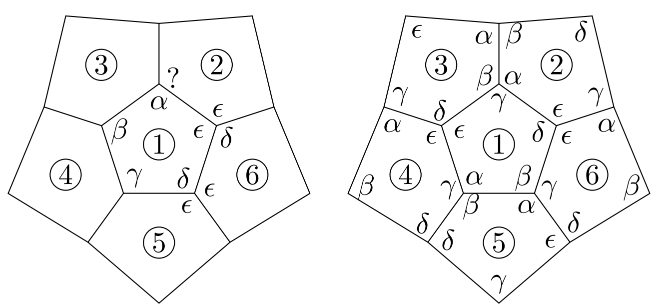

Figure 13: Neighborhood tiling for $\text{AVC}_3 = \{\alpha\beta\gamma, \delta\epsilon^2\}$.

Up to the symmetry of $\text{AVC}_3 = \{\alpha\beta\gamma, \delta\epsilon^2\}$, we only need to consider the first and third angle arrangements. We assume the center tiles of Figure 13 have the two arrangements.

On the left, by $V_{\delta,1} = \delta\epsilon^2$, we get ϵ_5, ϵ_6. By $V_{\epsilon,1} = \delta\epsilon^2$ and the fact that P_6 has only one ϵ, we get δ_6, ϵ_2. Then the angle $A_{2,13}$ adjacent to ϵ_2 is either α_2 or δ_2. This implies either $\alpha^2\cdots$ or $\alpha\delta\cdots$ belongs to AVC_3, a contradiction.

On the right, by $V_{\delta,1} = \delta\epsilon^2$, we get ϵ_2, ϵ_6. By $V_{\gamma,1} = \alpha\beta\gamma$ and the fact that ϵ_2 is adjacent only to α_2, γ_2, we get α_2, β_3. Then α_2, ϵ_2 determine P_2. By $V_{\epsilon,1} = \delta\epsilon^2$ and the fact that β_3 is adjacent only to α_3, δ_3, we get δ_3, ϵ_4. Then β_3, δ_3 determine P_3. By $V_{\alpha,1} = \alpha\beta\gamma$ and the fact that ϵ_4 is adjacent only to α_4, γ_4, we get γ_4, β_5. Then γ_4, ϵ_4 determine P_4. By $V_{\beta,1} = \alpha\beta\gamma$ and the fact that β_5 is adjacent only to α_5, δ_5, we get α_5, γ_6. Then α_5, β_5 determine P_5 and ϵ_6, γ_6 determine P_6.

Next we will argue that the number of tiles $f \le 24$. Since f is even, it is sufficient to show that $f < 26$. We note that AVC_3 implies

$$\alpha + \beta + \gamma + \delta + \epsilon - 3\pi = \frac{1}{2}\delta = \frac{4}{f}\pi, \quad \delta = \frac{8}{f}\pi.$$

Since $f \ge 16$, we have $\delta \le \frac{1}{2}\pi$. We will have two inequality restrictions on f.

Consider the pentagon in Figure 11. We have $a < \pi$ because otherwise any two adjacent edges would intersect at two points. We may determine arcs x and y by the cosine laws

$$\begin{aligned}
\cos x &= \cos^2 a + \sin^2 a \cos\delta, \\
\cos y &= \cos^2 a + \sin^2 a \cos\epsilon = \cos^2 a - \sin^2 a \cos\frac{\delta}{2}.
\end{aligned}$$

The inequality $y - x \le a$ then defines a region on the rectangle $(a, \delta) \in (0, \pi) \times (0, \frac{1}{2}\pi]$.

For $\frac{1}{2}\pi < a < \pi$, another inequality may be obtained by estimating the area of the pentagon. Since $\delta \le \frac{1}{2}\pi$, $\triangle BCD$ lies outside $\square ABCE$. Therefore

$$\frac{4}{f}\pi = \text{Area}(\pentagon ABDCE) \ge \text{Area}(\square ABCE).$$

The area of the quadrilateral can be further estimated

$$\text{Area}(\square ABCE) \ge \text{Area}(\triangle ACE) - \text{Area}(\triangle ABC).$$

By the assumption $\frac{1}{2}\pi < a < \pi$, we have

$$\text{Area}(\triangle ACE) \geq \epsilon = \pi - \frac{\delta}{2} = \pi - \frac{4}{f}\pi.$$

Moreover, Area$(\triangle ABC) + \pi$ is the sum $\sum$ of the three angles of $\triangle ABC$. Combining all the inequalities together, we get

$$\sum \geq 2\left(\pi - \frac{4}{f}\pi\right).$$

The sides of $\triangle ABC$ are x, y, a, and its three angles can be calculated by the cosine law. Then $\sum$ may be explicitly expressed as a function of (a, δ).

To show that $f \geq 26$ leads to contradiction, we note that $f \geq 26$ implies $\sum \geq \frac{22}{13}\pi$ by the estimation above. In Figure **??**, the solid curve separates the regions $y - x < a$ and $y - x > a$, and the dashed curve separates the regions $\sum > \frac{22}{13}\pi$ and $\sum < \frac{22}{13}\pi$. Moreover, the horizontal dotted line corresponds to $f = 26$, and the vertical dotted line corresponds to $a = \frac{1}{2}\pi$. We see that, for $f \geq 26$, the condition $y - x < a$ is not satisfied for $a \in (0, \frac{1}{2}\pi]$, and the condition $\sum \geq \frac{22}{13}\pi$ is not satisfied for $a \in [\frac{1}{2}\pi, \pi)$. Thus we conclude that $f \leq 24$.

By the second vertex counting equation (3.2) and [7, Theorem 6], $f \leq 24$ implies that either the tiling has vertices of degree 4 or 5, or $f = 24$ and the tiling is the earth map tiling with exactly two degree 6 vertices. The former case is covered by the calculation of the cases 1.$*$ and 2.$*$ and will be discussed in Sections 6.4 and 6.5. So we will only study the earth map tiling.

There are five families of earth map tilings, corresponding to distances $5, 4, 3, 2, 1$ between the only two high degree vertices called "poles". They are obtained by glueing copies of the "timezones" in Figure 14 (three timezones are shown for distance 5) along the "meridians". The vertical edges at the top meet at the north pole, and the vertical edges at the bottom meet at the south pole. For $f = 24$, the tiling consists of two time zones for distances $4, 3, 2, 1$ and six timezones for distance 5.

Next we carry out the propagation argument in [3, Section 2] leading to the proof of Proposition 4 of that paper. The neighborhood of a 3^5-tile is given by the right of Figure 13. If a nearby tile is still a 3^5-tile, then its neighborhood is again given by the right of Figure 13. To see whether this is possible, we simplify the presentation of the neighborhood tiling on the right of Figure 13 by keeping only γ and the orientations of the angle

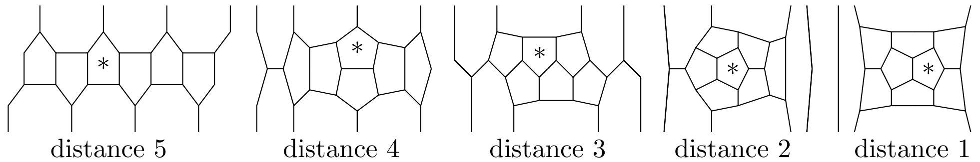

Figure 14: Timezones for earth map tilings.

arrangement. This gives the left picture in Figure 15. The middle picture is the mirror flipping of the left picture.

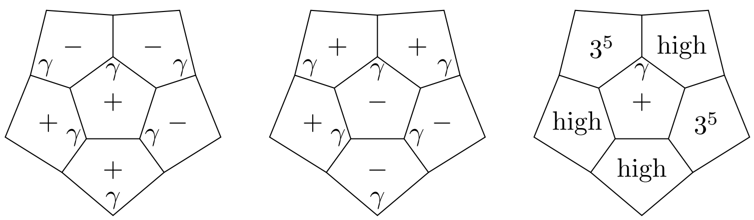

Figure 15: Propagation of neighborhood tiling.

Now each nearby tile is adjacent to three tiles in the neighborhood tiling. We may compare the location of γ and the orientations of the three tiles with the left or the middle picture (depending on whether the nearby tile is positively or negatively oriented). If everything matches, then the tile can be (but is not necessarily) a 3^5-tile, and we indicate the tile by 3^5 on the right of Figure 15. If there is a mismatch, then the tile must have a high degree vertex, and we indicate the tile by "high".

We apply the propagation to the $*$-labeled 3^5-tiles in Figure 14. For distances $4, 3, 2, 1$, all $*$-labeled tiles have at least three nearby 3^5-tiles. Since the right of Figure 15 has only two nearby 3^5-tiles, it cannot be the neighborhoods of the $*$-labeled tiles. For distance 5, we note that only the two tiles on the left and right of the $*$-labeled tile are 3^5-tiles. These two must be the two nearby 3^5-tiles on the right of Figure 15. Guided by this observation, it is easy to derive the unique earth map tiling of distance 5 in Figure 16 (only three of the six timezones are shown). In particular, we find that δ^6 must be a vertex.

So we calculate the equilateral pentagon with the third angle arrangement and three known vertices $\alpha\beta\gamma, \delta\epsilon^2, \delta^6$. We get

$$\gamma = 2\pi - \alpha - \beta,\ \delta = \frac{1}{3}\pi,\ \epsilon = \frac{5}{6}\pi,\ f = 24,$$

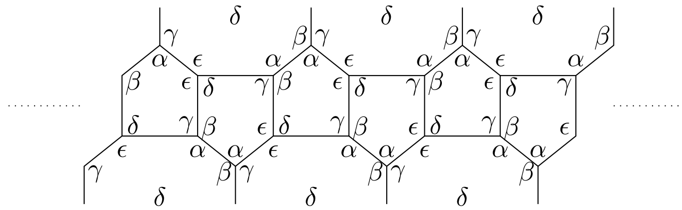

Figure 16: Tiling for $\text{AVC}_3 = \{\alpha\beta\gamma, \delta\epsilon^2\}$, $v_4 = v_5 = 0$.

and two solutions

$$(\alpha, \beta) = (0.1440\pi, 1.3333\pi),\ (0.1192\pi, 1.3807\pi).$$

Using the exact values of δ and ϵ, we may further get the following data for the first and second solutions

$$a = 0.2501\pi, \quad x = 0.2301\pi, \quad y = 0.4788\pi, \quad \phi = 0.3766\pi, \quad \psi = 0.1153\pi;$$
$$a = 0.2614\pi, \quad x = 0.2385\pi, \quad y = 0.5000\pi, \quad \phi = 0.3807\pi, \quad \psi = 0.1192\pi.$$

Similar to Section 5.3, we can verify that the inequalities between a, x, y and between $\alpha, \beta, \phi, \psi$ are satisfied for the first solution. The data for the second solution suggests $y - x = a$, $\alpha = \psi$ and $\beta = \pi + \phi$. Since the existence of the pentagon depends on these exact equalities, the approximate numerical calculation is not enough to verify the existence. We will use symbolic calculations to exactly verify the equalities. The details will be given in Section 6.4. In fact, the data also suggests $\beta = \frac{4}{3}\pi$ for the first solution. We will also verify this by symbolic calculation. The details will be given in Section 6.3.

6.3 Case 5.5

By the calculation in [2], only the fifth arrangement admits solution. To make the arrangement consistent with the 4.$*$ cases and the exceptional case, we exchange α and β to translate the fifth arrangement to the third arrangement. So we consider

$$\text{AVC}_3 = \{[\alpha, \beta, \delta, \gamma, \epsilon] \colon \alpha\beta\gamma, \beta\delta^2, \delta\epsilon^2\},$$

and after the exchange of α and β, the solution becomes

$$\alpha = 0.1440\pi,\ \beta = \frac{4}{3}\pi,\ \gamma = 0.5226\pi,\ \delta = \frac{1}{3}\pi,\ \epsilon = \frac{5}{6}\pi,\ f = 24.$$

Note that at the moment, all the values are only approximate, and the exact value of β, δ, γ will be justified by symbolic calculation.

By the method in Section 5.2, from the approximate values of five angles, we get all the possible angle combinations at vertices

$$\text{AVC} \subset \{[\alpha, \beta, \delta, \gamma, \epsilon]\colon \alpha\beta\gamma, \beta\delta^2, \delta\epsilon^2, \alpha^3\gamma^3, \alpha^2\gamma^2\delta^2, \alpha\gamma\delta^4, \delta^6\}.$$

Since "possible" does not necessarily mean appearing, the actual full AVC may be smaller than the right side.

By the AVC, the vertices can only have degree 3 or 6. By $f = 24$, the vertex counting equation (3.2) and [7, Theorem 6], the tiling is the earth map tiling with exactly two degree 6 vertices.

In Section 6.2, we explained that every earth map tiling has 3^5-tiles. So we study the possible ways of assigning the angles in the neighborhood of a 3^5-tile subject to our AVC. We will use the notations P_i, θ_i, $V_{\theta,i}$, $A_{i,jk}$ introduced in Sections 5.2 and 6.2.

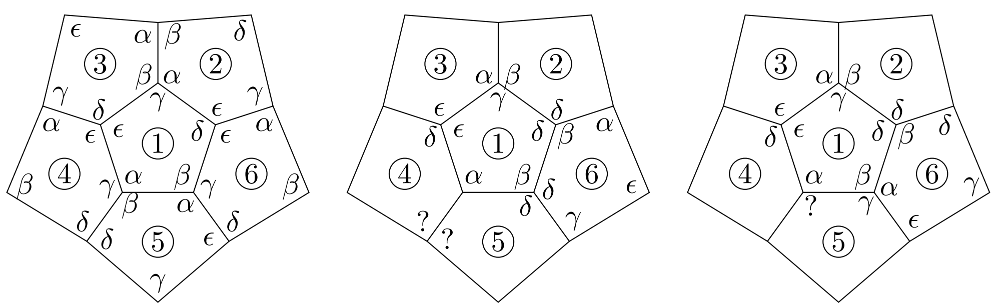

Figure 17: Neighborhood tiling for $\{[\alpha, \beta, \delta, \gamma, \epsilon]\colon \alpha\beta\gamma, \beta\delta^2, \delta\epsilon^2, \alpha^3\gamma^3, \alpha^2\gamma^2\delta^2, \alpha\gamma\delta^4, \delta^6\}$.

We have $V_{\epsilon,1} = \delta\epsilon^2$. This implies either $A_{3,14} = \delta_3, A_{4,13} = \epsilon_4$, or $A_{3,14} = \epsilon_3, A_{4,13} = \delta_4$. The left of Figure 17 describes the case $A_{3,14} = \delta_3, A_{4,13} = \epsilon_4$. By $V_{\gamma,1} = \alpha\beta\gamma$, the angle $A_{3,12}$ adjacent to δ_3 must be β_3. Then β_3, δ_3 determine P_3. The same reason determines P_4. By $V_{\gamma,1} = V_{\beta,3} = V_{\alpha,1} = V_{\gamma,4} = \alpha\beta\gamma$, we get α_2, β_5. Since $\alpha\epsilon\cdots$ is prohibited by the AVC, we get the angle ϵ_2 adjacent to α_2. Then α_2, ϵ_2 determine P_2. By $V_{\delta,1} = V_{\epsilon,2} = \delta\epsilon^2$, we get ϵ_6. Since δ_6, ϵ_6 are not adjacent, we get $V_{\beta,1} = \alpha\beta\gamma$. Since β_5, γ_5 are not adjacent, we get α_5, γ_6. Then α_5, β_5 determine P_5 and γ_6, ϵ_6 determine P_6. The tiling is the same as the right of Figure 13.

The middle and right of Figure 17 describe the case $A_{3,14} = \epsilon_3, A_{4,13} = \delta_4$. By $V_{\gamma,1} = \alpha\beta\gamma$ and the non-adjacency of β_3, ϵ_3, we get α_3, β_2. By $V_{\delta,1} = \beta\delta^2$

or $\delta\epsilon^2$, and the fact that β_2 is adjacent only to α_2, δ_2, we get $V_{\delta,1} = \beta\delta^2$ and δ_2, β_6. Then β_6 implies two possible angle arrangements for P_6, illustrated by the middle and right of Figure 17.

In the middle, by $V_{\beta,1} = V_{\delta,6} = \beta\delta^2$, we get δ_5. By $V_{\alpha,1} = \alpha\beta\gamma$ and δ_4, δ_5, we find that the two ?-labeled angles are α and ϵ, contradicting to $\alpha\epsilon\cdots \notin$ AVC. On the right, by $V_{\beta,1} = V_{\alpha,6} = \alpha\beta\gamma$, we get γ_5. By $V_{\alpha,1} = \alpha\beta\gamma$, we find that either β_5, γ_5 are adjacent, or P_5 has two γ. Both are contradictions.

So we conclude that the right of Figure 13 (i.e., the left of Figure 17) is the only neighborhood tiling fitting the AVC. Then the propagation argument in Section 6.2 (which no longer uses the AVC) shows that the tiling is the earth map tiling in Figure 16. We find that the actual full AVC at the end is $\{\alpha\beta\gamma, \delta\epsilon^2, \delta^6\}$. Since the AVC does not include $\beta\delta^2$, the pentagon and the tiling is for the exceptional case in Section 6.2, and is not for Case 5.5. Furthermore, the approximate value of α shows that the tiling is for the first solution of the exceptional case.

In Section 6.2, we already verified the existence of the pentagon, and obtained the approximate value $a = 0.2501\pi$. It is given by Figure 18, with the left being the scheme and the right being the actual shape. However, the numerical calculation cannot imply that $f(a) = 0$ exactly matches $\beta = \frac{4}{3}\pi$.

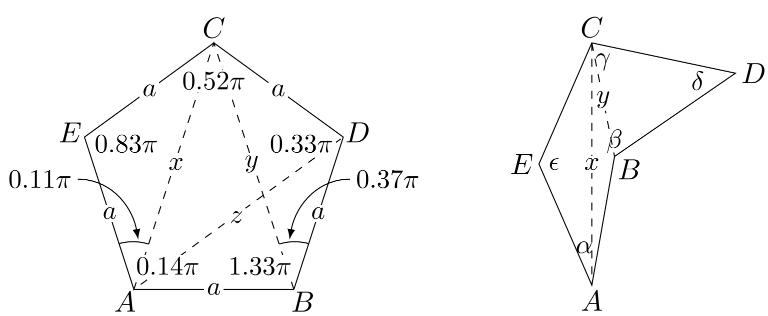

Figure 18: Pentagon for $\{[\alpha, \beta, \delta, \gamma, \epsilon]\colon \alpha\beta\gamma, \beta\delta^2, \delta\epsilon^2\}$.

To prove the exact values of β, δ, γ, we reconstruct the pentagon by starting with these exact values. This is possible because an equilateral pentagon allows three free variables. The goal is to verify that $\alpha+\beta+\gamma = 2\pi$ is exactly satisfied for the pentagon, so that the original assumption on the appearance of the vertices $\alpha\beta\gamma, \beta\delta^2, \delta\epsilon^2$ is satisfied.

The two ways of calculating $\cos x$ by using $\triangle ACE$ and $\square ABDC$ imply a quadratic equation for $\cos a$

$$
\begin{aligned}
0 = &\left(1-\cos\frac{4}{3}\pi\right)\left(1-\cos\frac{1}{3}\pi\right)\cos^2 a \\
&+\left[\cos\frac{5}{6}\pi+\cos\left(\frac{4}{3}+\frac{1}{3}\right)\pi-\cos\frac{4}{3}\pi-\cos\frac{1}{3}\pi\right]\cos a \\
&+\left[\cos\frac{5}{6}\pi-\sin\frac{4}{3}\pi\sin\frac{1}{3}\pi\right].
\end{aligned}
$$

By $a = 0.2501\pi \in (0, \frac{1}{2}\pi)$, the exact value of $\cos a$ is

$$
\cos a = \frac{1}{3}\left(-1+\sqrt{3}+\sqrt{-5+4\sqrt{3}}\right).
$$

We may construct $\triangle ACE$ using this a and $\epsilon = \frac{5}{6}\pi$, and construct $\square ABDC$ using this a and $\beta = \frac{4}{3}\pi$ and $\delta = \frac{1}{3}\pi$. The validity of the quadratic equation above means that the triangle and the quadrilateral have matching $AC = x$ edge. Therefore they can be glued together to form a pentagon.

We can use a, δ, ϵ to calculate $\triangle AEC$, $\triangle BCD$ and then further use β to calculate $\triangle ABC$. Then we can confirm the approximate values $\alpha = 0.1440\pi$ and $\gamma = 0.5226\pi$. In the subsequent calculations of the exact values of α and γ, we will only choose the exact values consistent with the approximate values.

The two ways of calculating $\cos y$ by using $\triangle BCD$ and $\square ABCE$ imply another quadratic equation for $\cos a$

$$
\begin{aligned}
0 = &\left(1-\cos\frac{5}{6}\pi\right)(1-\cos\alpha)\cos^2 a \\
&+\left[\cos\frac{1}{3}\pi+\cos\left(\frac{5}{6}\pi+\alpha\right)-\cos\frac{5}{6}\pi-\cos\alpha\right]\cos a \\
&+\left[\cos\frac{1}{3}\pi-\sin\frac{5}{6}\pi\sin\alpha\right].
\end{aligned}
$$

Substituting the value of $\cos a$ into the equation, we get a linear equation relating $\cos\alpha$ and $\sin\alpha$

$$\begin{aligned}&\left(7+6\sqrt{3}+8\sqrt{-5+4\sqrt{3}}+5\sqrt{3}\sqrt{-5+4\sqrt{3}}\right)\cos\alpha\\&+3\left(2+\sqrt{3}+\sqrt{-5+4\sqrt{3}}\right)\sin\alpha\\=\ &19+3\sqrt{3}+5\sqrt{-5+4\sqrt{3}}+5\sqrt{3}\sqrt{-5+4\sqrt{3}}.\end{aligned}$$

The equation has two solutions, and the one consistent with the approximate value $\alpha = 0.1440\pi$ is

$$\begin{aligned}\alpha &= \arctan\frac{1}{33}\left(4+3\sqrt{3}-2\sqrt{-5+4\sqrt{3}}+4\sqrt{3}\sqrt{-5+4\sqrt{3}}\right)\\&= 0.14400988468593670938539230388\pi.\end{aligned}$$

Similarly, the two ways of calculating $\cos z$ by using $\triangle ABD$ and $\square ADCE$ imply a linear equation relating $\cos\gamma$ and $\sin\gamma$

$$\begin{aligned}&\left(7+6\sqrt{3}+8\sqrt{-5+4\sqrt{3}}+5\sqrt{3}\sqrt{-5+4\sqrt{3}}\right)\cos\gamma\\&+3\left(2+\sqrt{3}+\sqrt{-5+4\sqrt{3}}\right)\sin\gamma\\=\ &7-3\sqrt{3}-\sqrt{-5+4\sqrt{3}}+5\sqrt{3}\sqrt{-5+4\sqrt{3}}.\end{aligned}$$

The solution consistent with the approximate value $\gamma = 0.5226\pi$ is

$$\begin{aligned}\gamma &= \pi-\arctan\frac{1}{3}\left(12+7\sqrt{3}+6\sqrt{-5+4\sqrt{3}}+4\sqrt{3}\sqrt{-5+4\sqrt{3}}\right)\\&= 0.52265678198072995728127436277\pi.\end{aligned}$$

Then we may symbolically verify

$$\tan(\pi-\alpha-\gamma)=\frac{\tan(\pi-\gamma)-\tan\alpha}{1+\tan(\pi-\gamma)\tan\alpha}=\sqrt{3}.$$

The only exact value of $\pi-\alpha-\gamma$ consistent with the approximate value is $\pi-\alpha-\gamma=\frac{1}{3}\pi=\beta-\pi$.

6.4 Cases 1.2e, 1.5a and 2.4b

By the calculation in [2], only the eleventh arrangement admits solution for the three cases. So the three cases can be summarised as

$$\{[\alpha,\delta,\beta,\gamma,\epsilon]\colon \alpha\beta\gamma, \delta\epsilon^2, \alpha\delta^2\epsilon \text{ or } \alpha^4 \text{ or } \alpha^2\delta^3\}.$$

Again we translate into the third arrangement by exchanging α and γ

$$\{[\alpha,\beta,\delta,\gamma,\epsilon]\colon \alpha\beta\gamma, \delta\epsilon^2, \gamma\delta^2\epsilon \text{ or } \gamma^4 \text{ or } \gamma^2\delta^3\}.$$

After exchanging α and γ, all three cases have one solution

$$\alpha = 0.1192\pi,\ \beta = \frac{3}{2}\pi - \alpha,\ \gamma = \frac{1}{2}\pi,\ \delta = \frac{1}{3}\pi,\ \epsilon = \frac{5}{6}\pi,\ f = 24.$$

The exact values of γ, δ, ϵ will be justified by symbolic calculation.

By the method in Section 5.2, from the approximate values of five angles, we get

$$\text{AVC} \subset \{[\alpha,\beta,\delta,\gamma,\epsilon]\colon \alpha\beta\gamma, \delta\epsilon^2, \gamma\delta^2\epsilon, \gamma^4, \gamma^2\delta^3, \delta^6\}.$$

Since β appears only at $\alpha\beta\gamma$, and the total number of times β appears in the tiling is f, we find that $\alpha\beta\gamma$ appears f times. This implies that γ already appears f times at $\alpha\beta\gamma$, and therefore cannot appear at any other vertex. Therefore $\gamma\delta^2\epsilon, \gamma^4, \gamma^2\delta^3$ actually cannot appear, and

$$\text{AVC} \subset \{[\alpha,\beta,\delta,\gamma,\epsilon]\colon \alpha\beta\gamma, \delta\epsilon^2, \delta^6\}.$$

Since the right side is contained in the collection of possible angle combinations studied in Section 6.3, the tiling is given by Figure 16. In particular, the actual AVC is $\{\alpha\beta\gamma, \delta\epsilon^2, \delta^6\}$, which includes none of $\gamma\delta^2\epsilon, \gamma^4, \gamma^2\delta^3$. Therefore the pentagon and the tiling is for the exceptional case in Section 6.2, and is not for Cases 1.2e, 1.5a and 2.4b. Furthermore, the approximate value of α shows that the tiling is for the second solution of the exceptional case.

It remains to verify the existence of the pentagon. The approximate values for the second solution in Section 6.2 suggest $\alpha = \psi$, which means that the pentagon is obtained by directly glueing $\triangle ACE$ and $\triangle BCD$ together, and $\triangle ABC$ is reduced to an arc. The situation is described in Figure 19.

To prove the configuration in Figure 19, we reconstruct the pentagon by starting with the exact values of γ, δ, ϵ, and the assumption $AE = CE = BD = CD = a$. The goal is to verify that $AC = BC + a$, so that glueing

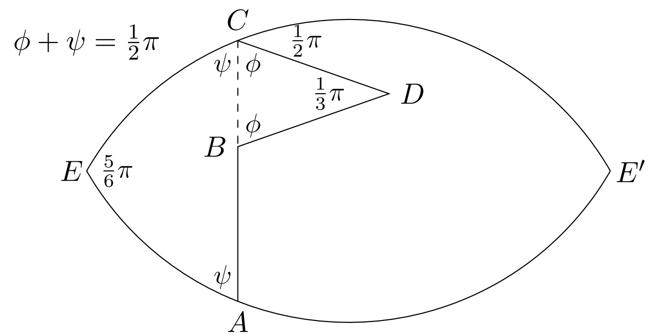

Figure 19: Pentagon for $\{[\alpha, \beta, \delta, \gamma, \epsilon]\colon \alpha\beta\gamma, \delta\epsilon^2, \gamma\delta^2\epsilon \text{ or } \gamma^4 \text{ or } \gamma^2\delta^3\}$.

the two isosceles triangles $\triangle ACE$ and $\triangle BCD$ gives a pentagon with equal sides.

We have

$$\tan\phi = \sec a \cot\frac{\delta}{2}, \quad \tan\psi = \sec a \cot\frac{\epsilon}{2}.$$

Then $\phi + \psi = \gamma = \frac{1}{2}\pi$ implies that

$$\sec a \cot\frac{\delta}{2} \cdot \sec a \cot\frac{\epsilon}{2} = 1.$$

Therefore

$$\cos a = \sqrt{-3 + 2\sqrt{3}}, \quad \sin a = \sqrt{1 - (-3 + 2\sqrt{3})} = -1 + \sqrt{3}.$$

Here we choose $\cos a$ to be positive because the approximate value $\psi = \alpha = 0.1192\pi$ implies $\tan\psi > 0$. Combined with $\cot\frac{\epsilon}{2} > 0$, we get $\sec a > 0$. The approximate value of a is

$$a = \arccos\sqrt{-3 + 2\sqrt{3}} = 0.2614366507506671650166836630\pi.$$

Now we have

$$\cos AC = \cos^2 a + \sin^2 a \cos \frac{5}{6}\pi$$
$$= (-3 + 2\sqrt{3}) - (4 - 2\sqrt{3})\frac{\sqrt{3}}{2} = 0,$$
$$\cos BC = \cos^2 a + \sin^2 a \cos \frac{1}{3}\pi$$
$$= (-3 + 2\sqrt{3}) + (4 - 2\sqrt{3})\frac{1}{2} = -1 + \sqrt{3} = \sin a.$$

The first equality implies $AC = \frac{1}{2}\pi$. The second equality implies $\cos BC > 0$, so that $0 < BC < \frac{1}{2}\pi$. Since we also have $0 < a < \frac{1}{2}\pi$, the second equality further implies $BC + a = \frac{1}{2}\pi = AC$.

The shape of the pentagon and the exact value of a imply the exact values of α and β

$$\alpha = \psi = \frac{1}{2}\pi - \phi = \frac{1}{2}\pi - \arctan\sqrt{3 + 2\sqrt{3}},$$
$$\beta = \pi + \phi = \pi + \arctan\sqrt{3 + 2\sqrt{3}}.$$

6.5 Cases 1.4f, 1.5b, 2.5f and 2.6b

For Case 1.5b, we have

$$\gamma = 2\pi - \alpha - \beta,\ \delta = \frac{1}{2}\pi,\ \epsilon = \frac{3}{4}\pi,\ f = 16,$$

By the calculation in [2], there are three solutions (the first and second solutions are for the first arrangement, and the third solution is for the third arrangement)

$$(\alpha, \beta) = (0.6338\pi, 0.5642\pi),\ (0.10133\pi, 1.56723\pi),\ (0.4536\pi, 0.8823\pi).$$

Then we use the method in Section 5.2 to find

$$\text{AVC} \subset \{\alpha\beta\gamma, \delta\epsilon^2, \delta^4\}.$$

For Cases 1.4f and 2.6b, we have

$$\gamma = 2\pi - \alpha - \beta,\ \delta = \frac{2}{5}\pi,\ \epsilon = \frac{4}{5}\pi,\ f = 20,$$

and two solutions (for the first and third arrangements, respectively)

$$(\alpha,\beta)=(0.6055\pi,0.5024\pi),\ (0.3095\pi,1.0615\pi).$$

From the solutions, we get

$$\text{AVC}\subset\{\alpha\beta\gamma,\delta\epsilon^2,\delta^3\epsilon,\delta^5\}.$$

For Case 2.5f, we have

$$\gamma=2\pi-\alpha-\beta,\ \delta=\frac{2}{7}\pi,\ \epsilon=\frac{6}{6}\pi,\ f=28,$$

and one solution (for the first arrangement)

$$(\alpha,\beta)=(0.5588\pi,0.4371\pi).$$

Then we find

$$\text{AVC}\subset\{\alpha\beta\gamma,\delta\epsilon^2,\delta^4\epsilon,\delta^7\}.$$

We first prove that the AVCs above do not admit tilings with first arrangement. The left of Figure 20 shows what happens at a high degree vertex, which means $\delta^3\epsilon,\delta^4\epsilon,\delta^4,\delta^5$ or δ^7 in our AVCs. We always have three consecutive δ at the vertex, and we may assume that the angles of P_2 are arranged as indicated. Since $V_{\epsilon,2}$ does not involve γ_1, we get the angle γ_1 adjacent to δ_1. Then γ_1,δ_1 determine P_1. By $V_{\epsilon,1}=V_{\epsilon,2}=\delta\epsilon^2$, we get P_3,δ_3. Then the angle ϵ_3 adjacent to δ_3 implies that $\alpha\epsilon\cdots$ is a vertex, contradicting to the AVC.

It remains to consider the third arrangement, which means the third solution of Case 1.5b and the second solution of Cases 1.4f and 2.6b. For Case 1.5b, the only high degree vertex δ^4 must appear. On the right of Figure 20, we start from such a vertex at the center and assume that the angles of P_1 are arranged as indicated. It is easy to use the AVC to determine all the angles of four tiles at the vertex. Then we can use the AVC to further get four tiles similar to P_2 and determine all their angles. Next we determine the four tiles similar to P_3 and all their angles. Finally we determine the four tiles similar to P_4 and all their angles. The result is the earth map tiling of distance 5, with 16 tiles.

For Cases 1.4f and 2.6b, if δ^5 appears, then we can carry out the argument similar to Case 1.5b, and also get the earth map tiling of distance 5, with 20

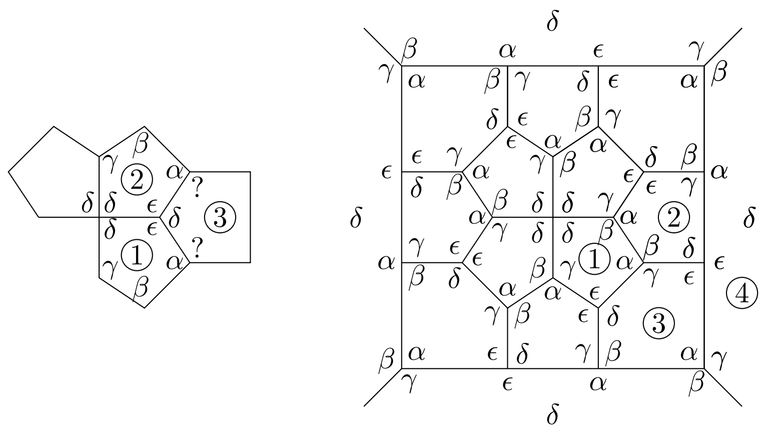

Figure 20: Tiling for $\{[\alpha,\beta,\delta,\gamma,\epsilon]\colon \alpha\beta\gamma, \delta\epsilon^2, \delta^4\}$, Case 1.5b.

tiles. Note that $\delta^3\epsilon$ is not a vertex of this tiling. Therefore the tiling is really for Case 2.6b, and not for Case 1.4f.

Finally, we study the remaining case that δ^5 does not appear. This means Case 1.4f, with

$$\text{AVC} = \{[\alpha,\beta,\delta,\gamma,\epsilon]\colon \alpha\beta\gamma, \delta\epsilon^2, \delta^3\epsilon\}.$$

We will show that the tiling is given by Figure 21. The tiling has two tiles with two degree 4 vertices and three degree 3 vertices, which we call 3^34^2-tiles. We draw the two tiles as the north and south "regions" (as opposed to the two poles in the earth map tiling) P_1, P_{10}. The tiling is obtained by glueing the left and right together.

To argue for the tiling in Figure 21, we start with a vertex $\delta^3\epsilon$ and four tiles P_1, P_2, P_3, P_4 around the vertex. We assume ϵ belongs to P_1, and the three δs belong to P_2, P_3, P_4, so that the vertex is really $V_{\epsilon,1} = V_{\delta,2} = V_{\delta,3} = V_{\delta,4}$. We also assume that the angles of P_1 are arranged as indicated, which is a straight line with left and right δ_1 identified.

By $V_{\gamma,1} = \alpha\beta\gamma$ and the non-adjacency of α_2, δ_2, we get β_2, P_5, α_5. Then β_2, δ_2 determine P_2. By $V_{\gamma,2} = \alpha\beta\gamma$ and the non-adjacency of α_3, δ_3, we get β_3, P_6, α_6. Then β_3, δ_3 determine P_3. By $V_{\gamma,3} = \alpha\beta\gamma$ and the non-adjacency of α_4, δ_4, we get β_4, P_7, α_7. Then β_4, δ_4 determine P_4.

By $V_{\alpha,2} = \alpha\beta\gamma$ and the non-adjacency of α_5, γ_5, we get β_5, P_8, γ_8. Then α_5, β_5 determine P_5. By $V_{\alpha,3} = \alpha\beta\gamma$ and the non-adjacency of α_6, γ_6, we get β_6, P_9, γ_9. Then α_6, β_6 determine P_6. By $V_{\alpha,4} = \alpha\beta\gamma$ and the non-adjacency of α_7, γ_7, we get $\beta_7, P_{10}, \gamma_{10}$. Then α_7, β_7 determine P_7. By $V_{\epsilon,3} = V_{\epsilon,7} = \delta\epsilon^2$,

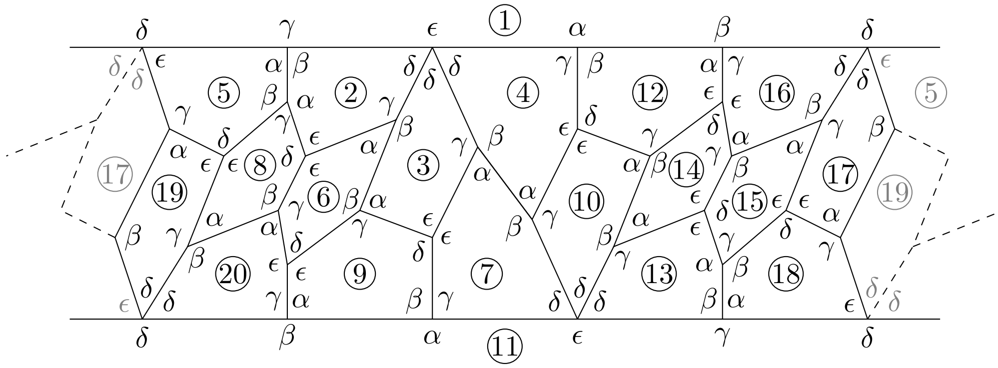

Figure 21: Tiling for $\{[\alpha,\beta,\delta,\gamma,\epsilon]\colon \alpha\beta\gamma, \delta\epsilon^2, \delta^3\epsilon\}$, Case 1.4f.

we find that P_3, P_7, P_9 meet at the degree 3 vertex. We also get δ_9, and γ_9, δ_9 determine P_9. By $V_{\gamma,7} = V_{\beta,9} = \alpha\beta\gamma$, we get P_{11}, α_{11}. By $V_{\delta,7} = \delta\epsilon^2$ or $\delta^3\epsilon$, and the fact that α_{11} is adjacent only to $\beta_{11}, \epsilon_{11}$, we get ϵ_{11}. Then $\alpha_{11}, \epsilon_{11}$ determine P_{11}.

By $V_{\alpha,1} = V_{\gamma,4} = \alpha\beta\gamma$, we get P_{12}, β_{12}. Now $V_{\epsilon,4}$ is either $\delta\epsilon^2$ or $\delta^3\epsilon$. We note that the first choice means that P_4 has one degree 4 vertex (i.e., a $3^4 4^1$-tile), and the second choice means that P_4 has two degree 4 vertices (i.e., a $3^3 4^2$-tile).

For the first choice $V_{\epsilon,4} = \delta\epsilon^2$, by the non-adjacency of $\beta_{12}, \epsilon_{12}$, we get $\delta_{12}, \epsilon_{10}$ at $V_{\epsilon,4}$. Then $\gamma_{10}, \epsilon_{10}$ determine P_{10}, and β_{12}, δ_{12} determine P_{12}. By $V_{\delta,7} = V_{\delta,10} = V_{\epsilon,11} = \delta^3\epsilon$, we get P_{13}, δ_{13}. By $V_{\beta,10} = \alpha\beta\gamma$ and the non-adjacency of α_{13}, δ_{13}, we get $\gamma_{13}, P_{14}, \alpha_{14}$. By $V_{\alpha,10} = V_{\gamma,12} = \alpha\beta\gamma$, we get β_{14}. Then γ_{13}, δ_{13} determine P_{13} and α_{14}, β_{14} determine P_{14}. By $V_{\epsilon,13} = V_{\epsilon,14} = \delta\epsilon^2$, we get P_{15}, δ_{15}. By $V_{\gamma,14} = \alpha\beta\gamma$ and the non-adjacency of α_{15}, δ_{15}, we get $\beta_{15}, P_{16}, \alpha_{16}$. Then β_{15}, δ_{15} determine P_{15}. By $V_{\epsilon,12} = V_{\delta,14} = \delta\epsilon^2$ or $\delta^3\epsilon$ and the non-adjacency of α_{16}, δ_{16}, we get $V_{\epsilon,12} = V_{\delta,14} = \delta\epsilon^2$ and ϵ_{16} at the vertex $V_{\epsilon,12} = V_{\delta,14}$. Then $\alpha_{16}, \epsilon_{16}$ determine P_{16}, and we also know that P_{16} shares a degree 3 vertex $\alpha\beta\gamma$ with P_1, P_{12}. Now we know $V_{\delta,1} = V_{\epsilon,5} = V_{\delta,16} = \delta^3\epsilon$ is a degree 3 vertex.

What we have proved so far can be interpreted as follows. If P_4 has only one degree 4 vertex (i.e., a $3^4 4^1$-tile), then P_1 has two degree 4 vertices (i.e., a $3^3 4^2$-tile). This means that there is at least one $3^3 4^2$-tile. So without loss of generality, we may assume at the very beginning that P_1 is a $3^3 4^2$-tile.

This means that we may also assume $V_{\delta,1} = \delta^3\epsilon$.

We review our proof and find that the proof up to getting P_{12}, β_{12} does not use the assumption $V_{\delta,1} = \delta^3\epsilon$. Therefore the proof remains valid until that point, and we may continue from that point with the additional assumption.

By $V_{\delta,1} = V_{\epsilon,5} = \delta^3\epsilon$, we get $P_{16}, P_{17}, \delta_{16}, \delta_{17}$. By $V_{\beta,1} = \alpha\beta\gamma$ and the non-adjacency of α_{16}, δ_{16}, we get γ_{16}, α_{12}. Then α_{12}, β_{12} determine P_{12} and γ_{16}, δ_{16} determine P_{16}. By $V_{\epsilon,12} = V_{\epsilon,16} = \delta\epsilon^2$, we get P_{14}, δ_{14}. By $V_{\gamma,12} = \alpha\beta\gamma$ and the non-adjacency of α_{14}, δ_{14}, we get β_{14} and an α outside P_{12}, P_{14}. The angle α is presumed to be α_{10}. But the claim depends on whether $V_{\epsilon,4} = V_{\delta,12}$ is $\delta\epsilon^2$ or $\delta^3\epsilon$. If the vertex is $\delta^3\epsilon$, then we find that α, δ are adjacent. The contradiction implies that $V_{\epsilon,4} = V_{\delta,12}$ is $\delta\epsilon^2$, which was the earlier first choice for the vertex. (Basically, we have just proved that P_4 is a $3^4 4^1$-tile if and only if P_1 is a $3^3 4^2$-tile.) Therefore all the subsequent argument become valid.

By $V_{\alpha,15} = V_{\beta,16} = \alpha\beta\gamma$, we get γ_{17}. Then γ_{17}, δ_{17} determine P_{17}. By $V_{\gamma,11} = V_{\beta,15} = V_{\alpha,13} = V_{\gamma,15} = \alpha\beta\gamma$, we get $P_{18}, \alpha_{18}, \beta_{18}$, and determine P_{18}. By $V_{\gamma,5} = V_{\beta,17} = V_{\alpha,17} = V_{\gamma,18} = \alpha\beta\gamma$, we get $P_{19}, \alpha_{19}, \beta_{19}$, and determine P_{19}. By $V_{\alpha,9} = V_{\beta,11} = \alpha\beta\gamma$ and $V_{\delta,11} = V_{\epsilon,18} = V_{\delta,19} = \delta^3\epsilon$, we get $P_{20}, \gamma_{20}, \delta_{20}$, and determine P_{20}. Then the angle arrangements of $P_2, P_5, P_6, P_{19}, P_{20}$ finally determine P_8.

Finally, we need to verify the existence of the pentagon. We may calculate the edge lengths and angles as in Section 5.3 and verify the inequalities needed for the existence. For the third solution of Case 1.5b, we have

$$\begin{aligned}
a &= 0.2155056950783077521179234617 26\pi, \\
\alpha &= 0.4536848189767118629447911059 35\pi, \\
\beta &= 0.8823880837972543968284667242 8\pi, \\
\gamma &= 0.6639270972260337402267421697 7\pi,
\end{aligned}$$

and

$$\phi = 0.289\pi, \quad \psi = 0.155\pi, \quad x = 0.292\pi, \quad y = 0.392\pi.$$

The inequalities for the existence can be verified and we get the pentagon on the left of Figure 22. For the second solution of Cases 1.4f and 2.6b, we have

$$\begin{aligned}
a &= 0.2168370613509100033513656616 54\pi, \\
\alpha &= 0.3095921182677239254157322478 69\pi, \\
\beta &= 1.0615243280895767593474563028 9\pi, \\
\gamma &= 0.6288835536426993152368114492 35\pi,
\end{aligned}$$

and

$$\phi = 0.336\pi, \quad \psi = 0.126\pi, \quad x = 0.241\pi, \quad y = 0.408\pi.$$

Again the pentagon exists and is depicted on the right of Figure 22.

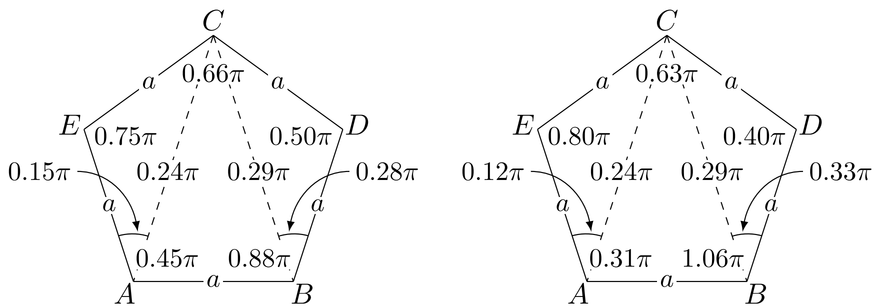

Figure 22: Pentagon for $\{[\alpha, \beta, \delta, \gamma, \epsilon]\colon \alpha\beta\gamma, \delta\epsilon^2, \delta^4 \text{ or } \delta^5 \text{ or } \delta^3\epsilon\}$.

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