Tiling of Sphere by Congruent Pentagons

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webpage for further reading:
http://www.math.ust.hk/∼mamyan/research/UROP.shtml

We consider tilings of the sphere by congruent pentagons. The basic example is dodecahedron, which has 12 pentagonal tiles.

To simplify the problem, we assume the tiling is edge-to-edge and all vertices have degree ≥ 3.

1 Numerical

Let v, e, f be the numbers of vertices, edges and faces in a spherical pentagonal tiling. Then we have

$$v - e + f = 2, \quad 5f = 2e.$$

Let v_i be the number of vertices of degree i. Then

$$v = v_3 + v_4 + v_5 + v_6 + \cdots, \quad 2e = 3v_3 + 4v_4 + 5v_5 + 6v_6 + \cdots.$$

It then easily follows that

$$v_3 = 20 + 2v_4 + 5v_5 + 8v_6 + \cdots, \quad \frac{f}{2} - 6 = v_4 + 2v_5 + 3v_6 + \cdots. \tag{1.1}$$

Exercise 1. Derive similar equalities for tiling of sphere by quadrilaterals. What about triangles? What about tiling of torus?

We conclude that vast majority of vertices have degree 3, and f must be an even number ≥ 12. We call vertices of degree ≥ 4 *high degree* vertices.

Theorem 1. *A pentagonal tiling of the sphere cannot have only one high degree vertex. If the tiling has exactly two high degree vertices, then the tiling is one of five families of* earth map tilings.

The theorem [8] is about the *topological structure* of the tiling. In other words, it is not concerned with edge lengths and angles.

Exercise 2. Show that the following are equivalent for a pentagonal spherical tiling.

1. The number of tiles is $f = 12$.
2. The number of vertices is $v = 20$.
3. All vertices have degree 3.
4. The tiling is the dodecahedron.

Theorem 2. *Any pentagonal tiling of the sphere has a tile, such that four vertices have degree* 3 *and the fifth vertex has degree* 3*,* 4 *or* 5*.*

The theorem [3, Proposition 1] implies that any pentagonal tiling of the sphere contains one of three neighborhood tilings around the special tile in the theorem in Figure 1. This provides a starting point for finding all the tilings of the sphere by congruent pentagons. We may first get lots of information by trying to tile such neighborhoods, and then try to tile beyond the neighborhoods.

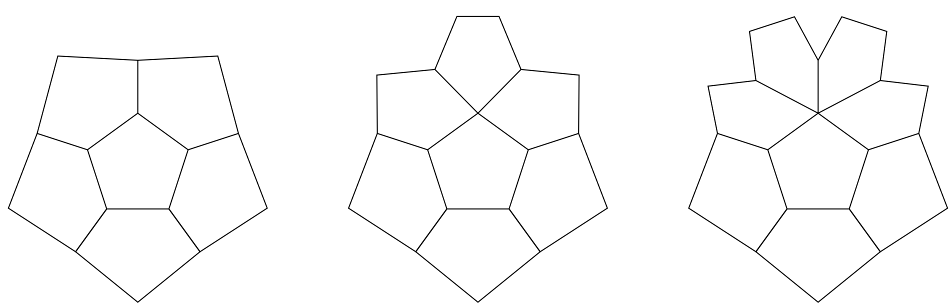

Figure 1: Three neighborhood tilings.

Exercise 3. If there is no special tile as described in Theorem 2, then every tile must be one of the following two types

1. It has at least one vertex of degree ≥ 6.
2. It has at least two vertices of degree 4 or 5.

Using the fact that a degree k vertex is shared by k tiles, prove Theorem 2.

Exercise 4. For quadrilateral tiling of sphere, show the following.

1. There are at least three tiles with degree 3 vertex.
2. It is possible for each tile to have at most one degree 3 vertex, and the minimal example has $f = 24$ tiles.

2 Combinatorial

By *combinatorial* tiling, we ignore edge length and angles. All the numerical results of the last section are combinatorial.

We know f is an even number ≥ 12. Moreover, Exercise 2 describes the minimal case of $f = 12$. The next number would be $f = 14$. By the second equality in (1.1), $f = 14$ implies $v_4 = 1$ and $v_{>4} = 0$. In particular, this means that there is one high degree vertex. Let us start with the unique high degree vertex 4, and try to draw the pentagonal tiling of the sphere in Figure 2. We note that, after the initial vertex of degree 4, all the subsequent vertices have degree 3.

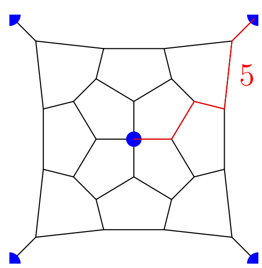

Figure 2: You cannot have only one high degree vertex.

We notice that the outer side of the picture must be four pentagons, and they all share a new vertex of degree 4 (the new vertex is indicated by four blue quarter disks). We also notice that the distance between the two high degree vertices is 5. This means that, if we assume that there are no high degree vertices within (and including) distance 4 of an existing vertex of degree 4, then we still get the same picture.

The observation leads to the following results [8].

Theorem 3. *A combinatorial tiling of the sphere by pentagons cannot have only* 1 *high degree vertex. If it has exactly* 2 *high degree vertices, then the tiling is an earth map tiling.*

Theorem 4. *In a combinatorial tiling of the sphere by pentagons, any high degree vertex has another high degree vertex within distance* 5*. Moreover, if one high degree vertex has no high degree vertex within distance* 4*, then the tiling is the earth map tiling of distance* 5*.*

The *earth map tiling* of distance 5 is given by Figure 3. Imagine that the dodecahedron is the earth with 12 pentagonal countries. The north and south poles are connected by the red line of length 5. If you cut through this line, then you gat a flat map of the earth. The map consists of three *timezones* (the solid parts on the right, each timezone consisting of four countries), the red cutting lines becoming *meridians*. If you repeat more such timezones, then you could imagine a worlds with n timezones and $4n$ countries. Such a world is the general earth map tiling of distance 5.

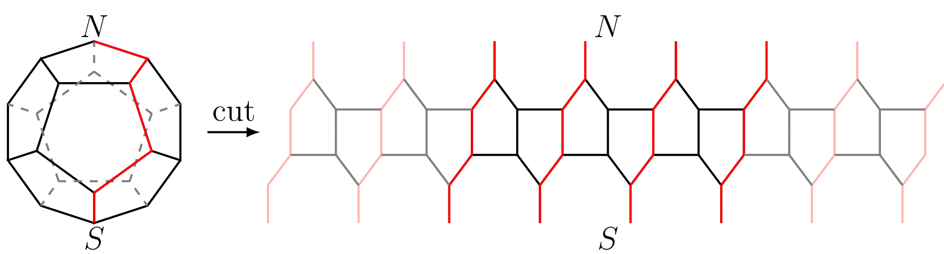

Figure 3: Earth map tiling of distance 5.

Instead of cutting a path of length 5, you may also cut a path of length 4, 3, 2, or 1. Then you can get other earth map tilings. See [8] for more details. We note that the two high degree vertices in the earth map tiling always have the same degree.

Exercise 5. Repeat the argument in Figure 2 by assuming there is one high degree vertex, and the high degree vertex has degree 6.

Exercise 6. Figure 2 is the map of earth from the polar viewpoint. Figure 3 is the map of earth from the equator viewpoint. Explain that they represent the same pentagonal tiling.

Exercise 7. Show that a tiling of sphere by 18 pentagons must have 3 vertices of degree 4 and no vertices of degree > 4. Then draw the picture of such a tiling.

Problem 1. Find all tilings with few high degree vertices. The next case beyond Theorem 1 is 3 high degree vertices of distance 3 from each other. The minimal example is given by Exercise 7.

Problem 2. Find all tilings such that high degree vertices are "evenly distributed". For example, each tile has exactly one high degree vertex. I have a construction for such tilings, and I believe my construction gives all such tilings.

Problem 3. Similar study for quadrilateral tilings.

Another way of constructing pentagonal tilings starts with any tiling (on any oriented surface, including the sphere). The tiles can be a mixture of various polygons. See the middle of Figure 4. Then we may divide each edge into three or four parts, and then construct the *pentagonal subdivision* (left of Figure 4, abbreviated PS) and the *double pentagonal subdivision* (right of Figure 4, abbreviated DPS).

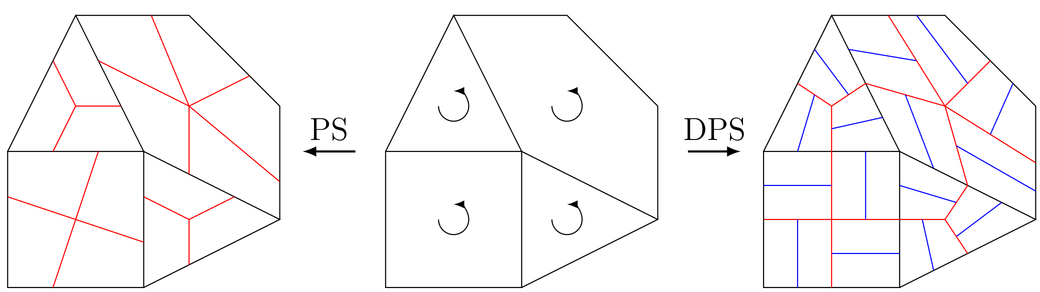

Figure 4: Pentagonal subdivision and double pentagonal subdivision.

In the pentagonal subdivision, each tile has the following vertices:

1. Three vertices obtained as partition points of the original (black) edges, or the vertices where black edges meet red edges. The three vertices have degree 3.

2. One vertex of the original tiling, or the vertex where the black edges meet. The degree of the vertex is the same as the original degree.

3. The center of a original tile, or the vertex where the red edges meet. If the original tile is an n-gon, then the degree of the vertex is n.

Exercise 8. Describe the degrees of the tiles in the double pentagonal subdivision.

Problem 4. Suppose a pentagonal tiling has the property that each tile has four degree 3 vertices and one high degree vertex. Show that the tiling is a pentagonal subdivision.

3 Edge and Angle

Two polygons are *congruent* if and only if one can be moved (and reflected if necessary) to another. This is equivalent to that the two polygons have the same edge lengths and angles. Here "same" means the same collection and the same arrangement.

As suggested by Theorem 2, we may consider putting edges and angles into three possible neighborhood tilings, such that all pentagons are congruent. If we ignore the angles and try to achieve *egde congruence*, then we get the following.

Proposition 5. *In a tiling of the sphere by (edge) congruent pentagons, the edges of the pentagon must be one of the five kinds:* $a^5, a^4b, a^3b^2, a^3bc, a^2b^2c$, *where* a, b, c *are distinct. Moreover, the edges are arranged in one of six ways.*

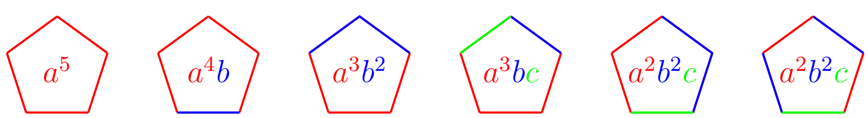

Figure 5: Six possible edge arrangements in the pentagon.

Exercise 9. Use Theorem 2 to prove that, in a spherical tiling by congruent pentagons, it is impossible for all five edges of the pentagon to have distinct lengths.

Exercise 10. Prove that, in a spherical tiling by congruent pentagons, if there is a tile with all five vertices having degree 3, then the sixth edge arrangement in Figure 5 is impossible.

The basic fact about angles is that the sum of angles (*angle sum*) at a vertex is 2π. Moreover, we have the angle sum equation for the pentagon (4π is the area of the sphere)

$$\sum(\text{five angles in pentagon}) - 3\pi = \text{Area(pentagon)} = \frac{4\pi}{f}. \tag{3.1}$$

It is also possible to count the distribution of angles in an *angle congruent* tiling.

Proposition 6. *In a tiling of sphere by angle congruent pentagons, if an angle appears at all degree* 3 *vertices, then the angle appears at least twice in the pentagon.*

Proposition 7. *In a tiling of sphere by angle congruent pentagons, if an angle* θ *does not appear at any degree* 3 *vertex, then there is only one such* θ*, and* θ *appears only once in the pentagon, and one of* $\alpha\theta^3, \theta^4, \theta^5$ *is a vertex.*

Exercise 11. Suppose an angle θ appears at every degree 3 vertex, and appears once in the pentagon. Then there are four non-θ angles, and the number of non-θ angles at each degree 3 vertex is ≤ 2. Use such estimations to prove Proposition 6.

4 Classification

Theorem 8. *If a tiling of the sphere by congruent pentagons has edge length* a^2b^2c*,* a^3bc*, or* a^3b^2*, then the tiling is either the pentagonal subdivision (with* $f = 12, 24, 60$*) or the double pentagonal subdivision (with* $f = 24, 48, 120$*) of platonic solids.*

The theorem is the combination of results in [4, 3, 7]. The three edge length combinations are called the cases of *variable edge lengths.*

In order for the pentagons in the pentagonal subdivision to be congruent, we need the tiles in the original tiling to be congruent regular polygons. This means that the original tiling should be platonic solids. It turns out that dual platonic solids give the same pentagonal subdivisions. Since tetrahedron is

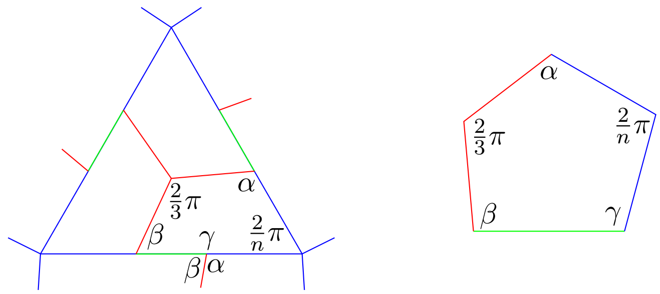

Figure 6: Pentagonal subdivision of platonic solid, $\alpha + \beta + \gamma = 2\pi$.

self dual, cube and octahedron are dual to each other, and dodecahedron and icosahedron are dual to each other, we get three families.

Figure 6 shows the pentagonal subdivision of the triangular tile in tetrahedron ($n = 3$), octahedron ($n = 4$), and icosahedron ($n = 5$). The only constraint is $\alpha + \beta + \gamma = 2\pi$. The pentagon allows two free parameters.

The double pentagonal subdivision is more complicated and allows no freedom. See [7].

In contrast to the variable edge length, the other extreme is the equilateral case, when all the edges have the same length.

Theorem 9. *There are exactly* 8 *tilings by congruent equilateral pentagons:*

- 3 *pentagonal subdivisions* ($f = 12, 24, 60$),
- 4 *earth map tilings* ($f = 16, 20, 24, 24$),
- 1 *special tiling* ($f = 20$).

The pentagonal subdivisions mean the reduced case $a = b = c$ of Theorem 8. The earth map tiling means the reduced case (red and black edges having the same length) of the first tiling in Figure 7.

The special tiling is given by Figure 8.

Progress have also been made for tilings by congruent pentagons with edge length a^4b (i.e., *almost equilateral* case). Besides the two tilings in Figure 7, we found more examples. But he classification is not yet complete.

Problem 5. Find all the earth map tilings with congruent almost equilateral pentagons.

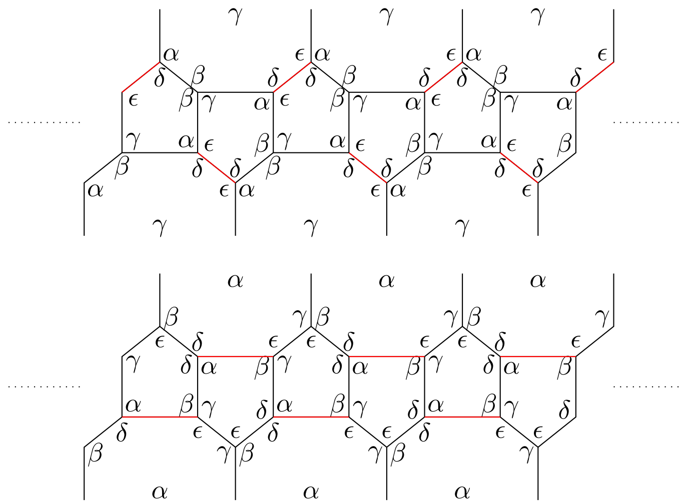

Figure 7: Almost equilateral earth map tiling.

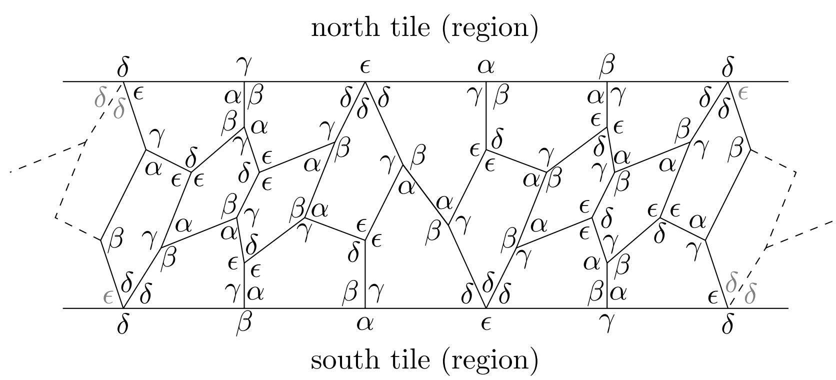

Figure 8: Special equilateral tiling.

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