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Spherical tiling by 12 congruent pentagons



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ABSTRACT

The tilings of the 2-dimensional sphere by congruent triangles have been extensively studied, and the edge-to-edge tilings have been completely classified. However, not much is known about the tilings by other congruent polygons. In this paper, we classify the simplest case, which is the edge-to-edge tilings of the 2-dimensional sphere by 12 congruent pentagons. We find one major class allowing two independent continuous parameters and four classes of isolated examples. The classification is done by first separately classifying the combinatorial, edge length, and angle aspects, and then combining the respective classifications together.

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1. Introduction

Tilings have been studied by mathematicians for more than 100 years. Some major achievements include the solution to Hilbert's 18th problem [5,14,17], the classification of wallpaper groups [11,15] and crystallographic groups [4,10,18], the classification of isohedral tilings of the plane [12], and the classification of edge-to-edge monohedral tilings of the 2-sphere by triangles [8,21,24]. We refer the reader for a more complete history of the subject to the excellent 1987 monograph by Grünbaum and Shephard [13].

In this paper, we study *monohedral* tilings, which are tilings that all tiles are geometrically congruent. For monohedral tilings of the plane by polygons, it is easy to see that any triangle or quadrilateral can be the tile. It is also known that convex tiles cannot have more than six edges, and there are exactly three classes of convex hexagonal tiles [16] and at least 14 classes of convex pentagonal tiles [13,20]. Our main result assumes straight line edges (actually great arcs) but not the convexity. Moreover, most of the other results in this paper allow any reasonably nice curves to be the edges.

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The tilings we study are also *edge-to-edge*, which means that no vertex of a tile lies in the interior of an edge of another tile. The assumption simplifies the classification of tiling patterns, or all the possible ways the tiles fit together to cover the whole sphere. It is easy to see that any triangle and quadrilateral can be the tile of an edge-to-edge monohedral tiling of the plane. Moreover, 8 of the 14 convex pentagon classes can be used for edge-to-edge tilings [3]. On the other hand, there is no general classification of plane tiling patterns, even for edge-to-edge and monohedral tilings by convex polygons. The known classifications often assume certain symmetry [12], or certain special geometric property of the tiles [22,23]. The difficulty with classifying the plane tiling patterns is that the number of tiles is infinite. By restricting to spherical tilings, the problem becomes finite and more manageable. Our classification does not assume any symmetry or special geometric property.

We also restrict the study to the tilings such that all vertices have *degree* ≥ 3 . This avoids some trivial examples obtained by artificially adding extra vertices to the edges, or getting new tilings by “wiggling modifications” of the edges. The assumption further simplifies the classification.

Now we come to the subject of this paper, the edge-to-edge monohedral tilings of the sphere by polygons, subject to degree ≥ 3 vertex condition. It is not hard to see that the tile can only be triangle, quadrilateral, or pentagon [25, Proposition 4]. The first work in this direction is Sommerville’s 1923 partial classification of edge-to-edge monohedral spherical tilings by triangles [21]. Davies [8] outlined a complete classification in 1967, and the complete proof was finally given by Ueno and Agaoka [24] in 2002. See [9] for the further study of monohedral non-edge-to-edge spherical tilings by triangles. As for quadrilateral spherical tilings, Ueno and Agaoka [25] showed that the classification can be very complicated. For works on the spherical tilings by special types of quadrilaterals, see [1,2].

As far as we know, there have been virtually no study of tilings of the sphere by congruent pentagons. However, we are of the opinion that the spherical pentagon tilings should be easier to study than the quadrilateral ones, because among triangle, quadrilateral and pentagon, pentagon is the “other extreme”. We feel that the spherical pentagon should be compared to the planar hexagon, and the spherical quadrilateral should be compared to the planar pentagon. To test our conviction, we classify the minimal case of the edge-to-edge monohedral tiling of the sphere by pentagons.

Main theorem. *Any edge-to-edge spherical tiling by 12 congruent pentagons must belong to one of the five classes in Fig. 1, subject to the condition that the sum of angles at any vertex is 2π .*

Here is how to read the tilings in Fig. 1: Combinatorially, the tilings are the dodecahedron, and are illustrated on the flat plane after punching a hole in one tile. The center tile shows the angles in a tile. The angles in the other tiles are determined by the location of δ and the angle arrangement orientation. As for the edge lengths, the usual solid lines have the same edge length a , the thick lines have the same length b , and the dotted lines have the same length c .

Although the classification contains five classes, only the fifth class is the essential one, because the other four classes only provide isolated examples (and perhaps the regular dodecahedron only). Fig. 2 is this essential class in its full glory.

The tile in the “essential class” T_5 is illustrated on the left of Fig. 3. The angle sum condition is

$$3\alpha = \beta + \gamma + \delta = 2\pi.$$

This implies that the area of the pentagonal tile is $\frac{\pi}{3}$, or one twelfth of the area of the sphere.

It is easy to see that the tile in T_5 allows two free parameters. Divide the tile as in the middle of Fig. 3. Since $\alpha = \frac{2\pi}{3}$ is already fixed, the area $A(a)$ of the triangle $\triangle ABC$ is completely determined by a . The function $A(a)$ is strictly increasing, and $A(b)$ is the area of $\triangle ADE$. When $A(a) + A(b) \leq \frac{\pi}{3}$ and $A(a) + A(b)$ is sufficiently close to $\frac{\pi}{3}$, we can always find a suitable angle $\angle CAD$ (equivalent to finding suitable $\delta = \angle BAE$) so that the area of the pentagon is $\frac{\pi}{3}$. This shows that we can freely choose a and b within certain range to get a tile that fits into the tiling, and the tile is completely determined by the choice.

Suppose we put the cube in Fig. 4 inside a sphere, so that the cube and the sphere have the same center. If we put a light source at the center, then the projection of the cube to the sphere is a T_5 class tiling with $\beta + \gamma = \delta = \pi$.

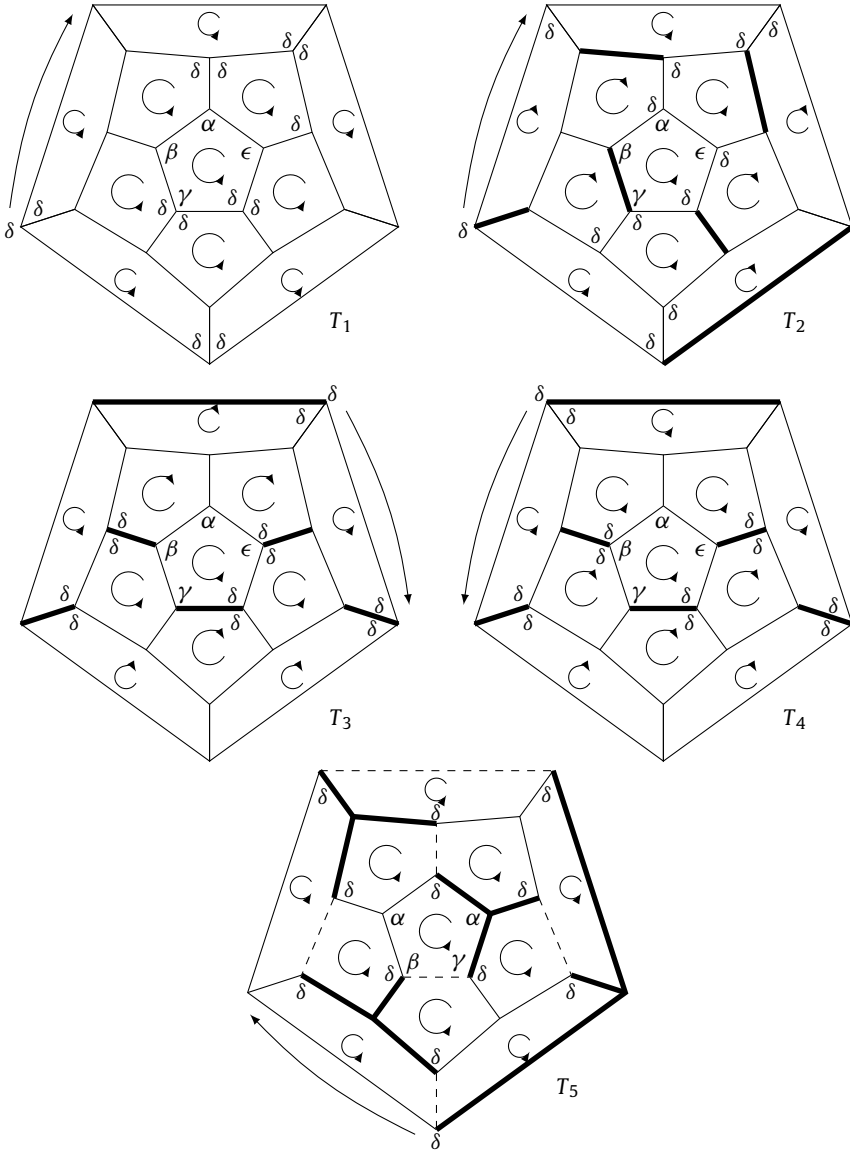


Fig. 1. All the possible spherical tilings by 12 congruent pentagons.

For the tiling class T_2 , the angle sum condition is $3\alpha = 2\beta + \epsilon = 2\gamma + \delta = \alpha + \delta + \epsilon = 2\pi$. This means

$$\alpha = \frac{2\pi}{3}, \quad \beta = \pi - \frac{1}{2}\epsilon, \quad \gamma = \frac{\pi}{3} + \frac{1}{2}\epsilon, \quad \delta = \frac{4\pi}{3} - \epsilon,$$

and should lead to only isolated examples. The reason is illustrated on the right of Fig. 3. Suppose a and ϵ are given. Fix a point A on the sphere and a direction for the edge AB . Starting from A and traveling along the given direction by distance a , we arrive at B . Turning an angle α at B and traveling by distance a , we arrive at C . Turning an angle ϵ at C and traveling by distance a , we arrive at D . Turning an angle $\delta = \frac{4\pi}{3} - \epsilon$ at D and traveling by distance a , we arrive at E . Turning

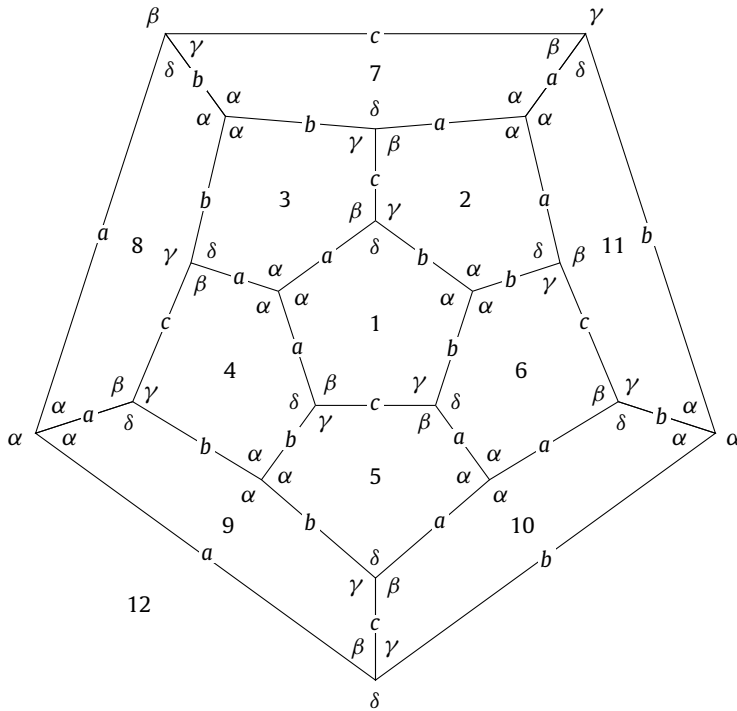


Fig. 2. The general class T_5 of spherical tilings by 12 congruent pentagons.

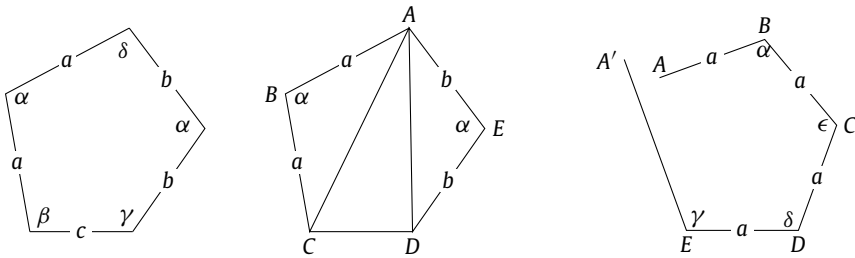


Fig. 3. The pentagonal tile.

an angle $\gamma = \frac{\pi}{3} + \frac{1}{2}\epsilon$ at E and traveling along, we get a line (actually a great arc) EA' . Then we want A to lie on the line EA' , which imposes one relation between a and ϵ , and reduces the number of free parameters to one. Once A lies on EA' , we further need the pentagon to have area $\frac{\pi}{3}$, which is the same as $\angle EAB = \beta = \pi - \frac{1}{2}\epsilon$. This imposes one more relation and we are left with no free parameter. Therefore the pentagonal tiles fitting into the tiling class T_2 must appear in isolated way.

Similar discussion can be made for T_3 and T_4 . For T_1 , we have even one less degree of freedom. The problem is crystalized below (the angles are renamed in four cases in order to get the same expressions for the angle sum equations).

Problem. Find spherical pentagons in Fig. 5 satisfying $3\alpha = 2\beta + \gamma = 2\delta + \epsilon = \alpha + \gamma + \epsilon = 2\pi$. Are there any examples besides the pentagon in the regular dodecahedron?

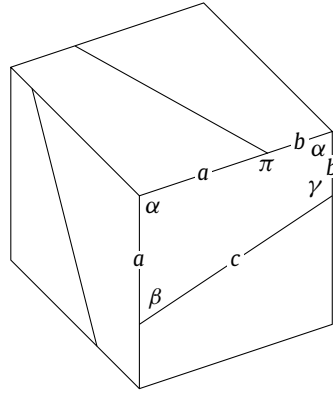


Fig. 4. A spherical pentagon tiling.

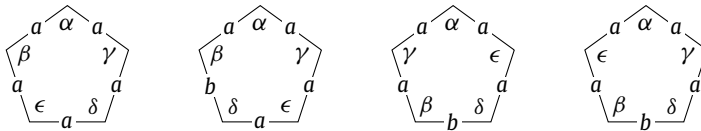


Fig. 5. Must these spherical pentagons be regular?

To prove the main theorem, we first separate and classify different aspects of tilings. At the most crude level, we study tilings by combinatorially congruent tiles in Section 2, which means that all tiles have the same number of edges, and edge lengths and angles are ignored. Proposition 5 says that the dodecahedron is the unique spherical tiling by 12 combinatorially congruent pentagons. Combinatorial tilings have been extensively studied. See [19] for a recent survey. Our study is actually a topological one, in the sense that the lines do not even need to be straight. See [26] for our further work for the non-minimal case.

In Sections 3 and 4, we study tilings by edge congruent tiles, which means that all tiles have the same edge length combination and arrangement. Propositions 9, 10, 11, 12 (plus the case all edges have the same length) completely classify spherical tilings by 12 edge congruent pentagons. Our further work in the non-minimal case can be found in [6].

In Sections 5 and 6, we study tilings by angle congruent tiles, which means that all tiles have the same angle combination and arrangement. Proposition 17 classifies the numerics in spherical tilings by 12 angle congruent pentagons. Propositions 18, 19, 20 then further classify the locations of the angles in three major cases. The complete classification would involve another major but rather complicated case. Since this case will not be needed in the proof of the main classification theorem, we only present some examples instead of the whole proof.

In Section 7, we combine the edge congruent and angle congruent classifications to prove the main theorem. While our edge congruent and angle congruent results do not require geometry, in that the edges are not necessarily great arcs, in combining the two, we use geometry to cut down the possible number of edge–angle combinations. So our main theorem does require the edges to be great arcs. For the non-minimal case, see [7] for further partial results.

Finally, we would like to thank the referees for the careful reading of the paper and many helpful suggestions.

2. Combinatorial conditions

In this section, we study the combinatorial aspect of tilings. This means that we ignore the edge lengths and the angles, and only require the tiles to have the same number of edges. Most results of the section are well known.

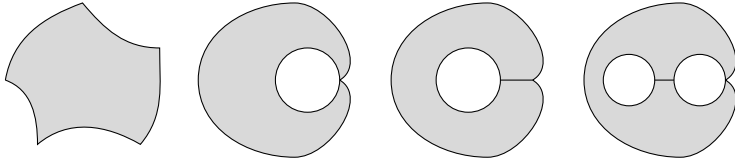


Fig. 6. Tiles.

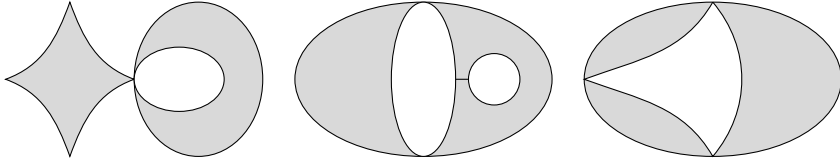


Fig. 7. Not tiles.

The results certainly apply to the edge-to-edge tilings as defined in [13]. However, some results can be applied to more general tilings. Specifically, we consider a connected graph nicely embedded in the sphere. We assume that all vertices in the graph have degree ≥ 3 . The interior of a tile is a connected component of the complement of the graph. A tile is the closure of its interior. The boundary of a tile is what is between the tile and its interior, and is a union of some edges in the graph.

The exclusion of vertices of degree 2 means that we do not view an n -gon as an $(n + k)$ -gon by artificially choosing another k points on the boundary as extra vertices. Therefore all tiles have naturally defined vertices and edges that are already given in the graph, and the tiling is naturally edge-to-edge.

Fig. 6 shows some possible tiles. The first one does not have any boundary points identified. The others have some boundary points (even boundary intervals) identified. In case some boundary intervals are identified, the interior of the tile is different from the topological version of the interior.

Fig. 7 shows some examples that are not tiles. They are unions of several tiles at finitely many boundary points.

Lemma 1. *In any (naturally generated) spherical tiling, there is at least one tile for which no boundary points are identified.*

Proof. We call a closed subset of the sphere a *simple region* if it is enclosed by a nicely embedded simple closed curve. The boundary points of a simple region are not identified. The lemma basically says that some tile is a simple region.

It is a topological fact that if R is the whole sphere or a simple region, and P is a tile in R with some boundary points identified, then the closure of some connected component of $R - P$ is a simple region.

Let P_1 be a tile, such that some boundary points are identified. Then the complement $S^2 - P_1$ has a connected component C_1 , such that the closure \bar{C}_1 is a simple region, and is actually a union of tiles. If \bar{C}_1 is a tile, then we are done. Otherwise, \bar{C}_1 contains a tile P_2 as a proper subset. If P_2 is a simple region, then we are also done. If P_2 is not a simple region, then some boundary points of P_2 are identified, and the complement $\bar{C}_1 - P_2$ has a connected component C_2 , such that the closure \bar{C}_2 is a simple region. Keep going. As long as we do not encounter tiles that are simple regions along the way, we find a strictly decreasing sequence of simple regions $\bar{C}_1 \supset \bar{C}_2 \supset \bar{C}_3 \supset \dots$. Since there are only finitely many tiles in the tiling, the sequence must stop. This means that some tile found in the process must be a simple region. \square

An immediate consequence of the lemma is that in case all tiles are congruent (full congruence, including edge lengths and angles), the boundary points of any tile are never identified. Since the property is needed in the subsequent discussion, we include it in the following definition.

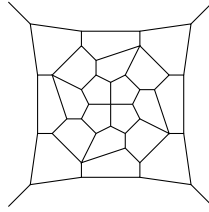


Fig. 8. Every tile has a vertex of degree > 3 .

Definition 2. A tiling is *combinatorially monohedral* if all tiles have the same number of edges and every tile has no boundary points identified. The tiles in such a tiling are *combinatorially congruent*.

Let f, e, v be the numbers of tiles (or faces), edges and vertices in a combinatorially monohedral tiling. Let n be the number of edges in a tile. Then we have the Euler equation and the Dehn–Sommerville equation

$$v - e + f = 2, \quad 2e = nf.$$

Note that the Dehn–Sommerville equation assumes that each edge is shared by two *different* faces, which follows from the condition that no boundary points of a tile are identified. (Actually we only need that no boundary intervals are identified.) Moreover, if v_i is the number of vertices of degree i , then

$$v = v_3 + v_4 + v_5 + \dots, \quad 2e = 3v_3 + 4v_4 + 5v_5 + \dots.$$

It is a simple consequence of all these equations that $n = 3, 4, 5$. See Proposition 4 of [25].

Now we concentrate on the case $n = 5$. The equations become

$$2e = 5f, \quad 2v = 3f + 4, \quad v_3 - 20 = 2v_4 + 5v_5 + 8v_6 + \dots. \tag{2.1}$$

Lemma 3. Any spherical tiling by combinatorially congruent pentagons contains a tile in which at least four vertices have degree 3.

Fig. 8 gives a combinatorial pentagon tiling of the sphere in which each tile has a degree 4 vertex.

Proof. Let t_j be the number of degree 3 vertices in the j th tile. Then $3v_3 = \sum t_j$. On the other hand, we have

$$\begin{aligned} 3f &= 2v - 4 = 2v_3 + 2v_4 + 2v_5 + 2v_6 + \dots - 4 \\ &\leq 2v_3 + 2v_4 + 5v_5 + 8v_6 + \dots - 4 \\ &= 2v_3 + (v_3 - 20) - 4 < 3v_3 = \sum t_j. \end{aligned}$$

This implies that some $t_i > 3$. \square

Lemma 4. A spherical tiling by combinatorially congruent pentagons has at least 12 tiles. Moreover, the minimum 12 is reached if and only if all vertices have degree 3, and if and only if the number of vertices is 20.

Proof. The third equation in (2.1) implies that $v \geq v_3 \geq 20$, and $v = 20$ if and only if all vertices have degree 3. On the other hand, by the second equation, $v \geq 20$ is the same as $f \geq 12$, and $v = 20$ is the same as $f = 12$. \square

Proposition 5. The spherical tiling by 12 combinatorially congruent pentagons is uniquely given by Fig. 9.

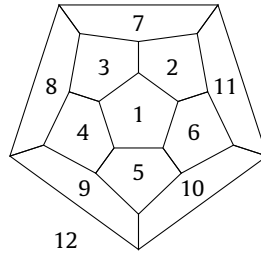


Fig. 9. Combinatorial structure of the minimal pentagon tiling.

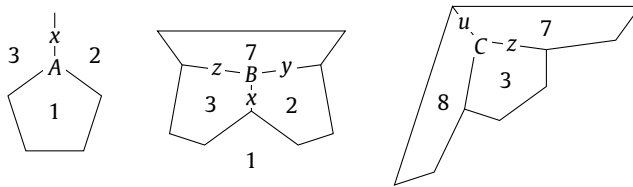


Fig. 10. Neighborhood of degree 3 vertex.

Proof. By Lemma 4, all vertices have degree 3.

We start with a tile P_1 . Each of the five edges of P_1 is shared by P_1 and exactly another tile. We denote these tiles by P_2, P_3, P_4, P_5, P_6 and assume that they are arranged in counterclockwise order.

Consider the vertex A of P_1 on the left of Fig. 10. It is also a vertex of P_2 and P_3 . Since A has degree 3, besides the two edges of P_1 connected to A , there is exactly one more edge x also connected to A . The three edges divide the neighborhood of A into three corners, each belonging to one tile. This local picture implies that x is a common edge of P_2 and P_3 . The similar situation happens between P_3 and P_4 , between P_4 and P_5 , between P_5 and P_6 , and between P_6 and P_2 . Therefore the first six tiles are glued together in the way described in Fig. 9. Of course this does not yet exclude the possibility that additional identification may happen among these tiles. But at least the relations described in Fig. 9 already exist between these tiles.

The edge x has another end vertex B besides A . The three edges connected to B are x, y, z , where y is an edge of P_2 adjacent to x and z is an edge of P_3 adjacent to x . See middle of Fig. 10. Now y is shared by P_2 and another tile P_7 , and z is shared by P_3 and another tile P'_7 . The three edges x, y, z divide the neighborhood of B into three corners, each belonging to one tile. Since two corners already belong to P_2 and P_3 , the third corner belongs to the same tile. This implies that $P_7 = P'_7$. Thus the 7th tile is established and is related to P_2 and P_3 as in Fig. 9. Similarly, P_8 can be defined from P_3 and P_4 , P_9 can be defined from P_4 and P_5 , P_{10} can be defined from P_5 and P_6 , and P_{11} can be defined from P_6 and P_2 .

The edge z has another end vertex C besides B . It is also a vertex of P_7 and P_8 . There are three edges connected to C . Two are edges of P_3 and the third one is u on the right of Fig. 10. The situation is the same as the left of Fig. 10, with P_3, P_7, P_8, C, u in place of P_1, P_2, P_3, A, x . We get the similar conclusion that the edge u is shared by P_7 and P_8 . We can deduce similar relations between P_8 and P_9 , etc. Then the tiles P_1 through P_{11} are glued together as in Fig. 9.

After constructing the first eleven tiles, we find that $P_7, P_8, P_9, P_{10}, P_{11}$ have one “free” edge each. The free edge of P_7 is shared by P_7 and another tile P_{12} . The free edge of P_8 is shared by P_8 and another tile P'_{12} . The situation is similar to the relation between P_2, P_3, P_7, P'_7 as described in the middle of Fig. 10. We get the similar conclusion that $P_{12} = P'_{12}$. Therefore the five possible 12th tiles are all the same.

Now we find twelve tiles that are related as in Fig. 9. If there are more tiles, then the additional tiles have to be glued to the existing twelve along some vertices or edges. This will either increase the degree of some vertex to be > 3 , or cause some edge to be shared by more than two tiles. Therefore the twelve are all the tiles in the tiling. Moreover, if there are additional identifications

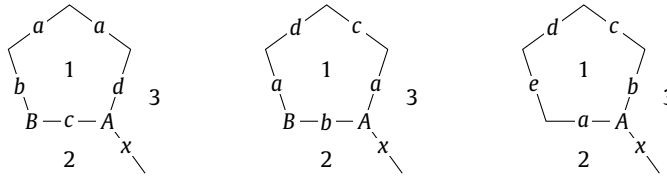


Fig. 11. Impossible edge length arrangements.

among these twelve tiles, then the identification of vertices will increase the degree of some vertex to be > 3 , the identification of edges will cause some edge to be shared by more than two tiles, and the identification of tiles will reduce the total number of tiles to be < 12 . Since all these should not happen, there is no more additional identification. We conclude that Fig. 9 gives the complete description of the tiling. \square

3. Edge congruent tiling

We study the distribution of edge lengths in a spherical tiling by congruent pentagons. Since angles are ignored, we introduce the following concept.

Definition 6. Two polygons are *edge congruent*, if there is a correspondence between the edges, such that the adjacencies of the edges are preserved and the edge lengths are preserved.

Two spherical (or planar) triangles with great arc (or straight line) edges are congruent if and only if they are edge congruent. However, this is no longer true for quadrilaterals and pentagons. Moreover, our results actually do not even require the edges to be great arcs (or straight).

Proposition 7. Let a, b, c, \dots , denote distinct numbers. Then in a spherical tiling by edge congruent pentagons, the edge lengths of the tile must form one of the following five combinations:

$$\begin{aligned}
 a^5 &= \{a, a, a, a, a\}, & a^4b &= \{a, a, a, a, b\}, \\
 a^3b^2 &= \{a, a, a, b, b\}, & a^3bc &= \{a, a, a, b, c\}, & a^2b^2c &= \{a, a, b, b, c\}.
 \end{aligned}$$

Proof. By purely numerical consideration, there are seven possible combinations of five numbers. What we need to show is that the combinations

$$a^2bcd = \{a, a, b, c, d\}, \quad abcde = \{a, b, c, d, e\},$$

are impossible. Up to symmetry, all the possible edge length arrangements of these two combinations are listed in Fig. 11.

In the first arrangement, by Lemma 3, one of the vertices A or B must have degree 3. By symmetry, we may assume $\deg A = 3$ without loss of generality. Let x be the only other edge at A besides c and d . Then we have tiles P_2 and P_3 described in the picture. The edge x is adjacent to an edge of length c in P_2 . Since P_2 is edge congruent to P_1 , and the edges adjacent to the unique edge of length c in P_1 are the edges of lengths b and d . Therefore, we have $x = b$ or $x = d$ as far as the edge length is concerned. Similarly, the adjacency of x and d in P_3 leads to $x = a$ or $x = c$ as far as the edge length is concerned. Since a, b, c, d are distinct, we get a contradiction.

We have been very careful with the wording in the argument above. We say “an edge of length c ” instead of “the edge c ” because this allows the possibility that the tile may have several edges of the same length c . This may happen when a, b, c, \dots are not assumed to be distinct. Since distinct letters indeed denote distinct values for the length in the current proposition, there is actually no confusion about the wording “the edge c ”. The wording in the subsequent discussion will be more casual.

In the second arrangement, by Lemma 3 and symmetry, we may still assume $\deg A = 3$. Then we may introduce x, P_2, P_3 as before. By considering x in P_2 and P_3 , we see that x is adjacent to a and b . Since there is no such edge in P_1 , we get a contradiction.

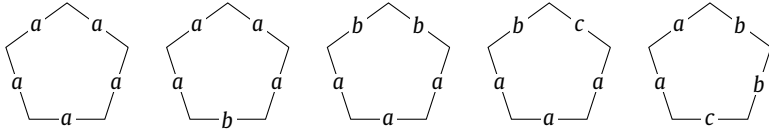


Fig. 12. Edge length arrangements when all vertices have degree 3.

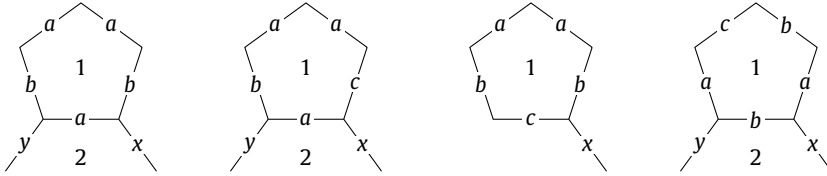


Fig. 13. Impossible edge length arrangements.

In the third arrangement, we may again assume $\deg A = 3$ and introduce x, P_2, P_3 . Then x is adjacent to both a and b from the viewpoint of P_2 and P_3 , yet there is no such edge in P_1 . \square

Proposition 8. *If a spherical tiling by edge congruent pentagons contains a tile with all vertices having degree 3, then the edge lengths must be arranged in one of the five ways in Fig. 12.*

Proof. By purely numerical consideration, the five possible edge length combinations $a^5, a^4b, a^3b^2, a^3bc, a^2b^2c$ in Proposition 7 have respectively 1, 1, 2, 2, 3 possible arrangements. Besides the five in Fig. 12, the remaining four are given in Fig. 13. We need to argue that these four are impossible.

In the first arrangement, x and y are adjacent to a and b . Since the only such edge in P_1 is a , we have $x = y = a$. Then in P_2 , we get a row xay of three consecutive edges of the same length a . Since there is no such row in P_1 , we get a contradiction.

In the second arrangement, x is adjacent to c , and y is adjacent to b . By the adjacency in P_1 , we have $x = y = a$. Then the edge row $xyb = aaa$ appears in P_2 but not in P_1 . We get a contradiction.

In the third arrangement, x is adjacent to b and c . Since there is no such edge in P_1 , we get a contradiction.

In the fourth arrangement, x and y are adjacent to a and b . By the adjacency in P_1 , we have $x = y = c$. Then c appears twice in P_2 but only once in P_1 . We get a contradiction. \square

4. Classification of edge congruent tiling

The discussion in Section 3 is local, because we ignored the global combinatorial structure given by Propositions 5. In this section, we discuss how the classification in Proposition 8 can fit the global structure.

We denote the tiles in Fig. 9 by P_1, P_2, \dots, P_{12} . We denote by E_{ij} the edge shared by the tiles P_i, P_j , and by V_{ijk} the vertex shared by P_i, P_j, P_k .

We also name an edge by its length and a vertex by the length of the edges at the vertex. For example, in Fig. 2, the edge E_{12} is a b -edge, and the edge E_{13} is an a -edge. Moreover, the vertex V_{123} is an abc -vertex, and the vertex V_{134} is an a^3 -vertex. The tiling has total of 12 a -edges, 12 b -edges, 6 c -edges, 4 a^3 -vertices, 4 b^3 -vertices, and 12 abc -vertices. We denote the *edgewise vertex combination* of the tiling by $\{4a^3, 4b^3, 12abc\}$.

For the edge length combination a^5 (the first pentagon in Fig. 12), we basically assign a to all edges in Fig. 9. All vertices are a^3 -vertices, and the edgewise vertex combination is $\{20a^3\}$. This can be realized by the regular dodecahedron tiling.

Proposition 9. *The spherical tilings by 12 edge congruent pentagons with edge lengths a, a, a, a, b , where $a \neq b$, are given by the tilings in Fig. 14 up to symmetries. The unlabeled edges have length a .*

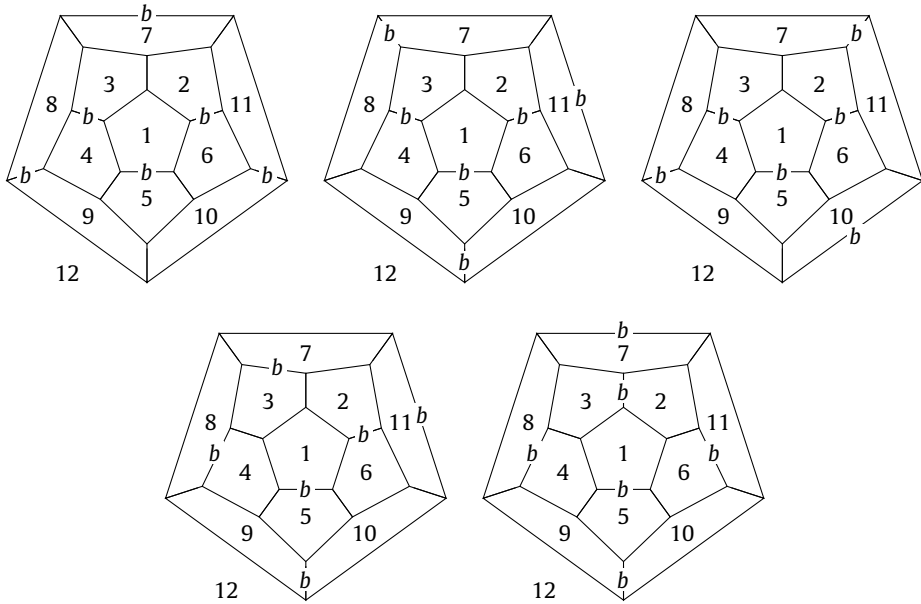


Fig. 14. Tilings for the edge length combination a^4b .

In proving the main theorem, we will only use the edgewise vertex combination $\{8a^3, 12a^2b\}$ derived in the proof.

Proof. We need to assign 6 b -edges to Fig. 9, such that each tile has exactly one b -edge. In such an assignment, there is no ab^2 -vertex and no b^3 -vertex. So the edgewise vertex combination is $\{ma^3, na^2b\}$. On the other hand, $\frac{1}{5}$ of all the 30 edges are b -edges. Therefore there are $N_a = 24$ a -edges and $N_b = 6$ b -edges. Then from the edgewise vertex combination, we get $3m + 2n = 2N_a = 48$ and $n = 2N_b = 12$. The solution is $m = 8$ and $n = 12$.

Any tile P in Fig. 9 and the five tiles around P form a tiling T_P of a “half sphere” (with wiggled boundary). Moreover, the remaining six tiles also form a tiling of the complementary “half sphere”. In fact, P has an “antipodal tile” P' (the antipodal of P_1 is P_{12} , for example), and the remaining six tiles form the tiling $T_{P'}$.

We say an assignment of b -edges is *splitting*, if there is a half sphere tiling T_P , such that if a b -edge is shared by P_i and P_j , then $P_i \in T_P$ if and only if $P_j \in T_P$. This implies that $P_i \in T_{P'}$ if and only if $P_j \in T_{P'}$. We also call the tile P a *splitting center*.

A splitting assignment is then the union of two independent assignments of 3 b -edges to T_P and the other 3 b -edges to $T_{P'}$. In this case, it is easy to see that the choice of the b -edge for P completely determines the other two b -edges in T_P . See T_{P_1} in the first three tilings in Fig. 14. Up to symmetry, we have three possible combinations of the choice of b -edges for P_1 and P_{12} , which give the first three tilings in Fig. 14. They are not equivalent because they have respectively 6, 2 and 4 splitting centers.

Next we search for non-splitting assignments. Assume there are edges connecting two b -edges. Without loss of generality, we may assume $E_{15} = E_{26} = b$ as in the fourth tiling in Fig. 14. Since P_1 and P_6 are not splitting centers, we get $E_{34} \neq b$ and $E_{10\bar{11}} \neq b$. If $E_{7\bar{11}} = b$, then the only possible b -edge of P_3 is $E_{38} = b$. This further implies that the b -edge of P_4 is $E_{49} = b$. Then P_4 becomes a splitting center. Therefore $E_{7\bar{11}} \neq b$, and the only possible b -edge of P_{11} is $E_{\bar{11}\bar{12}} = b$. This further successively implies that the b -edge of P_{10} is $E_{9\bar{10}}$, the b -edge of P_4 is E_{48} , and $E_{37} = b$.

Finally we look for non-splitting assignments such that no edge connects two b -edges. Without loss of generality, we may assume $E_{15} = b$. Since $E_{26}, E_{34}, E_{49}, E_{6\bar{10}}$ are connected to E_{15} by one edge, they are not b -edges. This implies that $E_{48} = E_{6\bar{11}} = b$. Then the 8 edges connected to E_{48} and

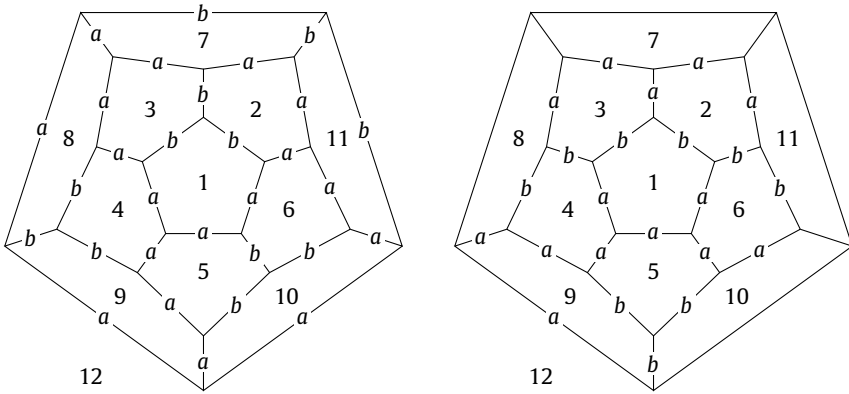


Fig. 15. Tilings for the edge length combination a^3b^2 .

$E_{6\bar{1}\bar{1}}$ by one edge cannot be b -edges. This is enough for us to determine the remaining 3 b -edges and get the fifth tiling in Fig. 14. \square

Proposition 10. *The spherical tiling by 12 edge congruent pentagons with edge lengths a, a, a, b, b , where $a \neq b$, is given by the left of Fig. 15 up to symmetries.*

Proof. By Proposition 8, we may start with P_1 , with the edges arranged as the third in Fig. 12. See Fig. 15. The length of E_{23} is either a or b .

The case $E_{23} = b$ is described on the left of Fig. 15. From $E_{12} = E_{23} = b$, we see that the other three edges of P_2 have length a . By the same reason, we get all the edge lengths of P_3 . Then among E_{78} and $E_{7\bar{1}\bar{1}}$, one is a and the other is b . By symmetry, we may assume $E_{78} = a$ and $E_{7\bar{1}\bar{1}} = b$. Then $E_{7\bar{1}\bar{2}} = b$. From $E_{2\bar{1}\bar{1}} = a, E_{7\bar{1}\bar{1}} = b$, we get all the edges of P_{11} . From $E_{16} = E_{26} = E_{6\bar{1}\bar{1}} = a$, we get all the edges of P_6 . From $E_{6\bar{1}\bar{0}} = b, E_{\bar{1}\bar{0}\bar{1}\bar{1}} = a$, we get all the edges of P_{10} . From $E_{56} = E_{5\bar{1}\bar{0}} = b$, we get all the edges of P_5 . From $E_{14} = E_{34} = E_{45} = a$, we get all the edges of P_4 . From $E_{49} = b, E_{59} = a$, we get all the edges of P_9 . From $E_{38} = a, E_{48} = b$, we get all the edges of P_8 . This concludes all the edge lengths.

The case $E_{23} = a$ is the right of Fig. 15. We can immediately get all the edges of P_2, P_3 . Afterwards, we get all the edges of P_4, P_6 . Now we know three edges of P_5 to be a , so that the other two edges are b . Then we get all the edges of P_9 , and find that P_8 has one b -edge adjacent to two a -edges, a contradiction. \square

Proposition 11. *The spherical tiling by 12 edge congruent pentagons cannot have edge lengths a, a, a, b, c , for distinct a, b, c .*

Proof. By Proposition 8, we may start with P_1 , with the edges arranged as the fourth in Fig. 12. See Fig. 16. Since E_{23} is adjacent to b and c , we get $E_{23} = a$. Then we successively get all the edges of $P_2, P_3, P_4, P_6, P_8, P_{11}$. Now we have four a -edges in P_7 , a contradiction. \square

Proposition 12. *The spherical tiling by 12 edge congruent pentagons with edge lengths a, a, b, b, c , where a, b, c are distinct, is given by the left of Fig. 17 up to symmetries.*

Proof. By Proposition 8, we may start with P_1 , with the edges arranged as the fifth in Fig. 12. See Fig. 17. Since E_{45} is adjacent to c , its length is either a or b .

The case $E_{45} = b$ is the left of Fig. 17. From $E_{14} = a, E_{15} = c, E_{45} = b$, we get all the edges of P_4, P_5 . Then we further get all the edges of P_6 . Since we cannot have three a -edges or three b -edges in a tile, we must have $E_{23} = c$ and then get all the edges of P_2, P_3 . By the same reason, we get

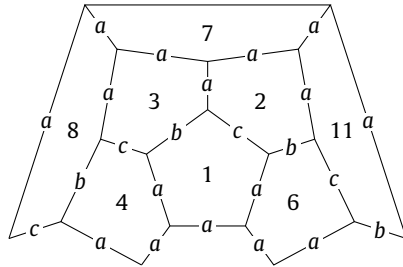


Fig. 16. Tilings for the edge length combination a^3bc .

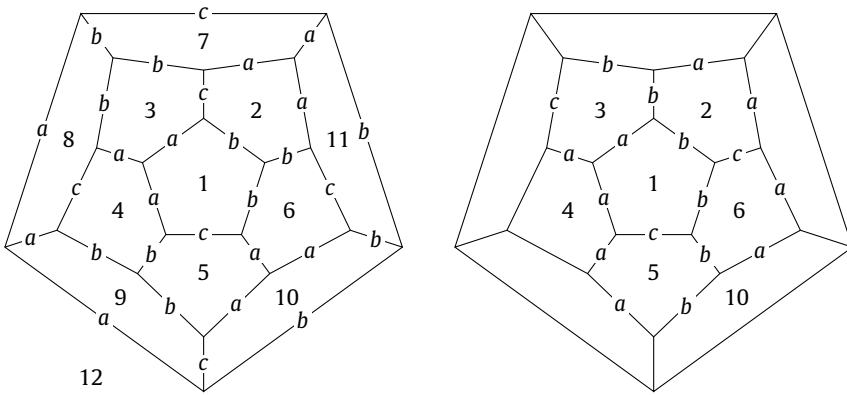


Fig. 17. Tilings for the edge length combination a^2b^2c .

$E_{9\overline{10}} = c$ and all the edges of P_9, P_{10} . Then we may get all the edges of P_7, P_8, P_{11} . This concludes all the edge lengths.

The case $E_{45} = a$ is the right of Fig. 17. We immediately get all the edges of P_5 . Then we have $E_{26} = a$ or c in P_6 . Either case implies $E_{23} = b$ in P_2 . Then we get all the edges of P_3 and find three a -edges in P_4 , a contradiction. \square

5. Angle congruent tiling

Now we turn to the distribution of angles in a spherical tiling by congruent pentagons. Since edge lengths are ignored, we introduce the following concept.

Definition 13. Two polygons are *angle congruent*, if there is a correspondence between the edges, such that the adjacencies of edges are preserved and the angles between the adjacent edges are preserved.

Two spherical triangles with great arc edges are congruent if and only if they are angle congruent. Two planar triangles with straight line edges are similar if and only if they are angle congruent. However, this is no longer true for quadrilaterals and pentagons.

In the regular dodecahedron tiling, all the angles are

$$\alpha = \frac{2\pi}{3}.$$

In what follows, we will always reserve α for this special angle, and let β, γ, \dots be angles not equal to α . In this section, we may occasionally allow some from β, γ, \dots to be equal. In the later sections, we will assume β, γ, \dots to be distinct.

Lemma 14. *In a spherical tiling by 12 angle congruent pentagons, the sum of five angles in a tile is $\frac{10\pi}{3} = 5\alpha$.*

In case the edges are great arcs, the lemma is a consequence of the formula for the area of a spherical pentagon. However, the lemma holds even if the edges are not great arcs.

Proof. The sum of angles at each vertex is 2π . By Lemma 4, there are 20 vertices. Therefore the total sum of all the angles in the tiling is 40π . On the other hand, the sum Σ of five angles in a tile is the same for all tiles because of the angle congruence. Since there are 12 tiles, we get $40\pi = 12\Sigma$. This implies $\Sigma = \frac{10\pi}{3}$. \square

In Section 4, we named a vertex by the lengths of edges at the vertex. Similarly, we can also name a vertex by the angles at the vertex. For example, in Fig. 2, V_{123} is a $\beta\gamma\delta$ -vertex, and V_{126} is an α^3 -vertex. The tiling has 8 α^3 -vertices and 12 $\beta\gamma\delta$ -vertices, and has the *anglewise vertex combination* $\{8\alpha^3, 12\beta\gamma\delta\}$. We also say that α^3 and $\beta\gamma\delta$ are vertices in the tiling, and the other combinations are not vertices.

We say a vertex is of $\alpha\beta\gamma$ -type, if the vertex is an $\alpha\beta'\gamma'$ -vertex, where β' and γ' may or may not be β and γ . Thus an $\alpha\beta\gamma$ -type vertex can be an $\alpha\beta\gamma$ -vertex, or an $\alpha\beta\delta$ -vertex, or an $\alpha\gamma\delta$ -vertex, etc. From purely symbolical viewpoint (recall that the symbol α is special), a degree 3 vertex must be one of the following types:

$$\alpha^3, \alpha^2\beta, \alpha\beta^2, \alpha\beta\gamma, \beta^3, \beta^2\gamma, \beta\gamma\delta.$$

Lemma 15. *A vertex of degree 3 must be one of the following types:*

1. α^3 .
2. $\alpha\beta\gamma$ with $\beta \neq \gamma$.
3. $\beta^2\gamma$ with $\beta \neq \gamma$.
4. $\beta\gamma\delta$ with β, γ, δ distinct.

Proof. We need to show that, if $\alpha \neq \beta$, then $\alpha^2\beta, \alpha\beta^2, \beta^3$ cannot be vertices. If $\alpha^2\beta$ is a vertex, then $2\alpha + \beta = 2\pi = 3\alpha$, which implies $\beta = \alpha$, a contradiction. If $\alpha\beta^2$ is a vertex, then $\alpha + 2\beta = 2\pi = 3\alpha$, which again implies $\beta = \alpha$. If β^3 is a vertex, then $3\beta = 2\pi = 3\alpha$, which still implies $\beta = \alpha$. \square

Lemma 16. *Suppose $\beta, \gamma, \delta, \epsilon$ are distinct angles.*

1. If $\beta^2\gamma$ is a vertex, then $\beta^2\delta, \beta\gamma^2, \beta\gamma\delta$ are not vertices.
2. If $\beta\gamma\delta$ is a vertex, then $\beta^2\gamma, \beta\gamma\epsilon$ are not vertices.

The lemma actually allows some angles to be α . Moreover, we get more exclusions by symmetry. For example, if $\beta\gamma\delta$ is a vertex, then the following cannot be vertices:

$$\beta^2\gamma, \beta^2\delta, \beta\gamma^2, \beta\delta^2, \gamma^2\delta, \gamma\delta^2, \beta\gamma\epsilon, \beta\delta\epsilon, \gamma\delta\epsilon.$$

Proof. If both $\beta^2\gamma$ and $\beta^2\delta$ are vertices, then $2\beta + \gamma = 3\alpha$ and $2\beta + \delta = 3\alpha$. This implies $\gamma = \delta$, a contradiction.

If both $\beta^2\gamma$ and $\beta\gamma^2$ are vertices, then $2\beta + \gamma = 3\alpha$ and $\beta + 2\gamma = 3\alpha$. This implies $\beta = \gamma$, a contradiction.

If both $\beta^2\gamma$ and $\beta\gamma\delta$ are vertices, then $2\beta + \gamma = 3\alpha$ and $\beta + \gamma + \delta = 3\alpha$. This implies $\beta = \delta$, again a contradiction.

The proof of the second statement is similar. \square

Proposition 17. *Let $\alpha = \frac{2}{3}\pi$ and let $\alpha, \beta, \gamma, \dots$ denote distinct angles. Then a spherical tiling by 12 angle congruent pentagons must have one of the following (unordered) angle combinations:*

1. α^5 : The angles in the pentagon are $\alpha, \alpha, \alpha, \alpha, \alpha$. The anglewise vertex combination is $\{20\alpha^3\}$.
2. $\alpha^3\beta\gamma$: The angles in the pentagon are $\alpha, \alpha, \alpha, \beta, \gamma$, satisfying $\beta + \gamma = 2\alpha$. The anglewise vertex combination is $\{8\alpha^3, 12\alpha\beta\gamma\}$.
3. $\alpha^2\beta^2\gamma$: The angles in the pentagon are $\alpha, \alpha, \beta, \beta, \gamma$, satisfying $2\beta + \gamma = 3\alpha$. The anglewise vertex combination is $\{8\alpha^3, 12\beta^2\gamma\}$.
4. $\alpha^2\beta\gamma\delta$: The angles in the pentagon are $\alpha, \alpha, \beta, \gamma, \delta$, satisfying $\beta + \gamma + \delta = 3\alpha$. The anglewise vertex combination is $\{8\alpha^3, 12\beta\gamma\delta\}$.
5. $\alpha\beta\gamma\delta\epsilon$: The angles in the pentagon are $\alpha, \beta, \gamma, \delta, \epsilon$, satisfying $\gamma = 3\alpha - 2\beta, \delta = 2\alpha - \beta, \epsilon = 2\beta - \alpha$. There are three possible anglewise vertex combinations: $\{4\alpha\beta\delta, 8\alpha\gamma\epsilon, 4\beta^2\gamma, 4\delta^2\epsilon\}, \{\alpha^3, 2\alpha\beta\delta, 7\alpha\gamma\epsilon, 5\beta^2\gamma, 5\delta^2\epsilon\}, \{2\alpha^3, 6\alpha\gamma\epsilon, 6\beta^2\gamma, 6\delta^2\epsilon\}$.

The fourth case degenerates into the second (when $\delta = \alpha$) and the third (when $\delta = \beta$) cases, which further degenerate into the first case. The tiling T_5 in Fig. 1 and Fig. 2 is the fourth case and its degenerate cases. The tilings T_1, T_2, T_3, T_4 in Fig. 1 belong to the third anglewise vertex combination in the fifth case.

Proof. We divide the proof by considering how many times the angle α appears in the pentagon.

Case 1. If the angles in the pentagon are $\alpha, \alpha, \alpha, \alpha, \beta$, then Lemma 14 tells us $4\alpha + \beta = 5\alpha$, so that the angles are really $\alpha, \alpha, \alpha, \alpha, \alpha$. This is the first case in the proposition.

Case 2. The angles in the pentagon are $\alpha, \alpha, \alpha, \beta, \gamma$, with $\beta, \gamma \neq \alpha$. Note that β, γ are not yet assumed to be distinct.

Lemma 14 becomes $\beta + \gamma = 2\alpha$. If $\beta^2\gamma$ is a vertex, then $2\beta + \gamma = 2\pi = 3\alpha$, and we get $\beta = \gamma = \alpha$, a contradiction. By the similar reason, $\beta\gamma^2$ is not a vertex. By Lemma 15, the only possible vertices are α^3 and $\alpha\beta\gamma$, and β, γ must be distinct.

Let $\{m\alpha^3, n\alpha\beta\gamma\}$ be the anglewise vertex combination. Since 12 tiles with angles $\alpha, \alpha, \alpha, \beta, \gamma$ in each tile have total of 36 angle α , 12 angle β and 12 angle γ , we get $3m + n = 36$ and $n = 12$. Therefore $m = 8$ and $n = 12$. This is the second case of the proposition.

Case 3. The angles in the pentagon are $\alpha, \alpha, \beta, \gamma, \delta$, with $\beta, \gamma, \delta \neq \alpha$. Note that β, γ, δ are not yet assumed to be distinct.

Lemma 14 becomes $\beta + \gamma + \delta = 3\alpha$. If $\alpha\beta\gamma$ is a vertex, then $\alpha + \beta + \gamma = 3\alpha$, and we get $\delta = \alpha$, a contradiction. The argument actually shows that there is no $\alpha\beta\gamma$ -type vertex.

If $\beta^2\gamma$ is a vertex, then $2\beta + \gamma = 3\alpha$ and $\beta = 3\alpha - \beta - \gamma = \delta$. Thus the $\beta^2\gamma$ -vertex is really a $\beta\gamma\delta$ -vertex. The argument actually applies to any $\beta^2\gamma$ -type vertex. By Lemma 15, we conclude that any vertex is either an α^3 -vertex or a $\beta\gamma\delta$ -vertex.

Let $\{m\alpha^3, n\beta\gamma\delta\}$ be the anglewise vertex combination. Without loss of generality, we may further assume that γ is different from α, β, δ . Since 12 tiles with angles $\alpha, \alpha, \beta, \gamma, \delta$ have total of 24 angle α and 12 angle γ , we get $3m = 24, n = 12$. Therefore $m = 8$ and $n = 12$. Depending on whether $\beta = \delta$, we get the third and the fourth cases of the proposition.

Case 4. The angles in the pentagon are $\alpha, \beta, \gamma, \delta, \epsilon$, with $\beta, \gamma, \delta, \epsilon \neq \alpha$. Again $\beta, \gamma, \delta, \epsilon$ are not yet assumed to be distinct.

Lemma 14 becomes $\beta + \gamma + \delta + \epsilon = 4\alpha$. We further divide the case by considering whether some among $\beta, \gamma, \delta, \epsilon$ are equal.

Case 4.1. $\beta = \gamma = \delta = \epsilon$.

Lemma 14 becomes $4\beta = 4\alpha$, and we get $\beta = \alpha$, a contradiction.

Case 4.2. $\beta = \delta = \epsilon \neq \gamma$.

Lemma 14 becomes $3\beta + \gamma = 4\alpha$. There is no $\alpha\beta\gamma$ -vertex because this implies $\alpha + \beta + \gamma = 3\alpha$, and we get $\beta = \gamma = \alpha$. Similar argument shows that $\beta^2\gamma, \beta\gamma^2$ are not vertices. By Lemma 15, the only vertex is α^3 , which is impossible.

Case 4.3. $\beta = \delta \neq \gamma = \epsilon$.

Lemma 14 becomes $\beta + \gamma = 2\alpha$. An argument similar to Case 4.2 shows that $\beta^2\gamma, \beta\gamma^2$ are not vertices. By Lemma 15, the only vertices are α^3 and $\alpha\beta\gamma$, so that the anglewise vertex combination is $\{m\alpha^3, n\alpha\beta\gamma\}$. Since 12 tiles with angles $\alpha, \beta, \beta, \gamma, \gamma$ have total of 12 angle α and 24 angle β , we get $3m + n = 12$ and $n = 24$. The system has no non-negative solution, a contradiction.

Case 4.4. $\beta = \epsilon$, and β, γ, δ are distinct.

Similar to Case 4.2, $\alpha\gamma\delta, \beta^2\gamma, \beta^2\delta, \beta\gamma\delta$ are not vertices. By the symmetry of γ and δ in the current case and the exclusion between $\alpha\beta\gamma$ and $\alpha\beta\delta$ (see Lemma 16), we may assume that the only possible $\alpha\beta\gamma$ -type vertex is $\alpha\beta\gamma$. Then by Lemma 15, the anglewise vertex combination is

$$\{m_1\alpha^3, m_2\alpha\beta\gamma, n_1\beta\gamma^2, n_2\beta\delta^2, k_1\gamma^2\delta, k_2\gamma\delta^2\}.$$

Since 12 tiles with angles $\alpha, \beta, \gamma, \delta$ have total of 12 angle α , 24 angle β , 12 angle γ and 12 angle δ , we get

$$\begin{aligned} 3m_1 + m_2 &= 12, \\ m_2 + n_1 + n_2 &= 24, \\ m_2 + 2n_1 + 2k_1 + k_2 &= 12, \\ 2n_2 + k_1 + 2k_2 &= 12. \end{aligned}$$

Adding the third and the fourth equations together, we get $m_2 + 2(n_1 + n_2) + 3(k_1 + k_2) = 24$. Compared with the second equation, we get $n_1 = n_2 = k_1 = k_2 = 0$. Then the second equation becomes $m_2 = 24$, contradicting to the first equation.

Case 4.5. $\beta, \gamma, \delta, \epsilon$ are distinct.

Similar to Case 4.2, there is no $\beta\gamma\delta$ -type vertex. By Lemma 15, the only possible types are $\alpha^3, \alpha\beta\gamma, \beta^2\gamma$. We further divide into two subcases.

Case 4.5.1. β appears in two vertices not involving α and with different angle combinations.

Without loss of generality, we may assume that the two vertices are $\beta^i\gamma^{3-i}$ and $\beta^j\delta^{3-j}$, $0 < i, j < 3$. By Lemma 16, we have $i \neq j$. Therefore up to symmetry, we may assume $\beta^2\gamma, \beta\delta^2$ are vertices. Solving $2\beta + \gamma = \beta + 2\delta = 3\alpha$ together with $\beta + \gamma + \delta + \epsilon = 4\alpha$ from Lemma 14, we get

$$3\beta - 2\epsilon = \alpha, \quad 3\gamma + 4\epsilon = 7\alpha, \quad 3\delta + \epsilon = 4\alpha.$$

Now the angle ϵ must appear at some vertex. Since there is no $\beta\gamma\delta$ -type vertex, by Lemmas 15 and 16, one of $\alpha\beta\epsilon, \alpha\gamma\epsilon, \alpha\delta\epsilon, \gamma^2\epsilon, \delta^2\epsilon, \gamma\epsilon^2, \delta\epsilon^2$ must appear as the vertex involving ϵ . If $\alpha\beta\epsilon$ is a vertex, then $\delta + \epsilon = 2\alpha$ and $3\beta - 2\epsilon = \alpha$ imply $\beta = \epsilon$, a contradiction. All other possibilities lead to similar contradictions.

Case 4.5.2. The opposite of Case 4.5.1 and its symmetric permutations.

This means that, without loss of generality, we may assume that the only possible vertices not involving α are $\beta^2\gamma$ and $\delta^2\epsilon$. Then the anglewise vertex combination is $\alpha^3, \beta^2\gamma, \delta^2\epsilon$ together with some $\alpha\beta\gamma$ -type vertices. By Lemma 16, we find only two possibilities:

$$\{m\alpha^3, n_1\alpha\beta\delta, n_2\alpha\gamma\epsilon, k_1\beta^2\gamma, k_2\delta^2\epsilon\}, \quad \{m\alpha^3, n_1\alpha\beta\epsilon, n_2\alpha\gamma\delta, k_1\beta^2\gamma, k_2\delta^2\epsilon\}.$$

In the first combination, counting the total number of each angle gives

$$3m + n_1 + n_2 = n_1 + 2k_1 = n_2 + k_1 = n_1 + 2k_2 = n_2 + k_2 = 12.$$

We get $k_1 = k_2 = m + 4, n_1 = 4 - 2m, n_2 = 8 - m$. In particular, we have $n_2, k_1, k_2 > 0$, so that $\alpha\gamma\epsilon, \beta^2\gamma, \delta^2\epsilon$ are vertices, and we get

$$\gamma + \epsilon = 2\alpha, \quad 2\beta + \gamma = 3\alpha, \quad 2\delta + \epsilon = 3\alpha.$$

The solution is

$$\gamma = 3\alpha - 2\beta, \quad \delta = 2\alpha - \beta, \quad \epsilon = 2\beta - \alpha.$$

On the other hand, to maintain $n_1 \geq 0$, the possible values for m are 0, 1, 2. Then we get the corresponding anglewise vertex combinations, which are the three combinations in the fifth case of the proposition.

The second combination gives

$$3m + n_1 + n_2 = n_1 + 2k_1 = n_2 + k_1 = n_2 + 2k_2 = n_1 + k_2 = 12.$$

The system has no non-negative solution.

Case 5. The angles in the pentagon are $\beta, \gamma, \delta, \epsilon, \zeta$, all different from α . But the five angles are not yet assumed to be distinct.

Lemma 14 becomes $\beta + \gamma + \delta + \epsilon + \zeta = 5\alpha$. We further divide the case by considering whether some angles are equal.

Case 5.1. $\beta = \gamma = \delta = \epsilon = \zeta$.

Lemma 14 implies $\beta = \alpha$, a contradiction.

Case 5.2. $\beta = \delta = \epsilon = \zeta \neq \gamma$.

The vertices must be $\beta^2\gamma$ or $\beta\gamma^2$, and Lemma 14 becomes $4\beta + \gamma = 5\alpha$. If $\beta^2\gamma$ is a vertex, then $2\beta + \gamma = 3\alpha$, and we get $\beta = \gamma$, a contradiction. If $\beta\gamma^2$ is a vertex, then we get the similar contradiction.

Case 5.3. $\beta = \delta = \epsilon \neq \gamma = \zeta$.

The vertices must be $\beta^2\gamma$ or $\beta\gamma^2$. We get contradiction similar to Case 5.2.

Case 5.4. $\beta = \epsilon = \zeta$, and β, γ, δ are distinct.

Lemma 14 becomes $3\beta + \gamma + \delta = 5\alpha$. This implies that $\beta\gamma\delta$ is not a vertex, so that all vertices are $\beta^2\gamma$ -type. Since there are three distinct angles, one angle must appear in two vertices with different angle combinations. By Lemma 16 and the symmetry between γ and δ , we only need to consider both $\beta^2\gamma, \beta\delta^2$ appear, or both $\beta^2\gamma, \gamma^2\delta$ appear, or both $\beta\gamma^2, \gamma\delta^2$ appear.

If $\beta^2\gamma, \beta\delta^2$ are vertices, then $2\beta + \gamma = 3\alpha$ and $\beta + 2\delta = 3\alpha$. Combined with $3\beta + \gamma + \delta = 5\alpha$, we get all angles equal, a contradiction. The other pairs of vertices lead to similar contradictions.

Case 5.5. $\beta = \epsilon, \gamma = \zeta$, and β, γ, δ are distinct.

Lemma 14 becomes $2\beta + 2\gamma + \delta = 5\alpha$. This implies that $\beta^2\delta, \gamma^2\delta, \beta\gamma\delta$ are not vertices. Since there are three distinct angles and all vertices are $\beta^2\gamma$ -type, one angle must appear in two vertices with different angle combinations. By Lemma 16, the symmetry between β and γ , and the fact that $\beta^2\delta$ and $\gamma^2\delta$ are not vertices, we may assume that $\beta^2\gamma, \beta\delta^2$ are vertices. This implies $2\beta + \gamma = 3\alpha$ and $\beta + 2\delta = 3\alpha$. Combined with $2\beta + 2\gamma + \delta = 5\alpha$, we get all angles equal, a contradiction.

Case 5.6. $\beta = \zeta$, and $\beta, \gamma, \delta, \epsilon$ are distinct.

Lemma 14 becomes $2\beta + \gamma + \delta + \epsilon = 5\alpha$. This implies that $\gamma\delta\epsilon$ is not a vertex.

If there is a vertex of $\beta\gamma\delta$ -type, then up to symmetry, we may assume $\beta\gamma\delta$ is a vertex. Then we get $\beta + \gamma + \delta = 3\alpha$ and $\beta + \epsilon = 5\alpha - (\beta + \gamma + \delta) = 2\alpha$. This implies that β, γ, δ cannot form $\beta^2\gamma$ -type vertices, and $\beta^2\epsilon, \beta\epsilon^2$ are not vertices. So the only possible vertices of $\beta^2\gamma$ -type are $\gamma^2\epsilon, \gamma\epsilon^2, \delta^2\epsilon, \delta\epsilon^2$. Moreover, by Lemma 16, the appearance of $\gamma^2\epsilon$ excludes $\gamma\epsilon^2$ and $\delta^2\epsilon$. Then by the symmetry between γ and δ in our current situation, we only need to consider the anglewise vertex combination $\{n\beta\gamma\delta, k_1\gamma^2\epsilon, k_2\delta\epsilon^2\}$. Counting the total number of each angle gives

$$n = 24, \quad n + 2k_1 = n + k_2 = k_1 + 2k_2 = 12.$$

The system has no non-negative solution.

So all vertices are of $\beta^2\gamma$ -type, and the anglewise vertex combination is

$$\{m_1\beta^2\gamma, m_2\beta^2\delta, m_3\beta^2\epsilon, n_1\beta\gamma^2, n_2\beta\delta^2, n_3\beta\epsilon^2, k_1\gamma^2\delta, k_2\gamma^2\epsilon, k_3\gamma\delta^2, k_4\gamma\epsilon^2, k_5\delta^2\epsilon, k_6\delta\epsilon^2\}.$$

Counting the total number of the angle β gives $2(m_1 + m_2 + m_3) + n_1 + n_2 + n_3 = 24$.

If $m_1 = m_2 = m_3 = 0$ and $n_1 > 0$, then by Lemma 16, we get $n_2 = n_3 = 0$ and $n_1 = 24$. Therefore the total number of the angle γ is $\geq 2n_1 = 48$, a contradiction. Similar argument rules out $n_2 > 0$ and $n_3 > 0$. Therefore m_1, m_2, m_3 cannot be all 0, and we may assume $m_1 > 0$ without loss of generality. By Lemma 16, this implies $m_2 = m_3 = n_1 = k_3 = k_4 = 0$, and we get $2m_1 + n_2 + n_3 = 24$.

If $n_2 = n_3 = 0$, then we get $m_1 = 12$. Thus the 12 $\beta^2\gamma$ -vertices already contain all 12 appearances of the angle γ . Therefore γ does not appear in any other vertex, and we are left with the anglewise vertex combination $\{12\beta^2\gamma, k_5\delta^2\epsilon, k_6\delta\epsilon^2\}$. By counting the total numbers of the angles δ and ϵ , we get $2k_5 + k_6 = k_5 + 2k_6 = 12$. This implies $k_5 = k_6 = 4 > 0$. Therefore both $\delta^2\epsilon$ and $\delta\epsilon^2$ are vertices, a contradiction to Lemma 16.

So we may assume $n_2 > 0$. By Lemma 16, we get $n_3 = k_5 = 0$, so that the anglewise vertex combination becomes $\{m_1\beta^2\gamma, n_2\beta\delta^2, k_1\gamma^2\delta, k_2\gamma^2\epsilon, k_6\delta\epsilon^2\}$. By counting the total number of each angle, we get

$$2m_1 + n_2 = 24, \quad m_1 + 2k_1 + 2k_2 = 2n_2 + k_1 + k_6 = k_2 + 2k_6 = 12.$$

By Lemma 16, we must have either $k_1 = 0$ or $k_2 = 0$. If $k_1 = 0$, the system gives $k_2 = \frac{4}{5}$. If $k_2 = 0$, the system gives $k_1 = \frac{2}{3}$. Since the numbers are not integers, we get a contradiction.

Case 5.7. $\beta, \gamma, \delta, \epsilon, \zeta$ are distinct.

Case 5.7.1. There are $\beta\gamma\delta$ -type vertices.

We may assume $\beta\gamma\delta$ is a vertex. This implies $\beta + \gamma + \delta = 3\alpha$, so that β, γ, δ cannot form $\beta^2\gamma$ -type vertices. By Lemma 14, we have $\epsilon + \zeta = 5\alpha - (\beta + \gamma + \delta) = 2\alpha$, so that ϵ, ζ cannot appear at the same vertex. By Lemma 16, the only $\beta\gamma\delta$ -type vertex is $\beta\gamma\delta$, and the only way for ϵ, ζ to appear is to get combined with angles from β, γ, δ .

If ϵ, ζ can be combined with the same angle from β, γ, δ , then up to symmetry, we may assume $\beta\gamma\delta, \beta^2\epsilon, \beta\zeta^2$ are vertices. We get $\beta + \gamma + \delta = 2\beta + \epsilon = \beta + 2\zeta = 3\alpha$. Together with $\epsilon + \zeta = 2\alpha$, we get $\delta = \alpha$, a contradiction.

So ϵ, ζ must be combined with different angles from β, γ, δ . By Lemma 16 and up to symmetry, there are three possible combinations.

In the first combination, $\beta\gamma\delta, \beta^2\epsilon, \gamma^2\zeta$ are vertices. We get $\beta + \gamma + \delta = 2\beta + \epsilon = 2\gamma + \zeta = 3\alpha$. Together with $\epsilon + \zeta = 2\alpha$, we get $\delta = \alpha$, a contradiction.

In the second combination, $\beta\gamma\delta, \beta\epsilon^2, \gamma\zeta^2$ are vertices. Similar to the first combination, we also get the contradiction $\delta = \alpha$.

In the third combination, $\beta\gamma\delta, \beta^2\epsilon, \gamma\zeta^2$ are vertices. We get $\beta + \gamma + \delta = 2\beta + \epsilon = \gamma + 2\zeta = 3\alpha$. Together with $\epsilon + \zeta = 2\alpha$, we do not get a contradiction as before. However, if $\beta\gamma\delta, \beta^2\epsilon, \gamma\zeta^2$ are the only vertices, then the total number of β will be strictly bigger than the total number of δ . Since both total numbers are actually 12, we conclude that these cannot be all the vertices. By Lemma 16, the only other possible vertices are $\delta\epsilon^2, \delta^2\zeta$. If $\delta\epsilon^2$ is a vertex, then $\delta + 2\epsilon = 3\alpha$. Adding this to the equations above, we get all angles equal, a contradiction. If $\delta^2\zeta$ is a vertex, then we get a similar contradiction.

Case 5.7.2. There is no $\beta\gamma\delta$ -type vertex.

We may assume $\beta^2\gamma$ is a vertex. Since there is no $\beta\gamma\delta$ -type vertex, by Lemma 16, the only other vertex involving β is $\beta\theta^2$ for a unique θ , and the only other vertex involving γ is $\gamma^2\rho$ for a unique ρ . Since each of three distinct angles δ, ϵ, ζ must appear at some vertex, there must be some pairing among δ, ϵ, ζ . Without loss of generality, therefore, we may assume both $\beta^2\gamma, \delta^2\epsilon$ are vertices. Moreover, up to symmetry, we may further assume that ζ is paired with β or γ . Given the existence of $\beta^2\gamma$, the only possibilities are $\beta\zeta^2$ and $\gamma^2\zeta$. So either $\beta^2\gamma, \delta^2\epsilon, \beta\zeta^2$ are vertices, or $\beta^2\gamma, \delta^2\epsilon, \gamma^2\zeta$ are vertices.

If $\beta^2\gamma, \delta^2\epsilon, \beta\zeta^2$ are vertices, then $2\beta + \gamma = 2\delta + \epsilon = \beta + 2\zeta = 3\alpha$. Combined with $\beta + \gamma + \delta + \epsilon + \zeta = 5\alpha$ from Lemma 14, we get the following pairwise relations

$$\begin{aligned} 3\beta + 2\delta &= 5\alpha, & 3\beta - \epsilon &= 2\alpha, & 3\gamma - 4\delta &= -\alpha, & 3\gamma + 2\epsilon &= 5\alpha, \\ \gamma - 4\zeta &= -3\alpha, & -\delta + 3\zeta &= 2\alpha, & \epsilon + 6\zeta &= 7\alpha. \end{aligned}$$

The relations exclude all other vertices, which must be of $\beta^2\gamma$ -type. Therefore the anglewise vertex combination is $\{m\beta^2\gamma, n\delta^2\epsilon, k\beta\zeta^2\}$. This implies that the total number of β is strictly bigger than the total number of γ . Since both numbers are 12, we get a contradiction.

If $\beta^2\gamma, \delta^2\epsilon, \gamma^2\zeta$ are vertices, then $2\beta + \gamma = 2\delta + \epsilon = 2\gamma + \zeta = 3\alpha$. Similar to the previous case, the equations together with $\beta + \gamma + \delta + \epsilon + \zeta = 5\alpha$ from Lemma 14 exclude all other vertices. Therefore the anglewise vertex combination is $\{m\beta^2\gamma, n\delta^2\epsilon, k\gamma^2\zeta\}$. Then the total number of δ is more than the total number of ϵ , a contradiction. \square

6. Classification of angle congruent tiling

Proposition 17 classifies the numerics of spherical tilings by 12 angle congruent pentagons. In this section, we study how the numerical information can fit the combinatorial structure in Fig. 9, similar to what we have done for edges in Section 4.

In Section 4, we introduced the notations P_i, E_{ij}, V_{ijk} for the tiles, the edges and the vertices in Fig. 9. In this section, we denote by $A_{i,jk}$ the angle of the tile P_i at the vertex V_{ijk} . We will even write $V_{ijk} = \alpha\beta\gamma$ if we know V_{ijk} is an $\alpha\beta\gamma$ -vertex.

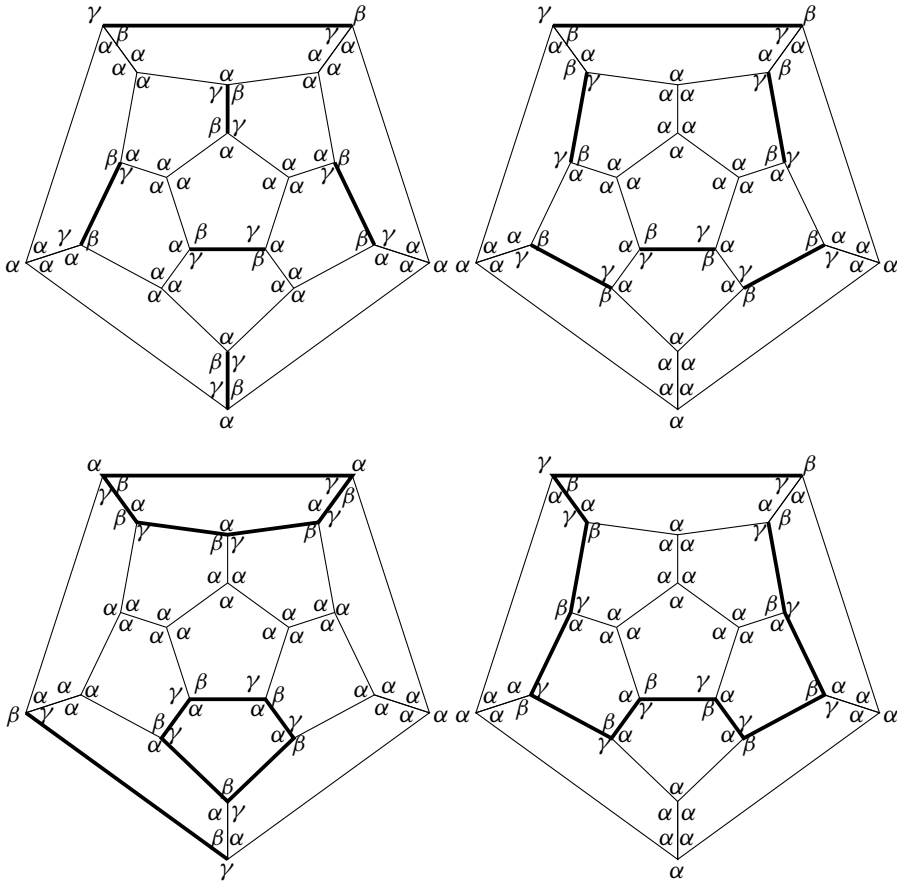


Fig. 18. Some tilings for the angle combination $\alpha^3\beta\gamma$.

Inside a specified tile, we can also name a vertex by its angle and an edge by the two angles at the ends. For example, in Fig. 2, $A_{1,23}$ is a δ -angle, $A_{1,26}$ and $A_{1,34}$ are α -angles. Moreover, E_{12} and E_{13} are $\alpha\delta$ -edges of P_1 .

In this section, we let $\alpha = \frac{2}{3}\pi$ and assume $\alpha, \beta, \gamma, \dots$ are always distinct.

There is not much to classify for the angle combination α^5 . Basically, we assign α to all angles in Fig. 9. This can be realized by the regular dodecahedron tiling.

There are many angle congruent tilings for the angle combination $\alpha^3\beta\gamma$. Fortunately, the classification is not needed in the proof of the main theorem. We simply present some examples in Fig. 18. Note that for any thick line in any tiling, we can exchange the angles β and γ along the line and still get an angle congruent tiling.

Proposition 18. *The spherical tilings by 12 angle congruent pentagons with angles $\alpha, \alpha, \beta, \beta, \gamma$, where $\alpha = \frac{2\pi}{3}$ and α, β, γ are distinct, are given by Fig. 21 up to symmetry. Note that in the tiling on the right, we can exchange the four angles β and γ around any thick line and still get a tiling.*

As an example of the exchange, on the right of Fig. 21, we can exchange β and γ around the edge E_{15} to get $A_{1,45} = A_{5,16} = \gamma$ and $A_{1,56} = A_{5,14} = \beta$. But we should keep $A_{4,15} = A_{6,15} = \beta$. The result is also an angle congruent tiling.

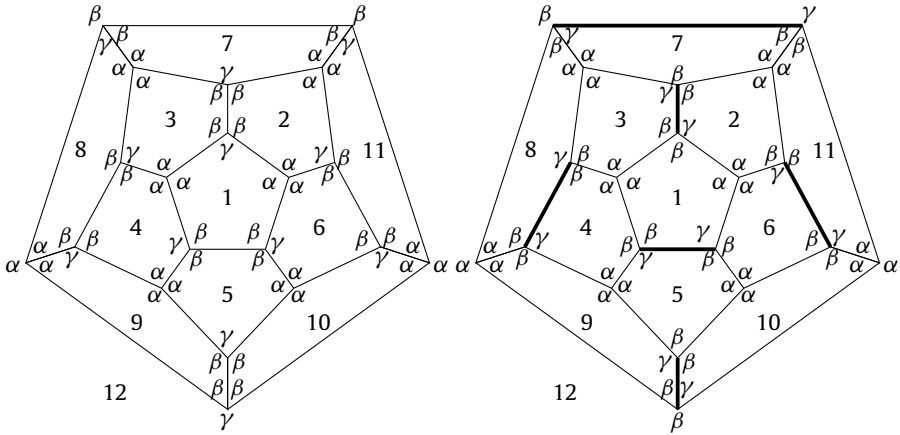


Fig. 21. Tilings for the angle combination $\alpha^2\beta^2\gamma$, continued.

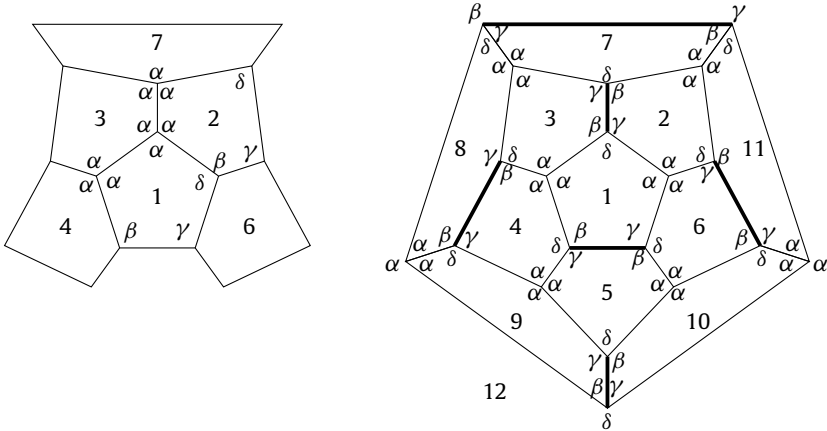


Fig. 22. Tilings for the angle combination $\alpha^2\beta\gamma\delta$.

We may successively carry out the same experiment around each of E_{48} , E_{611} , E_{910} , E_{712} . We find that we can make similar independent choices of β, γ , and each choice determines all the angles of the pair of tiles on the two sides of the edge. \square

Proposition 19. *The spherical tiling by 12 angle congruent pentagons with angles $\alpha, \alpha, \beta, \gamma, \delta$, where $\alpha = \frac{2\pi}{3}$ and $\alpha, \beta, \gamma, \delta$ are distinct, is given by the right of Fig. 22 up to symmetry. Note that in the tiling on the right, we can exchange the four angles β and γ around any thick line and still get a tiling.*

Proof. We are in the fourth case of Proposition 17. The anglewise vertex combination is $\{8\alpha^3, 12\beta\gamma\delta\}$, which implies the following AVC condition.

AVC(for $\alpha^2\beta\gamma\delta$): Any vertex is either α^3 or $\beta\gamma\delta$.

Up to symmetry, there are two possible ways of arranging the angles in the pentagon.

Arrangement 1. In this arrangement, the two α are adjacent. See the tile P_1 on the left of Fig. 22. By AVC, we get $V_{123} = V_{134} = \alpha^3$ and $V_{126} = \beta\gamma\delta$. Thus $A_{2,16} = \beta$ or γ , and is adjacent to α in P_2 .

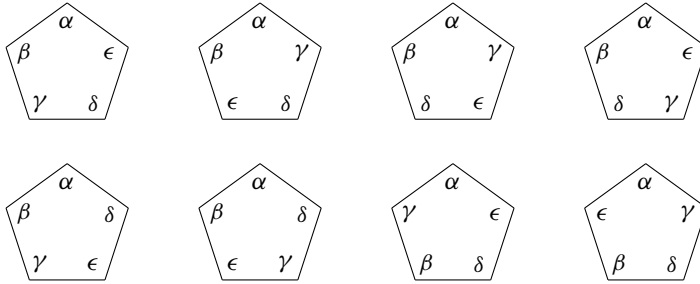


Fig. 23. Eight possible arrangements for the angle combination $\alpha\beta\gamma\delta\epsilon$.

Therefore $A_{2,16} = \beta$. Together with $A_{2,13} = \alpha$, we get all the angles of P_2 . Then $A_{2,37} = \alpha$ implies $V_{237} = \alpha^3$, and there are three α in P_3 , a contradiction.

Arrangement 2. In this arrangement, the two α are not adjacent. See the tile P_1 on the right of Fig. 22. By AVC, we get $V_{126} = V_{134} = \alpha^3$ and $V_{123} = \beta\gamma\delta$. If $A_{2,13} = \gamma$, $A_{3,12} = \beta$, then we get all the angles of P_2 , P_3 and all the angles at V_{237} , V_{2711} , V_{378} . If $A_{2,13} = \beta$, $A_{3,12} = \gamma$, then we get the similar result, except that the four angles β, γ around E_{23} are exchanged. Such exchange does not affect the subsequent discussion.

By $A_{3,48} = \delta$, we know $A_{4,38}, A_{8,34}$ are β, γ . We may make either choice for $A_{4,38}, A_{8,34}$ and then get all the angles of P_4, P_8 . Similar discussions can then be successively carried out around $E_{611}, E_{910}, E_{712}$ and determine all the angles. \square

Proposition 20. *If the angles in a spherical tiling by 12 angle congruent pentagons are $\alpha, \beta, \gamma, \delta, \epsilon$, with $\alpha = \frac{2\pi}{3}$ and all angles distinct, then up to symmetry, the tiling must have anglewise vertex combination $\{2\alpha^3, 6\alpha\gamma\epsilon, 6\beta^2\gamma, 6\delta^2\epsilon\}$ and is given by one of the tilings on the left of Figs. 25, 27, 32, 34.*

Proof. We are in the last case of Proposition 17. The following AVC condition applies to all three possible anglewise vertex combinations in the case.

AVC(for $\alpha\beta\gamma\delta\epsilon$): Any vertex is one of $\alpha^3, \alpha\beta\delta, \alpha\gamma\epsilon, \beta^2\gamma, \delta^2\epsilon$.

A consequence we will often use is that $\beta\epsilon, \gamma\delta, \gamma^2, \epsilon^2$ are forbidden at any vertex.

All three anglewise vertex combinations are symmetric with respect to the simultaneous exchange of β with δ and γ with ϵ . Modulo such symmetry, there are eight possible arrangements of the angles in the pentagon, described in Fig. 23. We discuss each case by assuming the angles of P_1 in Fig. 9 are arranged as in Fig. 23. Since the vertex $\alpha\gamma\epsilon$ appears in all three anglewise vertex combinations, we will also assume $V_{123} = \alpha\gamma\epsilon$. This leads to the possibilities $A_{2,13} = \gamma, A_{3,12} = \epsilon$ and $A_{2,13} = \epsilon, A_{3,12} = \gamma$.

Since all angles are distinct, there is no ambiguity for us to call the orientation of angles given by Fig. 23 counterclockwise, and call the other orientation clockwise.

Arrangement 1, Case 1. $A_{2,13} = \gamma, A_{3,12} = \epsilon$.

The case is the left of Fig. 24. Since $A_{2,16}$ is adjacent to $A_{2,13} = \gamma$ in P_2 , we get $A_{2,16} = \beta$ or δ . By AVC, we get $A_{2,16} = \delta$ and all the angles of P_2 .

By $A_{1,26} = \epsilon, A_{2,16} = \delta$ and AVC, we get $A_{6,12} = \delta$. Since $A_{6,15}$ is adjacent to δ in P_6 , we get $A_{6,15} = \gamma$ or ϵ . By AVC, we get $A_{6,15} = \epsilon$ and all the angles of P_6 .

By $A_{1,56} = \delta, A_{6,15} = \epsilon$ and AVC, we get $A_{5,16} = \delta$. Since $A_{5,14}$ is adjacent to δ in P_5 , we get $A_{5,14} = \gamma$ or ϵ . By AVC, we get $A_{5,14} = \epsilon$ and all the angles of P_5 .

By $A_{1,45} = \gamma, A_{5,14} = \epsilon$ and AVC, we get $A_{4,15} = \alpha$. Since $A_{4,13}$ is adjacent to α in P_4 , we get $A_{4,13} = \beta$ or ϵ . By AVC, we get $A_{4,13} = \beta$ and all the angles of P_4 .

By AVC, we further get $A_{3,14} = \gamma$, which is adjacent to $A_{3,12} = \epsilon$ in P_3 . This is a contradiction.

We remark that the conclusion of the case is the following: If the angles in a tile (of first arrangement) are counterclockwise oriented, and the vertex at α is $\alpha\gamma\epsilon$, then α, γ, ϵ must be clockwise

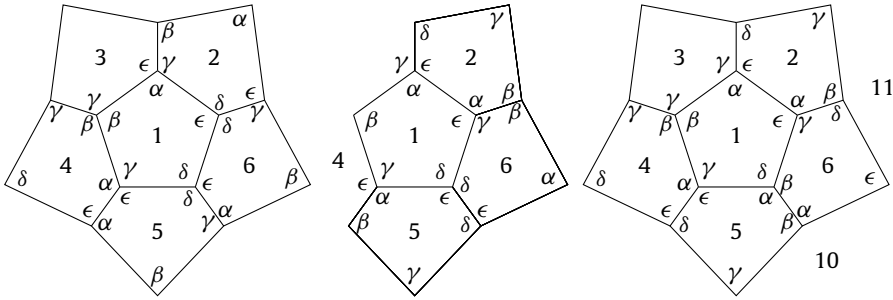


Fig. 24. Tilings for the first arrangement.

oriented at the vertex. By symmetry, we also know that if the angles in a tile are clockwise oriented, then α, γ, ϵ must be counterclockwise oriented at the vertex.

Arrangement 1, Case 2. $A_{2,13} = \epsilon, A_{3,12} = \gamma$.

The case is the middle and the right of Fig. 24. Since $A_{2,16}$ is adjacent to ϵ in P_2 , we get $A_{2,16} = \alpha$ or δ .

If $A_{2,16} = \delta$, then we get all the angles of P_2 and $A_{2,6\bar{1}1} = \gamma$ in particular. By AVC, we also get $A_{6,12} = \delta$. Being adjacent to $A_{6,12} = \delta$ in P_6 , one of $A_{6,15}, A_{6,2\bar{1}1}$ is γ . This contradicts to AVC at V_{156} or $V_{26\bar{1}1}$.

Therefore we must have $A_{2,16} = \alpha$. Then we get all the angles of P_2 and $A_{6,12} = \gamma$. Now consider the two possible orientations of P_6 .

In the middle of Fig. 24, P_6 is counterclockwise oriented. By AVC, we get $A_{5,16} = \epsilon$, and the angle $A_{5,14}$ adjacent to ϵ in P_5 must be α . This determines all the angles of P_5 . Now P_5 is counterclockwise oriented, and α, γ, ϵ is also counterclockwise oriented at V_{145} . This contradicts to the conclusion of Case 1.

On the right of Fig. 24, P_6 is clockwise oriented. By AVC, we get $A_{5,16} = \alpha$. Then the angle $A_{5,6\bar{1}0}$ adjacent to α in P_5 is β or ϵ . If $A_{5,6\bar{1}0} = \epsilon$, then P_6 is clockwise oriented, and α, γ, ϵ are also clockwise oriented at $V_{56\bar{1}0}$. This contradicts to the conclusion of Case 1. Therefore we must have $A_{5,6\bar{1}0} = \beta$. Then we get all the angles of P_5 and by AVC, we further get $A_{4,15} = \alpha$. The clockwise orientation of α, γ, ϵ at V_{145} and the conclusion of Case 1 imply that P_4 must be counterclockwise oriented. This determines all the angles of P_4 . Then by AVC, we get $A_{3,14} = \gamma$, and there are two γ in P_3 , a contradiction.

Arrangement 2, Case 1. $A_{2,13} = \gamma, A_{3,12} = \epsilon$.

Consider the two possible orientations of P_3 . The counterclockwise case is the left of Fig. 25. By AVC, we may successively determine all the angles of $P_2, P_7, P_{11}, P_6, P_5, P_{10}, P_{12}, P_8, P_4, P_9$. There is no contradiction, and we get an angle congruent tiling with anglewise vertex combination $\{2\alpha^3, 6\alpha\gamma\epsilon, 6\beta^2\gamma, 6\delta^2\epsilon\}$.

On the right of Fig. 25, P_3 is clockwise oriented. By AVC, we get $A_{4,13} = \alpha$, then get all the angles of P_4 , and further get $A_{5,14} = \alpha$. Now one of $A_{5,16}, A_{5,49}$ adjacent to α in P_5 is γ . This contradicts to AVC at V_{156} or V_{459} .

Arrangement 2, Case 2. $A_{2,13} = \epsilon, A_{3,12} = \gamma$.

By AVC, we get all the angles of P_2 and $A_{6,12} = \beta$. If P_6 is counterclockwise oriented, as on the left of Fig. 26, then by AVC, we get $A_{11,26} = \alpha$. Now one of $A_{11,27}, A_{11,6\bar{1}0}$ adjacent to α in P_{11} is γ . This contradicts to AVC at $V_{27\bar{1}1}$ or $V_{6\bar{1}0\bar{1}1}$. If P_6 is clockwise oriented, as on the right of Fig. 26, then by AVC, we get all the angles of P_5, P_4 and further get $A_{3,14} = \alpha$. Together with $A_{3,12} = \gamma$, we get $A_{3,84} = \beta$. This contradicts to AVC at V_{348} .

Arrangement 3, Case 1. $A_{2,13} = \gamma, A_{3,12} = \epsilon$.

If P_2 is clockwise oriented, as on the upper right of Fig. 27, then by AVC, we may successively determine all the angles of P_6, P_5, P_4 and get $A_{3,14} = \beta$. Thus we find β and ϵ adjacent in P_3 , a contradiction. Similar argument shows that P_3 cannot be counterclockwise oriented. Thus we get all the angles of P_2, P_3 as described on the left of Fig. 27. Then by AVC, we may successively determine

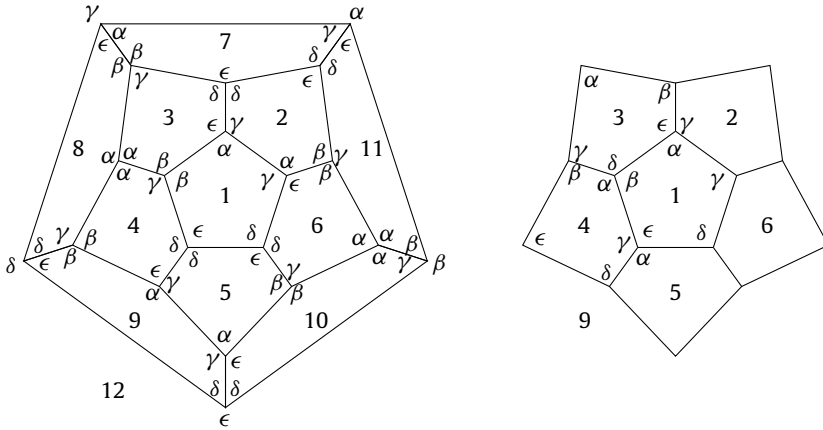


Fig. 25. Tilings for the second arrangement.

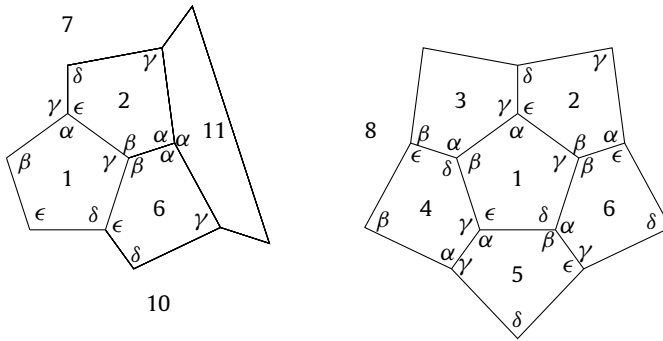


Fig. 26. Tilings for the second arrangement, continued.

all the angles of $P_7, P_{11}, P_6, P_{10}, P_5, P_4, P_8, P_9, P_{12}$. There is no contradiction, and we get an angle congruent tiling with anglewise vertex combination $\{2\alpha^3, 6\alpha\gamma\epsilon, 6\beta^2\gamma, 6\delta^2\epsilon\}$.

Arrangement 3, Case 2. $A_{2,13} = \epsilon, A_{3,12} = \gamma$.

The case is the lower right of Fig. 27. By AVC, we get all the angles of P_3, P_4 and further get $A_{5,14} = \delta$. Then the angle $A_{5,16}$ adjacent to δ in P_5 is either β or ϵ . Either way contradicts to AVC at V_{156} .

Arrangement 4, Case 1. $A_{2,13} = \gamma, A_{3,12} = \epsilon$.

By AVC, we get all the angles of P_2, P_6 and $A_{5,16} = \beta$. If P_5 is counterclockwise oriented, as on the left of Fig. 28, then by AVC, we get all the angles of P_4 and $A_{3,14} = \beta$. Then β and ϵ are adjacent in P_3 , a contradiction. If P_5 is clockwise oriented, as on the right of Fig. 28, then by AVC, we get all the angles of P_{10} and $A_{11,6\bar{1}0} = \delta$. Then one of $A_{11,26}, A_{11,\bar{1}0\bar{1}2}$ adjacent to $A_{11,6\bar{1}0}$ in P_{11} must be γ . This contradicts to AVC at $V_{26\bar{1}1}$ or $A_{\bar{1}0\bar{1}1\bar{1}2}$.

Arrangement 4, Case 2. $A_{2,13} = \epsilon, A_{3,12} = \gamma$.

If P_2 is counterclockwise oriented, as on the left of Fig. 29, then by AVC, we successively get all the angles of P_3, P_6, P_{11} and $A_{7,23} = \alpha, A_{7,2\bar{1}1} = \delta$. We find α and δ adjacent in P_7 , a contradiction. If P_2 is clockwise oriented, as on the right of Fig. 29, then by AVC, we successively get all the angles of P_3, P_7, P_8 and $A_{4,13} = A_{4,38} = \alpha$. We find two α in P_4 , a contradiction.

Arrangement 5, Case 2. $A_{2,13} = \epsilon, A_{3,12} = \gamma$.

We treat the second case first because the conclusion will be useful for the first case. By AVC, we get all the angles of P_2, P_3 . If P_4 is clockwise oriented, as on the left of Fig. 30, then by AVC, we successively get all the angles of P_5, P_6, P_9, P_{10} . Then we get a contradiction to AVC at $V_{6\bar{1}0\bar{1}1}$.

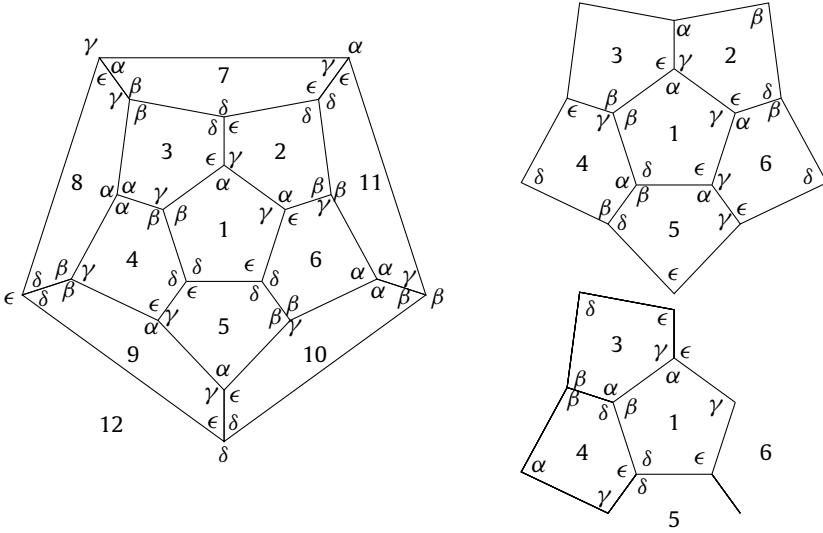


Fig. 27. Tilings for the third arrangement.

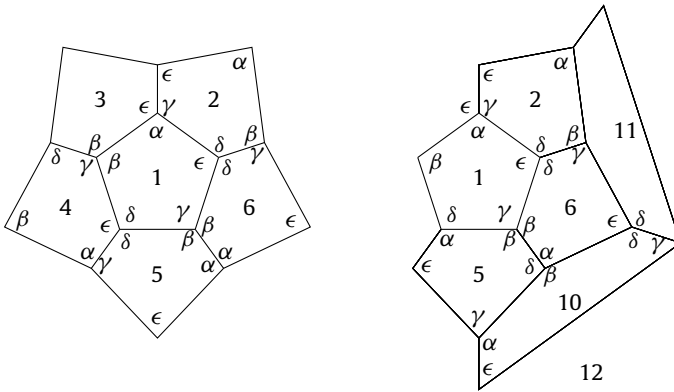


Fig. 28. Tilings for the fourth arrangement.

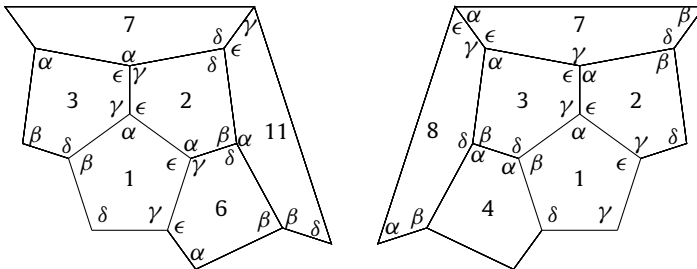


Fig. 29. Tilings for the fourth arrangement, continued.

In the argument above, we started from clockwise P_4 and derived counterclockwise P_6 . The same argument can be reversed, so that starting from counterclockwise P_6 , we can derive clockwise P_4 . On the other hand, if on the left of Fig. 30, we exchange β with δ , ϵ with γ , and then flip the picture

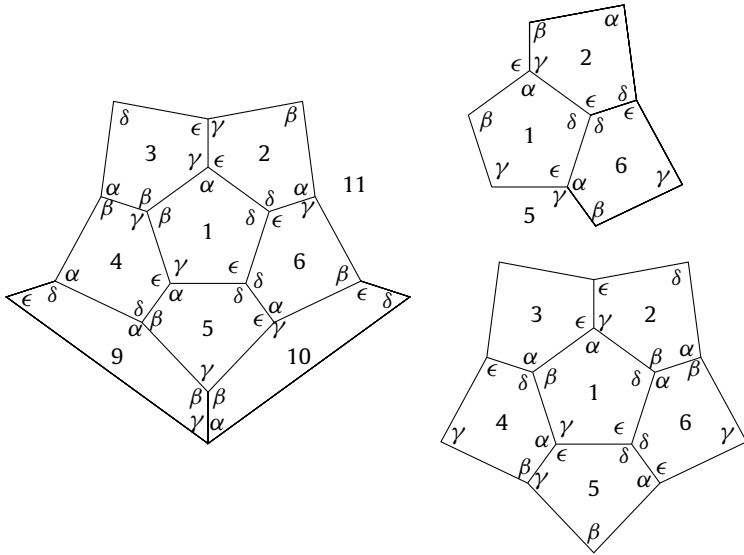


Fig. 30. Tilings for the fifth arrangement.

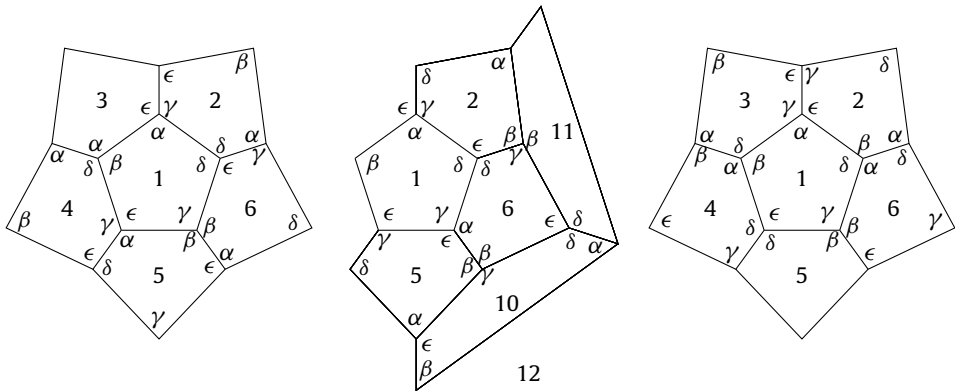


Fig. 31. Tilings for the sixth arrangement.

horizontally, we see that the argument also tells us that P_6 cannot be clockwise. We conclude that P_6 can be neither counterclockwise nor clockwise, a contradiction.

We remark that the conclusion of the case is the following: If the angles in a tile (of fifth arrangement) are counterclockwise oriented, and the vertex at α is $\alpha\gamma\epsilon$, then α, γ, ϵ must be counterclockwise oriented at the vertex.

Arrangement 5, Case 1. $A_{2,13} = \gamma, A_{3,12} = \epsilon$.

If P_2 is counterclockwise oriented, as on the upper right of Fig. 30, then by AVC, we get all the angles of P_6 and $A_{5,16} = \gamma$. Now P_6 is counterclockwise oriented and α, γ, ϵ are clockwise oriented at V_{156} . This contradicts to the conclusion of Case 2. If P_2 is clockwise oriented, as on the lower right of Fig. 30, then by AVC, we successively get all the angles of P_6, P_5, P_4 and $A_{3,14} = \alpha$. We find α, ϵ adjacent in P_3 , a contradiction.

Arrangement 6, Case 1. $A_{2,13} = \gamma, A_{3,12} = \epsilon$.

If P_2 is counterclockwise oriented, as on the left of Fig. 31, then by AVC, we successively get all the angles of P_6, P_5, P_4 and $A_{3,14} = \alpha$. We find α, ϵ adjacent in P_3 , a contradiction. If P_2 is clockwise

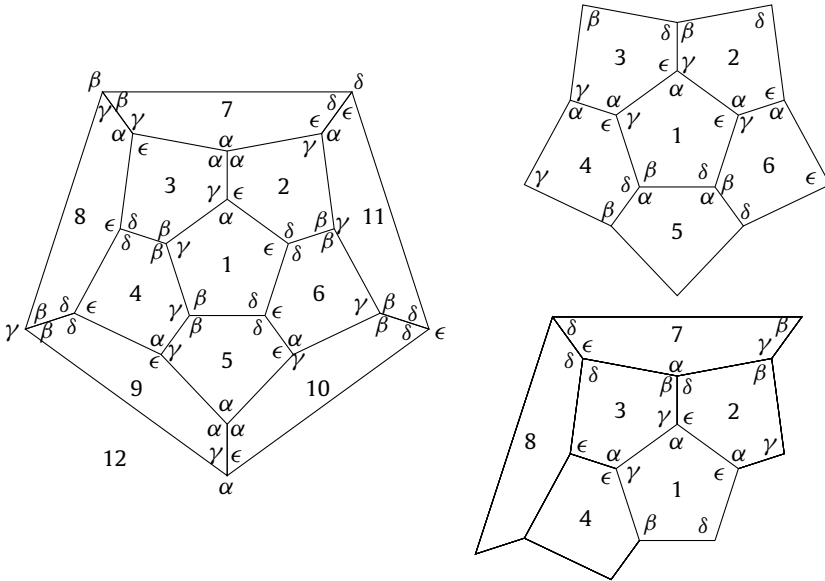


Fig. 32. Tilings for the seventh arrangement.

oriented, as in the middle of Fig. 31, then by AVC, we successively get all the angles of P_6, P_5, P_{10} and $A_{11,26} = \beta, A_{11,6\bar{1}0} = \delta$. We find β, δ adjacent in P_{11} , a contradiction.

Arrangement 6, Case 2. $A_{2,13} = \epsilon, A_{3,12} = \gamma$.

The case is the right of Fig. 31. By AVC, we successively get all the angles of P_2, P_3, P_4, P_6 and $A_{5,14} = \delta, A_{5,16} = \beta$. We find β, δ adjacent in P_5 , a contradiction.

Arrangement 7, Case 1. $A_{2,13} = \gamma, A_{3,12} = \epsilon$.

The case is the upper right of Fig. 32. By AVC, we successively get all the angles of P_2, P_3, P_4, P_6 and $A_{5,14} = A_{5,16} = \alpha$. We find two α in P_5 , a contradiction.

Arrangement 7, Case 2. $A_{2,13} = \epsilon, A_{3,12} = \gamma$.

If P_2 is clockwise oriented, as on the left of Fig. 32, then by AVC, we get all the angles of P_6, P_5 and $A_{11,26} = A_{10,56} = \gamma$. Since γ, ϵ are not adjacent, neither $A_{10,6\bar{1}1}$ nor $A_{11,6\bar{1}0}$ can be ϵ . Then by $A_{6,10\bar{1}1} = \gamma$ and AVC, we get $A_{10,6\bar{1}1} = A_{11,6\bar{1}0} = \beta$ and then all the angles of P_{10}, P_{11} . By the angles we know so far and AVC, we successively get all the angles of $P_9, P_4, P_3, P_7, P_8, P_{12}$. There is no contradiction, and we get an angle congruent tiling with anglewise vertex combination $\{2\alpha^3, 6\alpha\gamma\epsilon, 6\beta^2\gamma, 6\delta^2\epsilon\}$.

By the same argument (in the left picture, exchanging β with δ, ϵ with γ , and then flipping horizontally), we also conclude that, if P_3 is clockwise oriented, then we get the same angle congruent tiling on the left of Fig. 32. Therefore the only remaining case is that both P_2, P_3 are counterclockwise oriented, as on the lower right of Fig. 32. By AVC, we get all the angles of P_7 and $A_{8,37} = \delta$. Since $A_{8,34}$ is adjacent to α in P_8 , we get $A_{8,34} = \beta$ or ϵ . Either way contradicts to AVC at V_{348} .

Arrangement 8, Case 1. $A_{2,13} = \gamma, A_{3,12} = \epsilon$.

The case is the left of Fig. 33. By AVC, we successively get all the angles of P_2, P_3, P_4, P_6 and $A_{5,14} = A_{5,16} = \alpha$. We find two α in P_5 , a contradiction.

Arrangement 8, Case 2. $A_{2,13} = \epsilon, A_{3,12} = \gamma$.

If P_2 is counterclockwise oriented and P_3 is clockwise oriented, as on the right of Fig. 33, then by AVC, we successively get all the angles of $P_7, P_8, P_{11}, P_4, P_6$ and $A_{5,14} = \alpha, A_{5,16} = \delta$. We find α, δ adjacent in P_5 , a contradiction. By the same argument (on the right of Fig. 33, exchanging β with δ, ϵ with γ , and then flipping horizontally), we also conclude that it is impossible for P_2 to be clockwise oriented and P_3 to be counterclockwise oriented. Therefore P_2 and P_3 must have the same orientation.

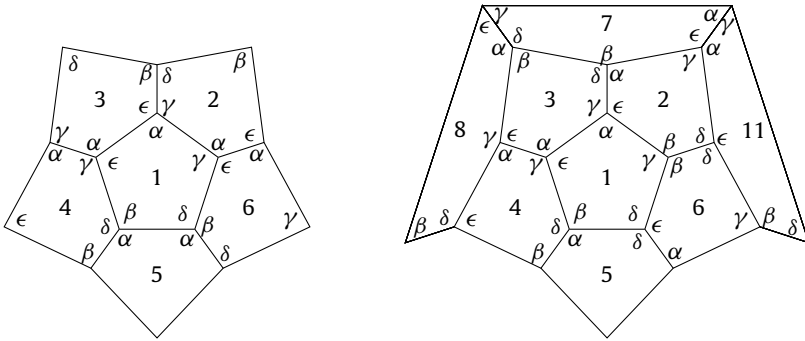


Fig. 33. Tilings for the eighth arrangement.

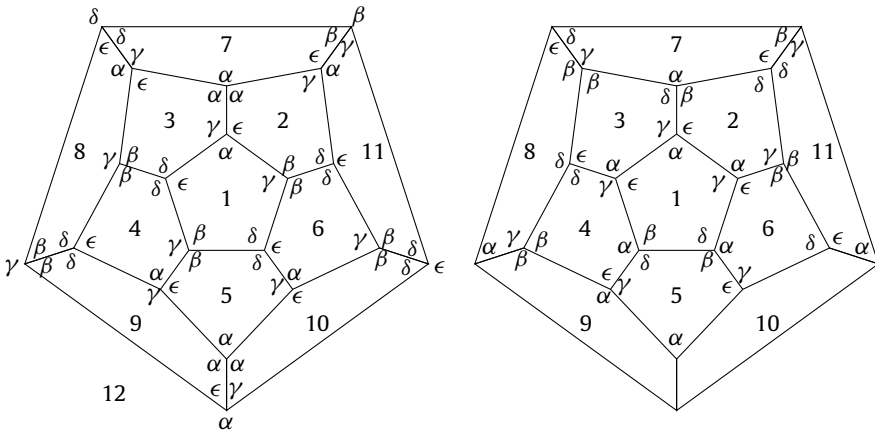


Fig. 34. Tilings for the eighth arrangement, continued.

If both P_2, P_3 are counterclockwise oriented, as on the left of Fig. 34, then by AVC, we successively get all the angles of $P_7, P_8, P_{11}, P_4, P_6, P_5, P_9, P_{10}, P_{12}$. There is no contradiction, and we get an angle congruent tiling with anglewise vertex combination $\{2\alpha^3, 6\alpha\gamma\epsilon, 6\beta^2\gamma, 6\delta^2\epsilon\}$.

If both P_2, P_3 are clockwise oriented, as on the right of Fig. 34, then by AVC, we successively get all the angles of $P_7, P_8, P_{11}, P_4, P_6, P_5$ and $A_{9,45} = \alpha, A_{9,48} = \beta$. We find α, β adjacent in P_9 , a contradiction. \square

7. Proof of main theorem

We have studied the spherical tilings by 12 congruent pentagons from the purely edge length viewpoint and the purely angle viewpoint. In this section, we combine the two together to prove the main theorem.

The following result shows that certain combinations of edge lengths and angles cannot happen for geometrical reasons.

Lemma 21. *Suppose in the spherical pentagon on the left of Fig. 35, three of the following four equalities hold*

$$a = b, \quad c = d, \quad \beta = \gamma, \quad \delta = \epsilon,$$

then all four equalities hold.

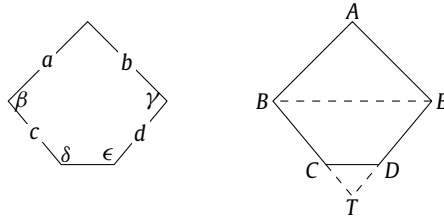


Fig. 35. Geometrical constraint.

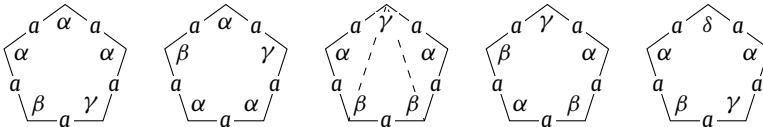


Fig. 36. Combinations of a^5 with $\alpha^3\beta\gamma$, $\alpha^2\beta^2\gamma$, $\alpha^2\beta\gamma\delta$.

Proof. Add auxiliary lines as on the right of Fig. 35.

Assume $a = b$ and $\beta = \gamma$. Then $a = b$ implies $\triangle ABE$ is an isosceles triangle, and $\beta = \gamma$ further implies that $\triangle TBE$ is also an isosceles triangle. Then $c = d$ if and only if TC and TD have the same length. This is further equivalent to $\triangle TCD$ being an isosceles triangle, which is the same as $\delta = \epsilon$.

Assume $c = d$ and $\epsilon = \delta$. Then $\epsilon = \delta$ implies $\triangle TCD$ is an isosceles triangle, and $c = d$ further implies $\triangle TBE$ is also an isosceles triangle. Then $\beta = \gamma$ is equivalent to $\angle ABE = \angle AEB$, which means $a = b$. \square

We recall the notations P_i, E_{ij}, V_{ijk} and the names a -edge, abc -vertex introduced in Section 4. We recall the names $\alpha\beta\gamma$ -vertex, $\alpha\beta\gamma$ -type vertex introduced in Section 5. We also recall the notation $A_{i,jk}$ and the names α -angle, $\alpha\beta$ -edge in a tile introduced in Section 6.

We can also name an angle by the two edges around the angle. For example, the angle $A_{1,23}$ in Fig. 2 is an ab -angle, and $A_{1,34}$ is an a^2 -angle.

We use a, b, c, \dots to denote distinct edge lengths. We use $\alpha, \beta, \gamma, \dots$ to denote distinct angles. Moreover, α will always be $\frac{2\pi}{3}$.

Proof of main theorem. The proof is divided according to the edge length combination in the pentagon.

Case a^5 . If the angle combination is α^5 , then the pentagon is regular, and the tiling is the regular dodecahedron.

Now consider the middle three cases in Proposition 17. Up to symmetry, the angle combination $\alpha^3\beta\gamma$ can be arranged as the first or the second in Fig. 36. By Proposition 18 and up to symmetry, the angle combination $\alpha^2\beta^2\gamma$ can be arranged as the third or the fourth in Fig. 36. By Proposition 19 and up to symmetry, the angle combination $\alpha^2\beta\gamma\delta$ can be arranged as the fifth in Fig. 36.

By Lemma 21, we have $\beta = \gamma$ in all except the third arrangement. In the third arrangement, divide the pentagon into three triangles. Note that the three triangles (and therefore the pentagon also) are completely determined by a . Moreover, the area of the two triangles on the side are strictly increasing in a . The length of the two dividing lines are strictly increasing in a , and the area of the triangle at the middle is also strictly increasing in a . Therefore the area of the pentagon is strictly increasing in a . So there is a unique a , such that the area of the pentagon is $\frac{1}{12}$ of the area of the sphere. Since this area is reached by the pentagon in the regular dodecahedron, we conclude that the third arrangement is the regular one.

Finally, for the angle combination $\alpha\beta\gamma\delta\epsilon$, we get four possible tilings from Proposition 20. Any one exists as long as there is a relevant pentagon with all sides equal. After suitable permutation among $\beta, \gamma, \delta, \epsilon$, these are included in the first four tilings in Fig. 1.

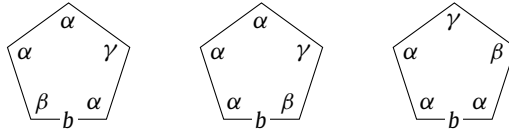


Fig. 37. Combinations of a^4b and $\alpha^3\beta\gamma$.

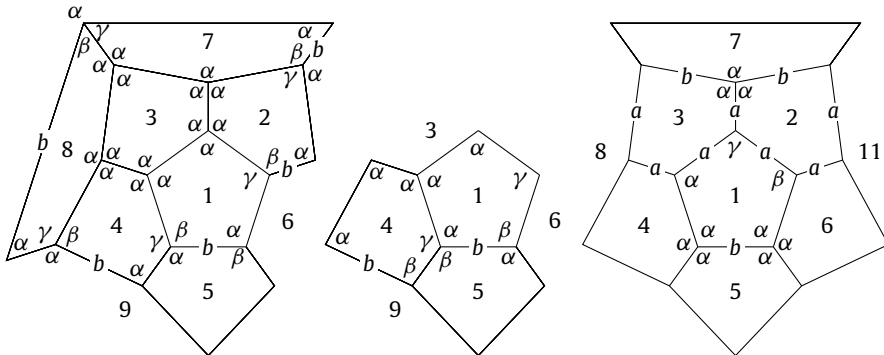


Fig. 38. Tilings for the combinations of a^4b and $\alpha^3\beta\gamma$.

Case a^4b . Recall that the edgewise vertex combination is $\{8a^3, 12a^2b\}$. We will not use the complete classification in Proposition 9. Instead, we will study the edge–angle combinations directly.

Subcase α^5 . In this case, b is completely determined by a , and the area of the pentagon is strictly increasing in a . Therefore there is only one a such that the area is $\frac{1}{12}$ of the area of the sphere. This is the regular dodecahedron tiling. Therefore we get $a = b$, a contradiction.

Subcase $\alpha^3\beta\gamma$. In addition to the edgewise vertex combination $\{8a^3, 12a^2b\}$, Proposition 17 also tells us the anglewise vertex combination $\{8\alpha^3, 12\alpha\beta\gamma\}$, which implies the following condition.

AVC(for $\alpha^3\beta\gamma$): Any vertex is either α^3 or $\alpha\beta\gamma$.

If β and γ are separated in the pentagon, as on the left of Fig. 37, then up to symmetry, there are three ways of assigning b , as the α^2 -edge or as one of two $\alpha\beta$ -edges. Lemma 21 implies that two such assignments lead to $\beta = \gamma$, a contradiction. The only assignment that does not lead to a contradiction is the left of Fig. 37. Similarly, if β and γ are adjacent in the pentagon, then among three possible ways (up to symmetry) of assigning b , one leads to a contradiction and we are left with the middle and the right of Fig. 37.

We start the further discussion by assuming the tile P_1 in Fig. 9 is given by one of the pentagons in Fig. 37.

The *first combination* is the left of Fig. 38. Since the b -edge is also an $\alpha\beta$ -edge of P_5 , the angles $A_{5,14}, A_{5,16}$ must be α, β . By the AVC condition above, we cannot have β^2 at a vertex. Therefore we get $A_{5,14} = \alpha, A_{5,16} = \beta$. By AVC, we further get $A_{4,15} = \gamma$, so that the b -edge of P_4 is either E_{34} or E_{49} . We note that, for the current combination, the choice of the b -edge together with the location of γ determine all the angles of a tile. The choice $E_{34} = b$ gives all the angles of P_4 and $A_{4,13} = \alpha, A_{4,38} = \beta$. By AVC, we get $A_{3,14} = \alpha$, and since b is an $\alpha\beta$ -edge of P_3 , we further get $A_{3,48} = \beta$. But $A_{4,38} = A_{3,48} = \beta$ contradicts to AVC at V_{348} . Therefore we must have $E_{49} = b$. Together with $A_{4,15} = \gamma$, we get all the angles of P_4 , as described in the picture.

So we see P_1 completely determines P_4 . Note that the edge E_{14} shared by the two tiles is opposite to γ in P_1 . Using this as a guidance, P_4 determines P_8, P_8 determines P_7 , and P_7 determines P_2 . Then by AVC, all vertices of P_3 are α^3 -vertices, so that all the angles of P_3 are α , a contradiction.

The *second combination* is the middle of Fig. 38. Similar to the first combination, by AVC and the fact that the b -edge is an $\alpha\beta$ -edge, we get $A_{5,16} = \alpha, A_{5,14} = \beta$. By AVC again, we get $A_{4,15} = \gamma$, and



Fig. 39. Angles around b -edge.

the b -edge of P_4 is either E_{34} or E_{49} . If $E_{34} = b$, then we get all the angles of P_4 and $A_{4,13} = \beta$. By AVC, we get $A_{3,14} = \gamma$, contradicting to the fact that b is an $\alpha\beta$ -edge in P_3 . Therefore we must have $E_{49} = b$. This determines all the angles of P_4 , as described in the picture.

So we see P_1 completely determines P_4 , just like the first combination. By the same guidance, we can successively get all the angles of P_8, P_7, P_2 and find all the angles of P_3 to be α , a contradiction.

The *third combination* is the right of Fig. 38. The combination has the special property that γ and b are opposite to each other in a tile and determine the location of each other. Moreover, since the b -edge is an α^2 -edge in P_5 , we get $A_{5,14} = A_{5,16} = \alpha$. By AVC, this implies that both ends of any b -edge are α^3 -vertices. By the edgewise vertex combination, this implies that all $\alpha\beta\gamma$ -vertices are a^3 -vertices. By AVC, we get $V_{123} = V_{126} = \alpha\beta\gamma$, so that $V_{123} = V_{126} = a^3$. Moreover, the angles $A_{2,13}, A_{3,12}$ are α, β , so that the opposite edges $E_{2\bar{1}1} = E_{38} = a$. From the four known a -edges of P_2 , we get $E_{27} = b$. As an end of this b -edge, we get $V_{237} = \alpha^3$. As the edge opposite to $A_{3,27} = \alpha$, we get $E_{34} = a$. The four known a -edges of P_3 imply $E_{37} = b$. Then we find two b -edges in P_7 , a contradiction.

Subcase $\alpha^2\beta^2\gamma$. We need to assign the b -edges to the two tilings in Fig. 21. The b -edge has to be a $\rho\tau$ -edge for some (not necessarily distinct) angles ρ and τ . Therefore the angles around the b -edge have to appear as one of the two scenarios in Fig. 39 (different b -edges may follow different scenario).

For the tiling on the left of Fig. 21, the only edge in P_1 fitting the scenarios in Fig. 39 is the β^2 -edge. By assigning the β^2 -edge as b in all the tiles, we get a special case of the tiling T_5 in Fig. 1 and Fig. 2, after suitable change of symbols.

For the tiling on the right of Fig. 21, the thick edge (which is the $\beta\gamma$ -edge) cannot be the b -edge because this would imply $\beta = \gamma$ by Lemma 21. On the other hand, any ordinary edge is adjacent to the thick edge in some tile. Therefore in some tile, the b -edge is adjacent to the thick edge. Since the congruence between tiles preserves the thick edge (because it is the $\beta\gamma$ -edge) and the b -edge, the b -edge is adjacent to the thick edge in every tile. Then E_{12}, E_{13} cannot be the b -edge because they are not adjacent to the thick edge E_{15} of P_1 . Moreover, E_{14}, E_{16} cannot be the b -edge because they are not adjacent to the thick edges in P_4, P_6 . Therefore no edge of P_1 can be the b -edge, a contradiction.

Subcase $\alpha^2\beta\gamma\delta$. We need to assign the b -edges to the right of Fig. 22. Again the angles around the b -edge must fit Fig. 39. The only such possible edge is the $\beta\gamma$ -edge. However, $E_{15} = b$ implies $\beta = \gamma$ by Lemma 21. We get a contradiction.

Subcase $\alpha\beta\gamma\delta\epsilon$. We need to assign the b -edges to the four tilings given by Proposition 20. The angles around the b -edge must fit Fig. 39.

In the tiling on the left of Fig. 25, no edge of P_3 fits Fig. 39.

In the tilings on the left of Figs. 27, 32, 34, by looking for edges of P_3 in the first two cases and of P_1 in the third case, we find the only edge fitting Fig. 39 is the $\beta\delta$ -edge. By assigning b to all the $\beta\delta$ -edges and suitably changing the symbols for the angles, we get the tilings T_2, T_3, T_4 of Fig. 1.

Case a^3b^2 . The tiling is given on the left of Fig. 15 and has the edgewise vertex combination $\{4a^3, 4b^3, 12a^2b\}$. The pentagon has 2 a^2 -angles θ_1 and θ_2 , 2 ab -angles ρ_1 and ρ_2 , and 1 b^2 -angle τ , described on the left of Fig. 40. Then a^3 -vertices, a^2b -vertices, b^3 -vertices are respectively $\theta_i\theta_j\theta_k$ -vertices, $\theta_i\rho_j\rho_k$ -vertices, τ^3 -vertices. All such vertices exist.

We have $\theta_1 + \theta_2 + \rho_1 + \rho_2 + \tau = 5\alpha$. Since there are τ^3 -vertices, we have $3\tau = 3\alpha$. Therefore

$$\tau = \alpha, \quad \theta_1 + \theta_2 + \rho_1 + \rho_2 = 4\alpha.$$

By Lemma 21, we have $\theta_1 = \theta_2$ if and only if $\rho_1 = \rho_2$. In this case, we have $\theta_1 + 2\rho_1 = 3\alpha$ at a $\theta_i\rho_j\rho_k$ -vertex. Combined with $2\theta_1 + 2\rho_1 = 4\alpha$, we conclude that all the angles are equal to α . By Lemma 21 again, we must have $a = b$, a contradiction. Therefore we have $\theta_1 \neq \theta_2$ and $\rho_1 \neq \rho_2$.

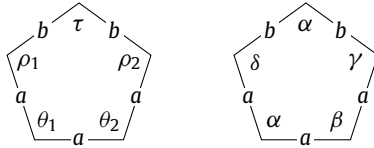


Fig. 40. Pentagon for the combination of a^3b^2 and $\alpha^2\beta\gamma\delta$.

At a $\theta_i\theta_j\theta_k$ -vertex, we have $\theta_i + \theta_j + \theta_k = 3\alpha$. Without loss of generality, we may assume the equality is either $2\theta_1 + \theta_2 = 3\alpha$ or $3\theta_1 = 3\alpha$.

Subcase $2\theta_1 + \theta_2 = 3\alpha$, $\theta_1 \neq \theta_2$. Since $\theta_1 \neq \theta_2$, the only a^3 -vertices (same as $\theta_i\theta_j\theta_k$ -vertices) are $\theta_1^2\theta_2$ -vertices. There are 4 a^3 -vertices, which involve 8 angle θ_1 and 4 angle θ_2 . Among the total of 12 angle θ_1 and 12 angle θ_2 , the remaining 4 angle θ_1 and 8 angle θ_2 must appear in the remaining 12 a^2b -vertices (same as $\theta_i\rho_j\rho_k$ -vertices).

Note that a $\theta_1\rho_1\rho_2$ -vertex implies $\theta_1 + \rho_1 + \rho_2 = 3\alpha$. Combined with $\theta_1 + \theta_2 + \rho_1 + \rho_2 = 4\alpha$ and $2\theta_1 + \theta_2 = 3\alpha$, we get $\theta_1 = \alpha = \theta_2$, a contradiction. Similarly, there is no $\theta_2\rho_1\rho_2$ -vertex. Therefore a^2b -vertices are $\theta_i\rho_j^2$ -vertices, and we have 4 $\theta_1\rho_1^2$ -vertices and 8 $\theta_2\rho_2^2$ -vertices. Since $j = k$ will lead to $\theta_1 = 3\alpha - 2\rho_j = 3\alpha - 2\rho_k = \theta_2$, we must have $j \neq k$. Without loss of generality, we may assume $j = 1, k = 2$ and get $\theta_1 + 2\rho_1 = 3\alpha$ and $\theta_2 + 2\rho_2 = 3\alpha$. Combined with $\theta_1 + \theta_2 + \rho_1 + \rho_2 = 4\alpha$ and $2\theta_1 + \theta_2 = 3\alpha$, we get all the angles equal to α , a contradiction.

Subcase $3\theta_1 = 3\alpha$, $\theta_1 \neq \theta_2$. We have $\theta_1 = \alpha$ and $\theta_2 + \rho_1 + \rho_2 = 3\alpha$. If any of $\theta_1\theta_2^2, \theta_1^2\theta_2, \theta_2^3$ is a vertex, then we will always get $\theta_2 = \alpha = \theta_1$, a contradiction. Therefore the only a^3 -vertices (same as $\theta_i\theta_j\theta_k$ -vertices) are θ_1^3 -vertices. Since all 12 angle θ_1 are concentrated at the 4 a^3 -vertices, the 12 a^2b -vertices (same as $\theta_i\rho_j\rho_k$ -vertices) must be $\theta_2\rho_j\rho_k$ -vertices. The possible $\theta_2\rho_j\rho_k$ -vertices are $\theta_2\rho_1^2, \theta_2\rho_2^2, \theta_2\rho_1\rho_2$. If any two appear as vertices, then we get $\rho_1 = \rho_2$, a contradiction. Since both ρ_1 and ρ_2 must appear, we conclude that all 12 a^2b -vertices are $\theta_2\rho_1\rho_2$ -vertices.

Denote $\theta_2 = \beta, \rho_1 = \delta, \rho_2 = \gamma$. Then $\alpha \neq \beta$ and $\delta \neq \gamma$, and the pentagon is given by the right of Fig. 40. The discussion in the last paragraph tells us that, as far as vertices are concerned, we have

$$a^3 = \alpha^3, \quad a^2b = \beta\gamma\delta \quad (\text{note that } \theta_2\rho_1\rho_2 = \beta\gamma\delta).$$

Now we try to fit the right of Fig. 40 into the left of Fig. 15, the only edge congruent tiling for a^3b^2 . The two a^2 -angles $A_{1,45}$ and $A_{1,56}$ in P_1 are $\theta_1 = \alpha$ and $\theta_2 = \beta$. Since $V_{145} = a^3 = \alpha^3$ and $V_{156} = a^2b = \beta\gamma\delta$, we get $A_{1,45} = \alpha$ and $A_{1,56} = \beta$. We also know the b^2 -angle of P_1 is $A_{1,23} = \tau = \alpha$. Knowing three angles of P_1 determines the other two angles. By the same argument, we get all the angles of all tiles. After rotating the result by angle $\frac{\pi}{5}$ in clockwise orientation and letting $c = a$, we get the tiling T_5 in Fig. 1 and Fig. 2.

Case a^2b^2c . The tiling is given on the left of Fig. 17. Since a, b, c are distinct, we can uniquely define $\alpha_1, \alpha_2, \beta, \gamma, \delta$ as the a^2 -angle, the b^2 -angle, the ac -angle, the bc -angle, the ab -angle in the pentagon. After assigning these angles, we find $3\alpha_1 = 3\alpha_2 = 2\pi$ at b^3 -vertices. Therefore $\alpha_1 = \alpha_2 = \alpha$, and we get the tiling T_5 in Fig. 1 and Fig. 2.

This completes the proof of the main classification theorem. \square

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