# An Elementary and Direct Proof of the Painlevé Property for the Painlevé Equations I, II and IV

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**Abstract**

We present a direct and elementary proof that all the solutions of the Painlevé Equations I, II and IV are meromorphic functions on the whole complex plane. The proof uses some ideas from the existing proofs but applies the ideas in a different setting.

# 1 Introduction

The six Painlevé equations have played a very important role in mathematics and physics. A fundamental property of these equations is that all solutions are meromorphic on the whole complex plane except at the fixed singularities. In particular, the equations have the Painlevé property, i.e., all the movable singularities are single-valued.

Although the fundamental property is widely accepted to be true, the history of the proofs has been messy (see the section on historical background in [1], for example). In the literature on the subject, one often finds conflicting comments on whether certain proofs are rigorous or whether the Painlevé property has been rigorously proved for certain Painlevé equations.

From a broader perspective, not only rigorous proofs, but good and clean proofs are also desirable since the property is considered as closely related to the integrability. In fact, the property is the heuristic reason behind the Painlevé test, the most widely used technique for detecting integrability. A good proof that relies mostly on the Painlevé test itself but not much else would help us better understand the relation. Among the existing proofs, Painlevé's original elementary and direct approach for the first equation in [5], which was later extended to the second equation in [1], the fourth equation in [6], and a modified version of the third equation in [2], is probably the closest to this purpose.

In this paper, we present another elementary and direct proof that all solutions of the following Painlevé equations

$$
\begin{aligned}
(P_{\mathrm{I}}) \quad u'' &= 6u^2 + x, \\
(P_{\mathrm{II}}) \quad u'' &= 2u^3 + xu + \alpha, \\
(P_{\mathrm{IV}}) \quad u'' &= \frac{1}{2u}u'^2 + \frac{3}{2}u^3 + 4xu^2 + 2(x^2 - \alpha)u + \frac{\beta}{u},
\end{aligned}
$$

are meromorphic functions defined on the whole complex plane. The key is the analysis of the behavior of the solution near movable singularities. The original Painlevé equations are no longer suitable for such analysis. Instead, some changes of variables that convert the Painlevé equations to regular systems at the singularities are used. These regular systems are well known and have been used in many proofs of the Painlevé property before, such as [1] [2] [5] [6]. However, the regular systems are used in different context in our proof. In the other proofs, the regular systems are used *at* the singularities. We use the regular systems *near* the singularities.

We would like to remark that it has been demonstrated in [3] [4] that the existence of the regularizations in the proofs is equivalent to passing the Painlevé test in a rigorous sense.

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## 2 The First Painlevé Equation

Let us consider a solution $u$ of the first Painlevé equation $u'' = 6u^2 + x$, with given finite initial data $u(x_0)$ and $u'(x_0)$ at a given location $x_0$. Let $\lambda$ be a straight line segment with $x_0$ as one end and length $B > \max\{|x_0|, 1\}$. We will prove that $u$ extends to a meromorphic function in a neighborhood of $\lambda$. By taking all choices of directions and lengths for $\lambda$, such neighborhoods cover the whole complex plane. Therefore we conclude that $u$ extends to a meromorphic function on the whole complex plane.

We fix a number $A$ sufficiently large compared with $|u(x_0)|$, $B$, and $|k|$, where

$$k = u'(x_0)^2 - 4u(x_0)^3 - 2x_0 u(x_0)$$

depends only on the solution $u$ and the initial location $x_0$. The specific size of $A$ will be clear from subsequent discussions. By $B > \max\{|x_0|, 1\}$, we have $|x| < 2B$ for $x \in \lambda$. We also denote by $\lambda_{x_0,x}$ the segment of $\lambda$ between $x_0$ an $x$.

Since the first Painlevé equation is regular, we may analytically extend $u$ along $\lambda$ as long as $u$ is bounded. If we can extend $u$ so that $|u| \leq A$ along whole $\lambda$, then we are done. Otherwise, we may analytically extend $u$ along $\lambda$ until $x_1$, such that

$$|x_1| \leq B, \quad |u(x_1)| = A; \quad \text{and} \quad |u(x)| \leq A \quad \text{for all } x \text{ on } \lambda_{x_0,x_1}. \tag{1}$$

Expecting $u$ to be near a pole in the vicinity of $x_1$, we introduce the indicial normalization $u = \theta^{-2}$ near $x_1$. There are two choices of $\theta$ that differ by a sign. We fix one choice now and may modify the choice later on if necessary.

Now we will carry out some estimations. We will often write $X < cY$ to indicate that $X$ is dominated by $Y$ with a specific number $c$ (fixed once a large $A$ is fixed) as the factor. See the remark after the inequality (4) for a concrete example. Basically this means that we abuse the notation by not writing $c_1$, $c_2$, $\cdots$, for different inequalities.

We already know $|\theta(x_1)| = A^{-1/2}$. To estimate $\theta'(x_1)$, we multiply $u'$ to the first Painlevé equation and integrate to get

$$u'^2 = 4u^3 + 2xu - 2\int_{x_0}^{x} udx + k. \tag{2}$$

Then we substitute $u = \theta^{-2}$ into (2)

$$\theta'^2 = 1 + \frac{x}{2}\theta^4 - \frac{1}{2}\theta^6 \int_{x_0}^{x} udx + \frac{k}{4}\theta^6. \tag{3}$$

By

$$\begin{aligned}
|x_1\theta(x_1)^4| &\leq BA^{-2}, \\
\left|\theta(x_1)^6 \int_{x_0}^{x_1} udx\right| &< A^{-3}(4BA) = 4BA^{-2}, \\
|k\theta(x_1)^6| &= |k|A^{-3},
\end{aligned}$$

we have

$$|\theta'(x_1)^2 - 1| \leq \frac{1}{2}BA^{-2} + 2BA^{-2} + \frac{1}{4}|k|A^{-3} < cBA^{-2}. \tag{4}$$

Specifically, if we have chosen $A > |k|$ at the beginning, then we actually have $|\theta'(x_1)^2 - 1| < 3BA^{-2}$.

The inequality (4) implies that either $|\theta'(x_1) - 1| < cBA^{-2}$ or $|\theta'(x_1) + 1| < cBA^{-2}$. By changing the sign of $\theta$ if necessary, we may assume $|\theta'(x_1) - 1| < cBA^{-2}$. Then we may introduce another variable $\xi$ by

$$u' = -2\theta^{-3} - \frac{x}{2}\theta - \frac{1}{2}\theta^2 + \xi\theta^3. \tag{5}$$

The transform $(u, u') \leftrightarrow (\theta, \xi)$ given by $u = \theta^{-2}$ and (5) converts the first Painlevé equation to the following regular system

$$\left\{\begin{aligned}
\theta' &= 1 + \frac{x}{4}\theta^4 + \frac{1}{4}\theta^5 - \frac{\xi}{2}\theta^6, \\
\xi' &= \frac{x^2}{8}\theta + \frac{3x}{8}\theta^2 + \left(\frac{1}{4} - x\xi\right)\theta^3 - \frac{5\xi}{4}\theta^4 + \frac{3\xi^2}{2}\theta^5.
\end{aligned}\right. \tag{6}$$

The system appeared first on page 229 of [5] and was used in essentially every elementary proof of the Painlevé property for the first Painlevé equation. See (3.2) on page 349 of [1] and (Ia), (Ib) on page 372 of [6]. What is new here is the context in which the system is used. We have carefully set up the location to apply the system, so that the subsequent argument becomes quite straightforward.

We estimate $\xi(x_1)$ by making use of (3):

$$\begin{aligned}
&\left|\theta'(x_1)^2 - \Big(1+\frac{x_1}{4}\theta(x_1)^4+\frac{1}{4}\theta(x_1)^5\Big)^2\right| \\
\le\ & \left|\theta'(x_1)^2 - 1 - \frac{x_1}{2}\theta(x_1)^4 - \frac{1}{2}\theta(x_1)^5\right| + \left|\frac{1}{4}x_1\theta(x_1)^4+\frac{1}{4}\theta(x_1)^5\right|^2 \\
\le\ & \frac{1}{2}\left|\theta(x_1)^6\int_{x_0}^{x_1} u dx\right| + \frac{1}{4}|k\theta(x_1)^6| + \frac{1}{2}|\theta(x_1)^5| + \frac{1}{16}\left(|x_1|+|\theta(x_1)|\right)^2|\theta(x_1)|^8 \\
\le\ & 2BA^{-2}+\frac{1}{4}|k|A^{-3}+\frac{1}{2}A^{-5/2}+\frac{1}{16}(B+A^{-1/2})^2A^{-4} \\
<\ & cBA^{-2}.
\end{aligned}$$

Since for large $A$, $\theta'(x_1)$ is very close to 1 and $\theta(x_1)$ is very small, the inequality above implies

$$\left|\theta'(x_1) - 1 - \frac{x_1}{4}\theta(x_1)^4 - \frac{1}{4}\theta(x_1)^5\right| < cBA^{-2}$$

and

$$|\xi(x_1)| = 2|\theta(x_1)|^{-6}\left|\theta'(x_1) - 1 - \frac{x_1}{4}\theta(x_1)^4 - \frac{1}{4}\theta(x_1)^5\right| < cBA.$$

Thus $u$ is converted to a solution of the system (6) around $x_1$, with

$$|\theta(x_1)| = A^{-1/2}, \quad |\xi(x_1)| < cBA. \tag{7}$$

This helps us to analyze the behavior of $(\theta,\xi)$ around $x_1$. The key for such an investigation is the following general result on ODE systems.

**Lemma** *Consider the initial value problem*

$$w' = f(x,w), \qquad w(x_0) = w_0,$$

*where* $f = (f_1,\cdots,f_n)$ *and* $w = (w_1,\cdots,w_n)$. *Suppose there are positive numbers* $\epsilon, \rho_i, L_i, M_i, N_{ij}, a_i$, *such that*

1. *If* $|x-x_0|\le\epsilon$ *and* $|w_i - w_{i0}|\le\rho_i$, *then*

$$|f_i(x,w)|\le L_i, \quad |\partial_x f_i(x,w)|\le M_i, \quad |\partial_{w_j} f_i(x,w)|\le N_{ij};$$

*2.* $\epsilon L_i \le \rho_i$, $\epsilon(\sum_i a_i N_{ij}) < a_j$.

*Then the initial value problem has a unique solution* $w(x)$ *for* $|x-x_0| \le \epsilon$. *Moreover, the solution satisfies*

$$|w_i(x) - w_{i0} - f_i(x_0, w_0)(x - x_0)| \le \frac{1}{2}(M_i + L_1 N_{i1} + \cdots + L_n N_{in})|x - x_0|^2.$$

The lemma may be proved by showing that

$$\Phi(w) = w_0 + \int_{x_0}^{x} f(x, w(x))dx$$

is a contracting operator on the complete metric space of analytic functions $w(x) = (w_1(x), \cdots, w_n(x))$ defined on the closed disk $|x - x_0| \le \epsilon$ and satisfying $|w_i(x) - w_{i0}| \le \rho_i$ for all $x$ in the disk, and with the metric

$$d(w, u) = \sum_i a_i \max_{|x-x_0|\le\epsilon} |w_i(x) - u_i(x)|.$$

The estimation for $|w_i(x)-w_{i0}-f_i(x_0,w_0)(x-x_0)|$ can be derived from the following remainder formula for Taylor series

$$g(x) - g(x_0) - g'(x_0)(x - x_0) = (x - x_0)^2 \int_0^1 tg''(tx_0 + (1 - t)x)dt.$$

and

$$|w_i''| = |\partial_x f_i + \sum_j (\partial_{w_j} f_i) f_j| < M_i + L_1 N_{i1} + \cdots + L_n N_{in}.$$

Now we apply the lemma to system (6) with initial data satisfying (7). We denote the two functions on the right side of (6) as $f_1(x, \theta, \xi)$ and $f_2(x, \theta, \xi)$. Moreover, we take

$$\epsilon = 3A^{-1/2}, \quad \rho_1 = c_1 A^{-1/2}, \quad \rho_2 = c_2 BA,$$

where $c_1$ and $c_2$ are absolute constants to be determined by (8) and (9).

For $|x - x_1| \le \epsilon$, $|\theta - \theta(x_1)| \le \rho_1$, and $|\xi - \xi(x_1)| \le \rho_2$, we have

$$|x| < 2B, \quad |\theta| \le (c_1 + 1)A^{-1/2}, \quad |\xi| < cBA.$$

This implies

$$
\begin{aligned}
|f_1| &< 1 + c\left(B((c_1+1)A^{-1/2})^4 + ((c_1+1)A^{-1/2})^5 + BA((c_1+1)A^{-1/2})^6\right) \\
&< 2, \\
|f_2| &< c\Big(B^2((c_1+1)A^{-1/2}) + B((c_1+1)A^{-1/2})^2 \\
&\quad +(1+BBA)((c_1+1)A^{-1/2})^3 + BA((c_1+1)A^{-1/2})^4 \\
&\quad +(BA)^2((c_1+1)A^{-1/2})^5\Big) \\
&< c\left((c_1+1) + (c_1+1)^3 + (c_1+1)^5\right) B^2 A^{-1/2}, \\
|\partial_x f_1| &< cA^{-2}, \\
|\partial_\theta f_1| &< cBA^{-3/2}, \\
|\partial_\theta f_2| &< cB^2, \\
|\partial_\xi f_1| &< cA^{-3}, \\
|\partial_\xi f_2| &< cBA^{-3/2},
\end{aligned}
$$

where the numbers $c$ in the first two inequalities do not depend on $c_1$ and $c_2$ (and the later ones may depend on $c_1$ and $c_2$). The first condition of the lemma is met by taking

$$L_1 = 2, \quad L_2 = c\left((c_1+1) + (c_1+1)^3 + (c_1+1)^5\right) B^2 A^{-1/2},$$

$$M_1 = cA^{-2}, \quad N_{11} = cBA^{-3/2}, \quad N_{21} = cB^2, \quad N_{12} = cA^{-3}, \quad N_{22} = cBA^{-3/2}$$

for various specific choices of $c$'s. With $a_1 = a_2 = 1$, the second condition will also be satisfied if the inequalities labeled ? in the following are satisfied

$$
\begin{aligned}
\epsilon L_1 &= 6A^{-1/2} \overset{?}{<} c_1 A^{-1/2} = \rho_1, && (8) \\
\epsilon L_2 &< 3c\left((c_1+1) + (c_1+1)^3 + (c_1+1)^5\right) B^2 A^{-1/2} \overset{?}{<} c_2 BA = \rho_2, && (9) \\
\epsilon(a_1 N_{11} + a_2 N_{21}) &< cA^{-1/2}(BA^{-3/2} + B^2) < cB^2 A^{-1/2} \overset{?}{<} 1 = a_1, \\
\epsilon(a_1 N_{12} + a_2 N_{22}) &< cA^{-1/2}(A^{-3} + BA^{-3/2}) < cBA^{-2} \overset{?}{<} 1 = a_2.
\end{aligned}
$$

The first is satisfied by choosing $c_1 = 7$. Then the second is satisfied by choosing $c_2 = 1$ and a sufficiently large $A$. After substituting $c_1$ and $c_2$ into the $c$'s, the third and the fourth inequalities are satisfied by choosing a sufficiently large $A$.

The conditions of the lemma are verified, and we conclude that $(\theta, \xi)$ is analytic (and $u = \theta^{-2}$ is meromorphic) on the disk $|x - x_1| \leq 3A^{-1/2}$. Moreover, from

$$M_1 + L_1 N_{11} + L_2 N_{12} < c\left(A^{-2} + BA^{-3/2} + (B^2 A^{-1/2})A^{-3}\right) < cBA^{-3/2},$$

we have

$$|\theta - \theta(x_1) - \theta'(x_1)(x - x_1)| < cBA^{-3/2}|x - x_1|^2$$

on the disk. Since $\theta'(x_1)$ is very close to 1 and $|\theta(x_1)| = A^{-1/2}$ (see (7)), if $|x - x_1| = 3A^{-1/2}$, then

$$|\theta| \geq |\theta'(x_1)| 3A^{-1/2} - A^{-1/2} - cBA^{-3/2}(3A^{-1/2})^2 > A^{-1/2}.$$

In terms of $u = \theta^{-2}$, this means $|u| < A$ for $|x - x_1| = 3A^{-1/2}$.

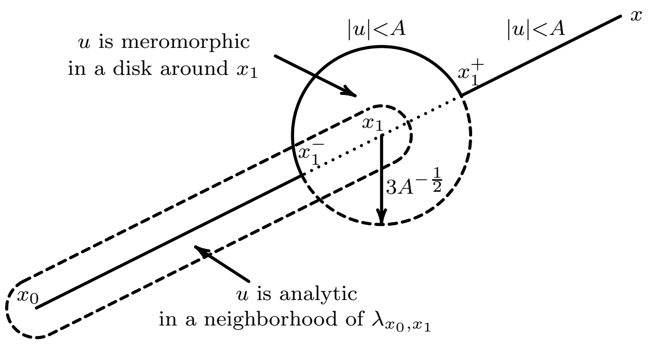
We have shown that $u$ is meromorphic on the disk $|x - x_1| \leq 3A^{-1/2}$. Moreover, we also know $|u| < A$ on the boundary of the disk. Denote by $x_1^+$ and $x_1^-$ the two points of intersection of $\lambda$ and the boundary $|x - x_1| = 3A^{-1/2}$, with $x_1^+$ further from $x_0$ than $x_1^-$. Then we can start from $x_1^+$, move along $\lambda$, and repeat the argument again. The only difference now is that when we take the integral from $x_0$ to $x$ (a point on $\lambda$ further than $x_1^+$), the integration path is not the straight line but modified by replacing the line segment connecting $x_1^-$ to $x_1^+$ with the half circle connecting the two points. Note that we still have $|u| < A$ along the modified path, and the length of the modified path is $\leq 2\pi B$. This means that the estimations we have carried out still work, except the bound for the length of the path of integration needs to be increased from $2B$ to $2\pi B$. For example, the estimation (4) will now become

$$|\theta'(x_1)^2 - 1| \leq \frac{1}{2}BA^{-2} + 2\pi BA^{-2} + \frac{1}{4}|k|A^{-3} < cBA^{-2}.$$

Clearly, such a modification will not affect the whole argument.

The repeat argument may lead us to another point $x_1'$, such that $|u| < A$ between $x_1^+$ and $x_1'$, and $|u(x_1')| = A$. The distance between $x_1$ and $x_1'$ is larger than the distance between $x_1$ and $x_1^+$, which is the fixed number $3A^{-1/2}$. Thus we only need to repeat the argument finitely many times before we cover the whole line $\lambda$.

We would like to emphasize that the estimation around $x_1$ should achieve two goals. First, it should provide a specific lower bound for the radius of convergence. Such a lower bound implies that the pole singularities cannot "accumulate" and the induction involves only finitely many steps. Second, it should enable us to get round $x_1$ along a specific alternative route (i.e., the half circle) on which $|u| < A$. This makes the induction possible.




# 3 The Second Painlevé Equation

In this section, we prove that all solutions of the second Painlevé equation $u'' = 2u^3 + xu + \alpha$ are defined on the whole complex plane and are meromorphic. The proof is almost identical to the one for the first Painlevé equation. We will present fairly detailed estimations and skip technical discussions.

Fix $u$, $x_0$, $B$, and $\lambda$ as in the case of the first Painlevé equation. We will also choose a large number $A$ determined entirely by $x_0$, $u(x_0)$, $u'(x_0)$, $B$, and $\alpha$. Similar to the first Painlevé equation (especially noting that the second Painlevé equation is regular), we continue our discussion by assuming that $x_1$ satisfies (1).

We multiply $u'$ to the second Painlevé equation and integrate to get

$$u'^2 = u^4 + xu^2 + 2\alpha u - \int_{x_0}^{x} u^2 dx + k, \tag{10}$$

where

$$k = u'(x_0)^2 - u(x_0)^4 - x_0 u(x_0)^2 - 2\alpha u(x_0).$$

We substitute the indicial normalization $u = \theta^{-1}$ into (10) and get

$$\theta'^2 = 1 + x\theta^2 + 2\alpha\theta^3 - \theta^4 \int_{x_0}^{x} u^2 dx + k\theta^4. \tag{11}$$

This implies

$$\begin{aligned} |\theta'(x_1)^2 - 1| &\leq |x_1\theta(x_1)^2| + 2|\alpha\theta(x_1)^3| + \left|\theta(x_1)^4 \int_{x_0}^{x_1} u^2 dx\right| + |k\theta(x_1)^4| \\ &\leq BA^{-2} + 2|\alpha|A^{-3} + A^{-4}(2B)A^2 + |k|A^{-4} \\ &< cBA^{-2}. \end{aligned}$$

Here we note that the constants $c$ in this section depend on $\alpha$, $x_0$, $u(x_0)$, and $u'(x_0)$. Thus we have $\theta'(x_1)$ very close to either 1 or $-1$. Assuming $\theta'(x_1)$ is close to 1, we introduce another variable $\xi$ by

$$u' = -\theta^{-2} - \frac{1}{2}x - \left(\frac{1}{2}+\alpha\right)\theta + \xi\theta^2.$$

Together with $u = \theta^{-1}$, the second Painlevé equation is converted into the system

$$\left\{\begin{array}{rcl} \theta' & = & 1 + \dfrac{1}{2}x\theta^2 + \left(\dfrac{1}{2}+\alpha\right)\theta^3 - \xi\theta^4, \\ \xi' & = & \left(\dfrac{1}{4}+\dfrac{\alpha}{2}\right)x + \left(\dfrac{1}{4}+\alpha+\alpha^2 - x\xi\right)\theta \\ & & -\left(\dfrac{3}{2}+3\alpha\right)\xi\theta^2 + 2\xi^2\theta^3. \end{array}\right. \tag{12}$$

The system (12) is also well known and appeared as (5.7), (5.8) (with $\mu = 1$) on page 369 of [1] and (IIa), (IIb) on page 372 of [6]. Again the difference here is the context where the system is used. If $\theta'(x_1)$ is close to $-1$ instead, we may introduce $\xi$ in a similar way (take $\mu = -1$ in [1], for example), so that the new system is still regular. The subsequent arguments will be the same, and we will not repeat them.

We estimate $\xi(x_1)$ by making use of (11):

$$\begin{aligned} & \left|\theta'(x_1)^2 - \left(1 + \frac{1}{2}x_1\theta(x_1)^2 + \left(\frac{1}{2}+\alpha\right)\theta(x_1)^3\right)^2\right| \\ \le\ & \left|\theta'(x_1)^2 - 1 - x_1\theta(x_1)^2 - (1+2\alpha)\theta(x_1)^3\right| + \left|\frac{1}{2}(x_1 + (1+2\alpha)\theta(x_1))\theta(x_1)^2\right|^2 \\ \le\ & |\theta(x_1)^3| + \left|\theta(x_1)^4\int_{x_0}^{x_1} u^2 dx\right| + |k\theta(x_1)^4| + \frac{1}{4}(|x_1| + (1+2|\alpha|)|\theta(x_1)|)^2|\theta(x_1)|^4 \\ \le\ & A^{-3} + A^{-4}(2B)A^2 + |k|A^{-4} + \frac{1}{4}(B + (1+2|\alpha|)A^{-1})^2A^{-4} \\ <\ & cBA^{-2}. \end{aligned}$$

This further implies

$$|\xi(x_1)| = |\theta(x_1)|^{-4}\left|\theta'(x_1) - 1 - \frac{1}{2}x_1\theta(x_1)^2 - \left(\frac{1}{2}+\alpha\right)\theta(x_1)^3\right| < cBA^2.$$

Then we need to carry out the estimations for the solution of the system (12) with the initial data satisfying

$$|\theta(x_1)| = A^{-1}, \quad |\xi(x_1)| < cBA^2.$$

Take

$$\epsilon = 3A^{-1}, \quad \rho_1 = c_1 A^{-1}, \quad \rho_2 = c_2 A,$$

where $c_1$ and $c_2$ are absolute constants to be determined. For $|x - x_1| \le \epsilon$, $|\theta - \theta(x_1)| \le \rho_1$, $|\xi - \xi(x_1)| \le \rho_2$, we have $|x| < 2B$, $|\theta| \le (c_1+1)A^{-1}$, $|\xi| < cBA^2$, and

$$\begin{aligned}
|f_1| &< 1 + c\Big(B((c_1+1)A^{-1})^2 + ((c_1+1)A^{-1})^3 + (BA^2)((c_1+1)A^{-1})^4\Big) \\
&< 2 = L_1, \\
|f_2| &< c\Big(B + (1 + B(BA^2))(c_1+1)A^{-1} + (BA^2)((c_1+1)A^{-1})^2 \\
&\quad +(BA^2)^2((c_1+1)A^{-1})^3\Big) \\
&< c\Big(1 + (c_1+1) + (c_1+1)^3\Big) B^2 A = L_2, \\
|\partial_x f_1| &< cA^{-2} = M_1, \\
|\partial_\theta f_1| &< cBA^{-1} = N_{11}, \\
|\partial_\theta f_2| &< cB^2A^2 = N_{21}, \\
|\partial_\xi f_1| &< cA^{-4} = N_{12}, \\
|\partial_\xi f_2| &< cBA^{-1} = N_{22},
\end{aligned}$$

where the numbers $c$ in the first two inequalities do not depend on $c_1$ and $c_2$. Then with $a_1 = 1$ and $a_2 = A^{-2}$, the conditions of our lemma will be satisfied if the inequalities labeled ? in the following are satisfied

$$\begin{aligned}
\epsilon L_1 &= 6A^{-1} \overset{?}{<} c_1 A^{-1} = \rho_1, \\
\epsilon L_2 &< 3c\Big(1 + (c_1+1) + (c_1+1)^3\Big) B^2 \overset{?}{<} c_2 A = \rho_2, \\
\epsilon(a_1 N_{11} + a_2 N_{21}) &< cA^{-1}\Big(BA^{-1} + A^{-2}(B^2A^2)\Big) \overset{?}{<} 1 = a_1, \\
\epsilon(a_1 N_{12} + a_2 N_{22}) &< cA^{-1}\Big(A^{-4} + A^{-2}(BA^{-1})\Big) \overset{?}{<} A^{-2} = a_2.
\end{aligned}$$

By choosing $c_1 = 7$, $c_2 = 1$, and a sufficiently large $A$, these are satisfied. The lemma tells us that $(\theta, \xi)$ is analytic in the disk $|x - x_1| \le 3A^{-1}$. Moreover, from

$$M_1 + L_1 N_{11} + L_2 N_{12} < c\Big(A^{-2} + BA^{-1} + (B^2A)A^{-4}\Big) < cBA^{-1},$$

we have

$$|\theta - \theta(x_1) - \theta'(x_1)(x - x_1)| < cBA^{-1}|x - x_1|^2.$$

Since $\theta'(x_1)$ is very close to 1 and $|\theta(x_1)| = A^{-1}$, if $|x - x_1| = 3A^{-1}$, then

$$|\theta| > |\theta'(x_1)|3A^{-1} - A^{-1} - cBA^{-1}(3A^{-1})^2 > A^{-1}.$$

In other words, $|u| < A$ along the circle $|x - x_1| = 3A^{-1}$.

# 4 The Fourth Painlevé Equation

In this section, we prove that all solutions of the fourth Painlevé equation

$$u'' = \frac{1}{2u}u'^2 + \frac{3}{2}u^3 + 4xu^2 + 2(x^2 - \alpha)u + \frac{\beta}{u}$$

are defined on the whole complex plane and are meromorphic. In contrast to the first two Painlevé equations, we need to consider two types of singularities, according to whether $u$ approaches $\infty$ or 0.

Fix $u$, $x_0$, $B$, and $\lambda$ as before and assume $u(x_0) \neq 0$. We will choose a large number $A$ determined entirely by $x_0$, $u(x_0)$, $u'(x_0)$, $B$, $\alpha$, and $\beta$. Now we start from $x_0$ and move along $\lambda$ to a point $x_1$, such that (1) is satisfied. Note that because of the possible singularity where $u$ approaches 0, the existence of $x_1$ has to be justified. For the moment, we will ignore the issue and carry out the argument at the point $x_1$ where $u$ is large.

We integrate the fourth Painlevé equation to get

$$u'^2 = u^4 + 4xu^3 + 4(x^2 - \alpha)u^2 - 2\beta - 4u\int_{x_0}^{x}(u^2 + 2xu)dx + ku, \tag{13}$$

where

$$k = \frac{u'(x_0)^2}{u(x_0)} - u(x_0)^3 - 4x_0u(x_0)^2 - 4(x_0^2 - \alpha)u(x_0) + \frac{2\beta}{u(x_0)}.$$

Then we substitute the indicial normalization $u = \theta^{-1}$ into (13) and get

$$\theta'^2 = 1 + 4x\theta + 4(x^2 - \alpha)\theta^2 - 2\beta\theta^4 - 4\theta^3\int_{x_0}^{x}(u^2 + 2xu)dx + k\theta^3. \tag{14}$$

This implies

$$\begin{aligned}
|\theta'(x_1)^2 - 1| &\leq 4|x_1\theta(x_1)| + 4|(x_1^2 - \alpha)\theta(x_1)^2| + 2|\beta\theta(x_1)^4| \\
&\quad + \left|4\theta(x_1)^3\int_{x_0}^{x_1}(u^2 + 2xu)dx\right| + |k\theta(x_1)^3| \\
&\leq 4BA^{-1} + 4(B^2 + |\alpha|)A^{-2} + 2|\beta|A^{-4} \\
&\quad + 4A^{-3}(2B)(A^2 + 2BA) + |k|A^{-3} \\
&< cBA^{-1}.
\end{aligned}$$

Therefore $\theta'(x_1)$ is very close to either 1 or $-1$. If $\theta'(x_1)$ is close to 1, then we introduce a new variable $\xi$ by

$$u' = -\theta^{-2} - 2x\theta^{-1} - 2 + 2\alpha + \xi\theta.$$

Together with $u = \theta^{-1}$, the fourth Painlevé equation is converted into the following system

$$\left\{ \begin{array}{rcl} \theta' & = & 1 + 2x\theta + 2(1-\alpha)\theta^2 - \xi\theta^3, \\ \xi' & = & 2 - 4\alpha + 2\alpha^2 + \beta - 2x\xi + 4(\alpha - 1)\xi\theta + \dfrac{3}{2}\xi^2\theta^2. \end{array} \right. \tag{15}$$

The system (15) also appeared in Case 2 on page 369 of [6]. If $\theta'(x_1)$ is close to $-1$, then we may introduce $\xi$ in a similar way (change signs as indicated in [6], for example). The subsequent arguments are also similar.

Now we estimate $\xi(x_1)$. From (14), we have

$$\begin{array}{rl} & \left|\theta'(x_1)^2 - \Big(1 + 2x_1\theta(x_1) + 2(1-\alpha)\theta(x_1)^2\Big)^2\right| \\ \leq & \left|\theta'(x_1)^2 - 1 - 4x_1\theta(x_1) - 4(1-\alpha)\theta(x_1)^2\right| + |2(x_1 + (1-\alpha)\theta(x_1))\theta(x_1)|^2 \\ \leq & 4|x_1^2 - 1||\theta(x_1)^2| + 2|\beta||\theta(x_1)^4| + 4|\theta(x_1)^3| \left|\int_{x_0}^{x_1} (u^2 + 2xu)dx\right| \\ & +|k||\theta(x_1)^3| + 4(|x_1| + |1-\alpha||\theta(x_1)|)^2|\theta(x_1)|^2 \\ \leq & 4(B^2+1)A^{-2} + 2|\beta|A^{-4} + 4A^{-3}(2B)(A^2 + BA) + |k|A^{-3} \\ & +4(B + |1-\alpha|A^{-1})^2A^{-2} \\ < & cBA^{-1}. \end{array}$$

This further implies

$$|\xi(x_1)| = |\theta(x_1)|^{-3}\left|\theta'(x_1) - 1 - 2x_1\theta(x_1) - 2(1-\alpha)\theta(x_1)^2\right| < cBA^2.$$

We are ready to apply our lemma to estimate the solution of the system (15) with the initial data satisfying

$$|\theta(x_1)| = A^{-1}, \quad |\xi(x_1)| < cBA^2.$$

For $|x - x_1| \leq \epsilon = 3A^{-1}$, $|\theta - \theta(x_1)| \leq \rho_1 = 7A^{-1}$, $|\xi - \xi(x_1)| \leq \rho_2 = c_2B^2A$, we have $|x| < 2B$, $|\theta| \leq 8A^{-1}$, $|\xi| < cBA^2$, and

$$|f_1| \ < \ 1 + c\Big(BA^{-1} + (1+|\alpha|)A^{-2} + BA^{-1}\Big) < 2 = L_1,$$

$$
\begin{aligned}
|f_2| &< 2+4|\alpha|+2|\alpha|^2+|\beta|+c\left(B^2A^2+(1+|\alpha|)BA+B^2A^2\right)\\
&< cB^2A^2=L_2,\\
|\partial_x f_1| &< cA^{-1}=M_1,\\
|\partial_\theta f_1| &< cB=N_{11},\\
|\partial_\theta f_2| &< cB^2A^3=N_{21},\\
|\partial_\xi f_1| &< cA^{-3}=N_{12},\\
|\partial_\xi f_2| &< cB=N_{22}.
\end{aligned}
$$

Then we choose $c_2=4c$, where $c$ is the constant in the bound $L_2$. With $a_1=1, a_2=A^{-3}$, and sufficiently large $A$, the second condition of the lemma is verified.

$$
\begin{aligned}
\epsilon L_1 &= 6A^{-1}<7A^{-1}=\rho_1,\\
\epsilon L_2 &= (3A^{-1})(cB^2A^2)<4cB^2A=\rho_2,\\
\epsilon(a_1N_{11}+a_2N_{21}) &< cA^{-1}(B+A^{-3}B^2A^3)<1=a_1,\\
\epsilon(a_1N_{12}+a_2N_{22}) &< cA^{-1}(A^{-3}+A^{-3}B)<A^{-3}=a_2.
\end{aligned}
$$

The lemma then tells us that $(\theta,\xi)$ is analytic in the disk $|x-x_1|\le 3A^{-1}$. Then

$$M_1+L_1N_{11}+L_2N_{12}<c(A^{-1}+B+B^2A^{-1})<cB$$

further tells us

$$|\theta-\theta(x_1)-\theta'(x_1)(x-x_1)|<cB|x-x_1|^2.$$

Thus

$$|\theta|\ge|\theta'(x_1)|3A^{-1}-A^{-1}-cB(3A^{-1})^2>A^{-1},$$

and $|u|=|\theta|^{-1}<A$ along the circle $|x-x_1|=3A^{-1}$.

It remains to justify the existence of $x_1$. The issue here is that before reaching $x_1$, $u$ may become small and the equation may become singular. We will prove that if $x_2$ is a point on $\lambda$, such that

1. $u$ exists and is analytic along the straight line $\lambda_{x_0,x_2}$ from $x_0$ to $x_2$;

2. $|u(x)|\le A$ on $\lambda_{x_0,x_2}$;

3. $|x_2|\le B, 0<|u(x_2)|\le A^{-4}$,

then $u$ is an analytic function on the disk $|x - x_2| \leq A^{-4}$. Since $A^{-4}$ is a definite lower bound for the radius of analyticity, after applying the conclusion at points where $|u| \leq A^{-4}$ for finitely many times, we are able to reach $x_1$.

By $|u(x_2)| \leq A^{-4}$ and (13), we have

$$\begin{aligned} |u'(x_2)^2 + 2\beta| &\leq c\Big((A^{-4})^4 + B(A^{-4})^3 \\ &\quad +(B^2+1)(A^{-4})^2 + A^{-4}(2B)(A^2+2BA) + |k|A^{-4}\Big) \\ &< cBA^{-2}. \end{aligned}$$

For large $A$, therefore, $u'(x_2)$ is very close to either $i\sqrt{2\beta}$ or $-i\sqrt{2\beta}$. Now we need to split our discussion into two cases.

**Case 1** $\beta \neq 0$.

Assuming $u'(x_2)$ is close to $i\sqrt{2\beta}$ (the case that $u'(x_2)$ is close to $-i\sqrt{2\beta}$ is similar and the discussion will be omitted), we introduce the change of variable

$$u = \theta, \quad u' = i\sqrt{2\beta} + \xi\theta$$

near $x_2$. This converts the fourth Painlevé equation into the following system

$$\left\{ \begin{aligned} \theta' &= i\sqrt{2\beta} + \xi\theta, \\ \xi' &= -2\alpha + 2x^2 - \frac{\xi^2}{2} + 4x\theta + \frac{3}{2}\theta^2. \end{aligned} \right. \tag{16}$$

We remark that the system also appeared as (10), (11) on page 369 of [6]. Replacing $u$ by $\theta$ in (13), we get

$$\begin{aligned} \left|\frac{\theta'(x_2)^2 + 2\beta}{\theta(x_2)}\right| &\leq |\theta(x_2)^3| + 4|x_2\theta(x_2)^2| + 4|(x_2^2 - \alpha)\theta(x_2)| \\ &\quad +4\left|\int_{x_0}^{x_2}(u^2 + 2xu)dx\right| + |k| \\ &\leq (A^{-4})^3 + 4B(A^{-4})^2 + 4(B^2 + |\alpha|)A^{-4} \\ &\quad +4(2B)(A^2 + 2BA) + |k| \\ &< cBA^2. \end{aligned}$$

Since $\theta'(x_2) = u'(x_2)$ is very close to $i\sqrt{2\beta}$ and $\beta \neq 0$, we then have

$$|\xi(x_2)| = \left|\frac{\theta'(x_2) - i\sqrt{2\beta}}{\theta(x_2)}\right| < cBA^2.$$

Combined with $|\theta(x_2)| = |u(x_2)| \le A^{-4}$ and the system (16), it is easy to estimate the radius of convergence for $u = \theta$ around $x_1$ in terms of an explicit function of $A$. Specifically, for $|x - x_2| \le \epsilon = A^{-4}$, $|\theta - \theta(x_2)| \le \rho_1 = 3\sqrt{|\beta|}A^{-4}$, $|\xi - \xi(x_2)| \le \rho_2 = cBA^2$, we have $|x| < 2B$, $|\theta| \le (1 + 3\sqrt{|\beta|})A^{-4}$, $|\xi| < cBA^2$, and

$$
\begin{aligned}
|f_1| &< \sqrt{2|\beta|} + cBA^{-2}(1 + 3\sqrt{|\beta|})A^{-4} < 2\sqrt{|\beta|} = L_1, \\
|f_2| &< 2|\alpha| + 2(2B)^2 + c\left((BA^2)^2 + B(1 + 3\sqrt{|\beta|})A^{-4} + (A^{-4})^2\right) \\
&< cB^2A^4 = L_2, \\
|\partial_x f_1| &= 0 = M_1, \\
|\partial_\theta f_1| &< cBA^2 = N_{11}, \\
|\partial_\theta f_2| &< cB = N_{21}, \\
|\partial_\xi f_1| &< cA^{-4} = N_{12}, \\
|\partial_\xi f_2| &< cBA^2 = N_{22}.
\end{aligned}
$$

Then the conditions of our lemma are satisfied with $a_1 = a_2 = 1$.

$$
\begin{aligned}
\epsilon L_1 &= 2\sqrt{|\beta|}A^{-4} < 3\sqrt{|\beta|}A^{-4} = \rho_1, \\
\epsilon L_2 &= cB^2 < \rho_2, \\
\epsilon(a_1N_{11} + a_2N_{21}) &< cA^{-4}(BA^2 + B) < 1 = a_1, \\
\epsilon(a_1N_{12} + a_2N_{22}) &< cA^{-4}(A^{-4} + BA^2) < 1 = a_2.
\end{aligned}
$$

Thus $(\theta, \xi)$ (as well as $u = \theta$) is analytic in the disk $|x - x_2| \le A^{-4}$. Moreover,

$$M_1 + L_1N_{11} + L_2N_{12} < c(\sqrt{|\beta|}BA^2 + B^2) < cBA^2$$

further tells us

$$|\theta - \theta(x_2) - \theta'(x_2)(x - x_2)| < cBA^2|x - x_2|^2.$$

Thus

$$|u| = |\theta| \le |\theta'(x_2)|A^{-4} + A^{-4} + cBA^2(A^{-4})^2 < (2\sqrt{|\beta|} + 1)A^{-4} < A$$

in the disk $|x - x_2| \le A^{-4}$.

**Case 2** $\beta = 0$.

In this case, the indicial normalization $u = \theta^2$ near $x_2$ (the assumption $u(x_2) \neq 0$ allows us to fix one branch of $\theta$) converts the fourth Painlevé equation into

$$\theta'' = (x^2 - \alpha)\theta + 2x\theta^3 + \frac{3}{4}\theta^5. \tag{17}$$

The equation also appeared in subcase 1(ii) on page 369 of [6]. We introduce $\xi = \theta'$ so that the second order equation becomes a system of two first order equations. An estimation similar to the $\beta \neq 0$ case gives us

$$|\xi(x_2)|^2 = \left|\frac{u'(x_2)}{2\theta(x_2)}\right|^2 = \left|\frac{u'(x_2)^2}{4u(x_2)}\right| < cBA^2.$$

It is then easy to estimate the radius of convergence for the solution of

$$\begin{cases} \theta' &= \xi, \\ \xi' &= (x^2 - \alpha)\theta + 2x\theta^3 + \frac{3}{4}\theta^5, \end{cases}$$

with the initial data satisfying $|\theta(x_2)| \leq A^{-2}$ and $|\xi(x_2)| < cBA$. Specifically, for $|x - x_2| \leq \epsilon = A^{-4}$, $|\theta - \theta(x_2)| \leq \rho_1 = A^{-2}$, $|\xi - \xi(x_2)| \leq \rho_2 = 1$, we have $|x| < 2B$, $|\theta| \leq 2A^{-2}$, $|\xi| < cBA$, and

$$\begin{aligned} |f_1| &< cBA = L_1, \\ |f_2| &< c\Big((B^2 + |\alpha|)A^{-2} + BA^{-6} + A^{-10}\Big) < 1 = L_2, \\ |\partial_x f_1| &= 0 = M_1, \\ |\partial_\theta f_1| &= 0 = N_{11}, \\ |\partial_\theta f_2| &< c(B^2 + |\alpha|) = N_{21}, \\ |\partial_\xi f_1| &= 1 = N_{12}, \\ |\partial_\xi f_2| &= 0 = N_{22}. \end{aligned}$$

Then the conditions of our lemma are satisfied with $a_1 = a_2 = 1$.

$$\begin{aligned} \epsilon L_1 &= cBA^{-3} < A^{-2} = \rho_1, \\ \epsilon L_2 &= A^{-4} < 1 = \rho_2, \\ \epsilon(a_1 N_{11} + a_2 N_{21}) &< cA^{-4}(B^2 + |\alpha|) < 1 = a_1, \\ \epsilon(a_1 N_{12} + a_2 N_{22}) &< cA^{-4} < 1 = a_2. \end{aligned}$$

Thus $(\theta, \xi)$ (as well as $u = \theta^2$) is analytic in the disk $|x - x_2| \leq A^{-4}$. Moreover,

$$M_1 + L_1 N_{11} + L_2 N_{12} = 1$$

further tells us

$$|\theta - \theta(x_2) - \theta'(x_2)(x - x_2)| \leq \frac{1}{2}|x - x_2|^2.$$

Thus (recall that $\theta' = \xi$ and $|\xi(x_2)|^2 < cBA^2$)

$$|\theta| \leq |\theta'(x_2)|A^{-4} + A^{-2} + \frac{1}{2}(A^{-4})^2 < 2A^{-2},$$

and $|u| = |\theta|^2 < 4A^{-4} < A$ in the disk $|x - x_2| \leq A^{-4}$.

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