Tilings of the Sphere by Geometrically Congruent Pentagons II

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Abstract

There are exactly seven edge-to-edge tilings of the sphere by congruent equilateral pentagons.

1 Introduction

This paper is the second in the series of our attempt at the classification of edge-to-edge tilings of the sphere by congruent pentagons. The first of the series is by Cheuk, Cheung and Yan [2], in which we showed how to classify such tilings when there is enough variety in edge lengths. Specifically, we proved that, if there is a tile with all vertices having degree 3, then there is no tiling by more than 12 tiles, such that the edge combination is a^3bc, a^2b^2c, or a^3b^2. The method should be sufficient for dealing with all the other cases of enough variety in edge lengths. When all edges have equal length (i.e., the tiles are equilateral pentagons), however, a completely different method is needed. This is developed in this paper.

For general discussions about spherical tilings, we refer the reader to the introduction of [2]. Here we only mention that the tile in an edge-to-edge tiling of the sphere by congruent polygons must be triangle, quadrilateral or pentagon. The triangular tilings are completely classified [5, 6]. We believe

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the pentagonal tilings are easier to study than the quadrilateral ones because 5 is the "other extreme" in $3, 4, 5$. After developing the method for the case that there is enough variety in the edge lengths in [2], and the case that there is no variety in the edge length in this paper, the remaining and the most difficult case is the edge combination a^4b, which means that four edges of the pentagon have equal length and the fifth edge has different length. This will be the subject of a future paper in the series.

The key idea of the paper is the following. A general pentagon is determined by the free choice of 4 edge lengths and 3 angles, yielding 7 degrees of freedom. The requirement that all 5 edges are equal imposes 4 equations, leaving $7 - 4 = 3$ degrees of freedom for equilateral pentagons. Therefore 3 more independent equations are enough to completely determine such pentagons.

On the other hand, the complete list of possible angle combinations at degree 3 vertices in a tiling is given by [4, Theorem 1]. Moreover, further restrictions on such combinations are given by [4, Section 3] (and Proposition 5 in particular). With one exception, this provides 3 independent equations among 5 angles. This means that, with one exception, the equilateral pentagon can be completely determined. Once we know the pentagon, it is then not difficult to find the tiling.

The minimal case of edge-to-edge tilings of sphere by 12 congruent pentagons is completely classified [1, 3]. In fact, the minimal tiling by congruent equilateral pentagons is the regular dodecahedron. Hence we will assume the number of tiles $f > 12$ in this paper. By [7], we actually know that f is an even number ≥ 16.

It turns out that we need to calculate more than 400 cases, including various angle arrangements in the pentagon. We use the MAPLE software to carry out all these calculations and find out that almost all cases either do not lead to equilateral pentagons, or lead to pentagons whose area is not 4π (the area of the unit sphere) divided by an even number ≥ 16 (the number of tiles). For the remaining limited number of cases, we find total of seven tilings. Together with the regular dodecahedron from [3], we have the following complete list.

Theorem. *There are eight tilings of the sphere by congruent equilateral pentagons. In the list below, the edge length is a, the angles are arranged as $[\alpha, \beta, \delta, \gamma, \epsilon]$ in the pentagon, and we always have $\alpha + \beta + \gamma = 2\pi$. Specifically, there are three pentagonal subdivisions:*

1. *$f=12$, $a=0.2322\pi$, $\alpha=\beta=\gamma=\delta=\epsilon=\frac{3}{2}\pi$. Regular dodecahedron, or subdivision of tetrahedron.*

2. *$f=24$, $a=0.1745\pi$, $\alpha=0.8010\pi$, $\beta=0.5113\pi$, $\gamma=0.6875\pi$, $\delta=\frac{2}{3}\pi$, $\epsilon=\frac{1}{2}\pi$. Subdivision of (cube, octahedron). See Case 4.2c in Section 3.4.*

3. *$f=60$, $a=0.1186\pi$, $\alpha=0.9059\pi$, $\beta=0.4093\pi$, $\gamma=0.6847\pi$, $\delta=\frac{2}{3}\pi$, $\epsilon=\frac{2}{5}\pi$, $f=60$. Subdivision of (dodecahedron, icosahedron). See Case 4.2d in Section 3.4.*

There are four earth map tilings (Figure 11):

4. *$f=16$, $a=0.2155\pi$, $\alpha=0.4536\pi$, $\beta=0.8823\pi$, $\gamma=0.6639\pi$, $\delta=\frac{1}{2}\pi$, $\epsilon=\frac{3}{4}\pi$. See Case 1.5b in Section 4.4.*

5. *$f=20$, $a=0.2168\pi$, $\alpha=0.3095\pi$, $\beta=1.0615\pi$, $\gamma=0.6288\pi$, $\delta=\frac{2}{5}\pi$, $\epsilon=\frac{4}{5}\pi$. See Case 2.6b in Section 4.4.*

6. *$f=24$, $a=0.2501\pi$, $\alpha=0.1440\pi$, $\beta=\frac{4}{3}\pi$, $\gamma=0.5226\pi$, $\delta=\frac{1}{3}\pi$, $\epsilon=\frac{5}{6}\pi$. See first solution in Section 4.1.*

7. *$f=24$, $a=0.2614\pi$, $\alpha=0.1192\pi$, $\beta=1.3807\pi$, $\gamma=\frac{1}{2}\pi$, $\delta=\frac{1}{3}\pi$, $\epsilon=\frac{5}{6}\pi$. See second solution in Section 4.1.*

And there is one special tiling (Figure 16):

8. *$f=20$, $a=0.2168\pi$, $\alpha=0.3095\pi$, $\beta=1.0615\pi$, $\gamma=0.6288\pi$, $\delta=\frac{2}{5}\pi$, $\epsilon=\frac{4}{5}\pi$. See Case 1.4b in Section 4.4.*

The decimal values are effective digits. For example, $a=0.2322\pi$ means $a\in[0.2322\pi,0.2323\pi]$. The convention will be adopted throughout the paper, and we provide enough digits so that the approximate values are enough for rigorous conclusions.

The pentagonal subdivisions are are given by [4, Section 8]. The earth map tilings are given by Figure 11, in which three "timezones" are depicted. The earth map tilings with $f=16,20,24$ tiles have respectively $4,5,6$ timezones.

We note that the fifth and eighth tilings have the same pentagon. Moreover, we know the exact values of all data in the first, sixth and seventh tilings. For example, we have $a=\arccos\frac{\sqrt{5}}{3}$ for the regular decahedron and $a=\arccos\sqrt{-3+2\sqrt{3}}$ for the seventh tiling. More exact values and more digits for approximate values will be presented in the paper.

2 Spherical Geometry of Equilateral Pentagon

Consider the spherical equilateral pentagon in Figure 1, with edge length a and five angles $\alpha, \beta, \gamma, \delta, \epsilon$. By [1] and [2, Section 3], we may calculate the great arc x connecting β and ϵ vertices, from the triangle above x as well as the quadrilateral below x

$$\begin{aligned}\cos x &= \cos^2 a + \sin\alpha \sin^2 a,\\ \cos x &= (1-\cos\gamma)(1-\cos\delta)\cos^3 a - \sin\gamma\sin\delta\cos^2 a\\ &\quad + (\cos\gamma + \cos\delta - \cos\gamma\cos\delta)\cos a + \sin\gamma\sin\delta.\end{aligned}$$

Equating the two formulae for $\cos x$ and dividing $1-\cos a$, we get a quadratic equation for $\cos a$

$$L\cos^2 a + M\cos a + N = 0,$$

where the coefficients depend only on α, γ, δ,

$$\begin{aligned}L &= (1-\cos\gamma)(1-\cos\delta),\\ M &= \cos\alpha + \cos(\gamma+\delta) - \cos\gamma - \cos\delta,\\ N &= \cos\alpha - \sin\gamma\sin\delta.\end{aligned}$$

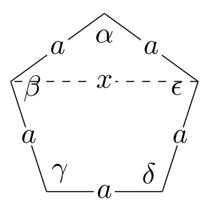

Figure 1: Spherical equilateral pentagon.

Let c_i be the arcs connecting respectively (β,ϵ), (α,γ), (β,δ), (γ,ϵ), (α,δ). The quadratic equation above is derived from the attempt to calculate c_1. By calculating each of the five arcs, we get five quadratic equations

$$L_i\cos^2 a + M_i\cos a + N_i = 0, \quad i = 1,2,3,4,5.$$

The five quadratic equations should share a common root $\cos a$. The sharing of a root among two quadratic equations can be detected by (and is equivalent to) the vanishing of the resultant

$$R_{ij} = (L_iN_j - L_jN_i)^2 - (L_iM_j - L_jM_i)(M_iN_j - M_jN_i).$$

The sharing of a root among five quadratic equations can be detected by (but may not be equivalent to) the vanishing of four resultants that relate all five equations together.

As pointed out in the introduction, the equilateral pentagon has three degrees of freedom. If we have three independent relations among the five angles, then the pentagon should be completely determined. Specifically, we may use the three independent relations to express five angles in terms of two. Then the resultants are functions of two free variables. We find the pentagon by looking for the common zero of four resultants.

We find the relations among angles by looking at possible angle combinations at vertices. For example, a vertex combination $\alpha\beta\gamma$ at a vertex implies a relation $\alpha+\beta+\gamma=2\pi$. The fact that all five angles must appear at some vertices imply certain number of such relations. The details are given by Table 1 extracted from [4]. The angles in the table are all dinstinct.

For any tiling, we call the collection of all the angle combinations at degree 3 vertices the *anglewise vertex combinations at degree* 3 *vertices*. We denote the collection by AVC_3. By [4, Theorem 1], for a tiling (not necessarily by pentagons) of any surface (not necessarily sphere) with at most 5 distinct angles appearing at degree 3 vertices, the AVC_3 contains the necessary part of a collection in the table, and is also contained in the necessary plus the optional part of the collection.

The table has five parts, corresponding to the number of distinct angles appearing at degree 3 vertices. For example, the first case of the three angle part is

$$\{\alpha\beta\gamma\} \subset \text{AVC}_3 \subset \{\alpha\beta\gamma, \alpha^3\}.$$

The consideration of pentagonal tilings of the sphere imposes more restrictions. For example, if $\text{AVC}_3 = \{\alpha\beta\gamma\}$, then $\alpha\beta\gamma$ is the only degree 3 vertex. By [4, Lemma 3], however, this would imply that each α, β, γ appears at least twice in the pentagon, contradicting to only five angles in a pentagon. Therefore α^3 must also be a vertex, and $\text{AVC}_3 = \{\alpha\beta\gamma, \alpha^3\}$. By [4, Lemma 3] again, since α appears at all degree 3 vertices, it must appear at least twice in the pentagon.

There is a simple inequality constraint on angles in a equilateral pentagon, given by [3, Lemma 21] (or [2, Lemma 3]). The constraint has been successfully used in [2] to eliminate quite a number of cases. By the constraint, it is easy to see that a quadrilateral pentagon can only allow odd number of distinct angles. Moreover, by [4, Lemma 4], there is at most one vertex not

<table>
<tr><th colspan="3">Necessary</th><th>Optional</th></tr>
<tr><td>α^3</td><td colspan="2"></td><td></td></tr>
<tr><td>$\alpha\beta^2$</td><td colspan="2"></td><td></td></tr>
<tr><td colspan="2">$\alpha\beta\gamma$</td><td rowspan="3"></td><td>α^3</td></tr>
<tr><td rowspan="2">$\alpha\beta^2$</td><td>$\alpha^2\gamma$</td><td rowspan="2"></td></tr>
<tr><td>γ^3</td></tr>
<tr><td rowspan="5" colspan="2">$\alpha\beta\gamma$</td><td rowspan="2">$\alpha\delta^2$</td><td>$\beta^2\delta$</td></tr>
<tr><td>β^3</td></tr>
<tr><td rowspan="2">$\alpha^2\delta$</td><td>$\beta\delta^2$</td></tr>
<tr><td>β^3</td></tr>
<tr><td>δ^3</td><td></td></tr>
<tr><td rowspan="2">$\alpha\beta^2$</td><td colspan="2">$\gamma\delta^2$</td><td>$\alpha^2\delta$</td></tr>
<tr><td colspan="2">$\alpha^2\gamma, \delta^3$</td><td></td></tr>
<tr><td rowspan="5">$\alpha\beta\gamma$</td><td rowspan="5" colspan="2">$\alpha\delta\epsilon$</td><td>$\beta\delta^2, \beta^2\epsilon$</td></tr>
<tr><td>$\beta\delta^2, \gamma\epsilon^2, \alpha^3$</td></tr>
<tr><td>$\beta\delta^2, \gamma^2\epsilon$</td></tr>
<tr><td>$\beta\delta^2, \gamma^3$</td></tr>
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<tr><th colspan="3">Necessary</th><th>Optional</th></tr>
<tr><td rowspan="19">$\alpha\beta\gamma$</td><td rowspan="12">$\alpha\delta^2$</td><td rowspan="3">$\alpha^2\epsilon$</td><td>$\beta\epsilon^2$</td></tr>
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<tr><td>β^3</td></tr>
<tr><td rowspan="3">$\beta\epsilon^2$</td><td>$\alpha^2\epsilon$</td></tr>
<tr><td>$\gamma^2\delta$</td></tr>
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<tr><td rowspan="3">$\beta^2\epsilon$</td><td>$\gamma\epsilon^2$</td></tr>
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<tr><td rowspan="2">$\delta\epsilon^2$</td><td>$\beta^2\epsilon$</td></tr>
<tr><td>β^3</td></tr>
<tr><td>ϵ^3</td><td>$\beta^2\delta$</td></tr>
<tr><td rowspan="6">$\alpha^2\delta$</td><td rowspan="3">$\beta^2\epsilon$</td><td>$\alpha\epsilon^2$</td></tr>
<tr><td>$\gamma\delta^2$</td></tr>
<tr><td>γ^3</td></tr>
<tr><td rowspan="2">$\delta^2\epsilon$</td><td>$\beta^2\epsilon$</td></tr>
<tr><td>β^3</td></tr>
<tr><td>ϵ^3</td><td>$\beta\delta^2$</td></tr>
<tr><td colspan="2">$\delta\epsilon^2$</td><td>α^3</td></tr>
<tr><td rowspan="3" colspan="2">$\alpha\beta^2, \gamma\delta^2$</td><td rowspan="2">$\alpha^2\epsilon$</td><td>$\beta\gamma^2$</td></tr>
<tr><td>$\delta\epsilon^2$</td></tr>
<tr><td>ϵ^3</td><td>$\alpha^2\delta$</td></tr>
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Table 1: Anglewise vertex combinations at degree 3 vertices, up to 5 angles.

appearing at degree 3 vertices. For the AVC_3 above, therefore, α, β, γ must be the only angles in the tiling. Then up to the symmetry of AVC_3 (i.e., exchanging β, γ), the angles in the pentagon is either $\alpha^2\beta^2\gamma$ or $\alpha^3\beta\gamma$. By [3, Lemma 21] again, we see that $\alpha^3\beta\gamma$ is impossible, and for $\alpha^2\beta^2\gamma$, the angles can have two possible arrangements in the pentagon

$$A\colon [\alpha, \alpha, \beta, \gamma, \beta],\ [\alpha, \beta, \beta, \alpha, \gamma].$$

We will denote the first arrangement by $A1$ and the second arrangement by $A2$.

Similar argument can be made for the other cases in the three angle part of Table 1. We get the following complete list (up to the permutation of symbols) of possible AVC$_3$s that also include the information on the angle combinations in the pentagon. For each AVC$_3$, we need to further consider two possible angle arrangements $A1$ and $A2$. The total number of cases is 10.

3.1 $\{\alpha^2\beta^2\gamma\colon \alpha\beta\gamma, \alpha^3\}$, 2 arrangements A.

3.2a $\{\alpha^2\beta^2\gamma\colon \alpha\beta^2, \alpha^2\gamma\}$, 2 arrangements A.

3.2b $\{\alpha^2\beta^2\gamma\colon \alpha^2\beta, \alpha\gamma^2\}$, 2 arrangements A.

3.3a $\{\alpha^2\beta^2\gamma\colon \alpha\beta^2, \gamma^3\}$, 2 arrangements A.

3.3b $\{\alpha^2\beta^2\gamma\colon \alpha\gamma^2, \beta^3\}$, 2 arrangements A.

In fact, we can also get the list above by using [4, Proposition 5] and [3, Lemma 21].

For the one angle part of Table 1, the same argument by using [3, Lemma 21] and [4, Lemma 4] shows that the pentagon must be α^5. This leads to the regular dodecahedron tiling.

For the two angle part of Table 1, the same argument leads to no possible tiling. In fact, by [4, Proposition 5], the pentagon must be $\alpha^2\beta^3$, contradicting to the requirement of odd number of angles.

The four angle part will be discussed in Section 3. The five angle part will be discussed in Section 4.

This section will use the spherical trigonometry to show that none of the cases from the three angle part gives pentagon fit for the tiling. We will discuss the four angle and five angle parts in the later sections.

In the later part of the paper, we may also get some information about angle combinations at vertices of degree > 3. Sometimes we know certain angle combinations must appear, and some other times we know all the possible (but not necessarily appearing) angle combinations at all vertices. We call such a collection *anglewise vertex combination* and denote by AVC. The AVC may also include the angle combination in the pentagon or even include the specific angle arrangement in the pentagon. Note that the AVC is only *partial* because it may not be equal to the actual collection of angle combinations. We will specify the relation between the partial AVC and the actual AVC.

In the remaining part of the section, we show that the three angle cases do not lead to tilings. These are actually reduced cases. By the symmetry of the pentagon, we have $c_1 = c_2$ and $c_3 = c_5$ in Figure 2. The picture depicts the arrangement $A1$, and we have similar equalities for the arrangement $A2$. Hence we only have three quadratic equations for $\cos a$. Moreover, the two vertices in AVC_3 enable us to express all three angles in terms of one angle. Therefore we only look for the common zero of two resultants that depend on single angle variable.

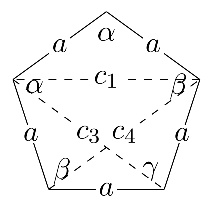

Figure 2: Three quadratic equations for three angle cases.

For example, in Case 3.1, the equations $\alpha + \beta + \gamma = 3\alpha = 2\pi$ imply that

$$\alpha = \frac{2}{3}\pi, \quad \gamma = \frac{4}{3}\pi - \beta, \quad 2\alpha + 2\beta + \gamma - 3\pi = \beta - \frac{1}{3}\pi = \frac{4}{f}\pi.$$

The condition $f \geq 16$ means $\frac{1}{3}\pi < \beta \leq \frac{7}{12}\pi$. For the arrangement $A1$, Figure 3 gives the graph of the resultants R_{13} (in red) and R_{14} (in blue) on the interval $[0.3\pi, 0.6\pi]$ containing $[\frac{1}{3}\pi, \frac{7}{12}\pi]$. In Figure 3, we find that the common zero of the two resultants is approximately $\beta = \frac{1}{3}\pi$. The exact value can be further confirmed by symbolic computation. Since this implies $f = \infty$, the solution is dismissed.

In Figure 3, we omit π in the coordinates values. So 0.6 for β really means $\beta = 0.6\pi$. We will adopt the same convention in Figures 4 and 8.

We carry out the similar calculation for all the three angle cases and two arrangements for each case. We find no pentagon suitable for tiling.

3 Four Angles at Degree 3 Vertices

We explained in Section 2 that, by [3, Lemma 21], the number of distinct angles in an equilateral pentagon must be odd. For the four angle and five angle parts of Table 1, therefore, the pentagon must have five distinct angles $\alpha, \beta, \gamma, \delta, \epsilon$. In particular, the angle ϵ appears only at vertices of degree > 3

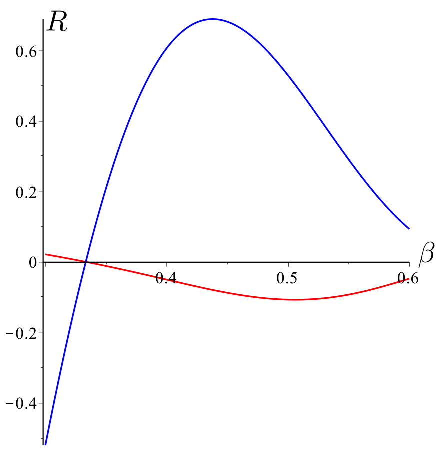

Figure 3: Case 3.1, arrangement $A1$, R_{13} and R_{14}.

in the four angle part. Moreover, up to the symmetry of flipping, there are generally twelve ways of arranging the angles in the pentagon

$$\begin{aligned} B\colon & [\alpha,\beta,\gamma,\delta,\epsilon],[\alpha,\beta,\gamma,\epsilon,\delta],[\alpha,\beta,\delta,\gamma,\epsilon],[\alpha,\beta,\delta,\epsilon,\gamma],\\ & [\alpha,\beta,\epsilon,\gamma,\delta],[\alpha,\beta,\epsilon,\delta,\gamma],[\alpha,\gamma,\beta,\delta,\epsilon],[\alpha,\gamma,\beta,\epsilon,\delta],\\ & [\alpha,\gamma,\delta,\beta,\epsilon],[\alpha,\gamma,\epsilon,\beta,\delta],[\alpha,\delta,\beta,\gamma,\epsilon],[\alpha,\delta,\gamma,\beta,\epsilon]. \end{aligned}$$

Of course, further symmetries in some cases may reduce the number of arrangements we need to consider.

For the first combination $\{\alpha\beta\gamma,\alpha\delta^2\}$ of the four angle part of the table (i.e., $\alpha\beta\gamma$ and $\alpha\delta^2$ belong to the actual AVC$_3$), if there are no more degree 3 vertices, then α appears at every degree 3 vertex. By [4, lemma 3], this implies that α appears at least twice in the pentagon. The contradiction shows that one of the optional vertices must appear, and we get two combinations

$$\{\alpha\beta\gamma,\alpha\delta^2,\beta^2\delta\},\quad \{\alpha\beta\gamma,\alpha\delta^2,\beta^3\}.$$

Similar argument for the second combination $\{\alpha\beta\gamma,\alpha^2\delta\}$ also gives two combinations

$$\{\alpha\beta\gamma,\alpha^2\delta,\beta\delta^2\},\quad \{\alpha\beta\gamma,\alpha^2\delta,\beta^3\}.$$

Since $\alpha\leftrightarrow\beta$ exchanges $\{\alpha\beta\gamma,\alpha\delta^2,\beta^2\delta\}$ and $\{\alpha\beta\gamma,\alpha^2\delta,\beta\delta^2\}$, the four combinations may be reduced to three. Up to the permutation of symbols, the actual AVC$_3$ must be one of the three combinations.

For the combination $\{\alpha\beta^2, \gamma\delta^2\}$, if there are no more degree 3 vertices, then by the proof of the case $\{\alpha\beta^2\}$ in the proof of [4, Proposition 5], we know that β and δ appear together at least three times in the pentagon. This contradicts to the five distinct angles in the pentagon. Therefore the optional vertex $\alpha^2\delta$ must appear, and we get $\text{AVC}_3 = \{\alpha\beta^2, \gamma\delta^2, \alpha^2\delta\}$.

For the combination $\{\alpha\beta\gamma, \delta^3\}$, we actually have $\text{AVC}_3 = \{\alpha\beta\gamma, \delta^3\}$. Then we make use of the fact that the fifth angle ϵ only appears at vertices of degree > 3. By [4, lemma 4], one of $\alpha\epsilon^3$, $\beta\epsilon^3$, $\gamma\epsilon^3$, $\delta\epsilon^3$, ϵ^4, ϵ^5 must be a vertex. Up to the symmetry of exchanging α, β, γ, we may omit $\beta\epsilon^3$ and $\gamma\epsilon^3$. Any of the remaining combinations $\alpha\epsilon^3$, $\delta\epsilon^3$, ϵ^4, ϵ^5 can be added to $\text{AVC}_3 = \{\alpha\beta\gamma, \delta^3\}$ to get a subset of the actual AVC.

In summary, for the case of four distinct angles appearing at degree 3 vertices, we get the following complete list of possible triples of angle combinations that must appear at vertices. The 12 arrangements are given by B, and the reductions of arrangements by further symmetries are also indicated. The total number of cases is 72.

4.1a $\{\alpha\beta\gamma\delta\epsilon\colon \alpha\beta\gamma, \alpha\delta^2, \beta^2\delta\}$. 12 arrangements B.

4.1b $\{\alpha\beta\gamma\delta\epsilon\colon \alpha\beta\gamma, \alpha\delta^2, \beta^3\}$. 12 arrangements B.

4.1c $\{\alpha\beta\gamma\delta\epsilon\colon \alpha\beta\gamma, \alpha^2\delta, \beta^3\}$. 12 arrangements B.

4.2a $\{\alpha\beta\gamma\delta\epsilon\colon \alpha\beta\gamma, \delta^3, \alpha\epsilon^3\}$. 6 arrangements ($\beta, \gamma$ exchange).

4.2b $\{\alpha\beta\gamma\delta\epsilon\colon \alpha\beta\gamma, \delta^3, \delta\epsilon^3\}$. 2 arrangements ($\alpha, \beta, \gamma$ exchange).

4.2c $\{\alpha\beta\gamma\delta\epsilon\colon \alpha\beta\gamma, \delta^3, \epsilon^4\}$. 2 arrangements ($\alpha, \beta, \gamma$ exchange).

4.2d $\{\alpha\beta\gamma\delta\epsilon\colon \alpha\beta\gamma, \delta^3, \epsilon^5\}$. 2 arrangements ($\alpha, \beta, \gamma$ exchange).

4.3 $\{\alpha\beta\gamma\delta\epsilon\colon \alpha\beta^2, \gamma\delta^2, \alpha^2\delta\}$. 12 arrangements B.

4.4 $\{\alpha\beta\gamma\delta\epsilon\colon \alpha\beta^2, \alpha^2\gamma, \delta^3\}$. 12 arrangements B.

3.1 Cases 4.1, 4.3, 4.4

We will follow the spherical trigonometry outlined in Section 2. For the concerned cases, we may express β, γ, δ in terms of α, so that the resultants become functions of α, ϵ. Note that among the five quadratic equations for $\cos a$, two do not involve ϵ, so that the resultant of these two is a function of

α only. We can find the value of α as the zero of this resultant. Then we can substitute the value of α into the other resultants, and find ϵ as the common zero of the other resultants.

For example, consider the arrangement $B1$ of Case 4.1b

$$\{[\alpha,\beta,\gamma,\delta,\epsilon]\colon \alpha\beta\gamma, \alpha\delta^2, \beta^3\}.$$

From the equations $\alpha+\beta+\gamma=\alpha+2\delta=3\beta=2\pi$, we get

$$\beta=\frac{2}{3}\pi,\quad \gamma=\frac{3}{4}\pi-\alpha,\quad \delta=\pi-\frac{1}{2}\alpha.$$

This implies

$$0<\alpha<\frac{3}{4}\pi,\quad \alpha\neq\frac{2}{3}\pi.$$

Moreover, the number of tiles $f\geq 16$ imposes a condition on the area of the pentagon

$$\alpha+\beta+\gamma+\delta+\epsilon-3\pi=\epsilon-\frac{1}{2}\alpha=\frac{4}{f}\pi\leq\frac{1}{4}\pi.$$

The resultant R_{14} does not involve ϵ. It is the product of three factors

$$\begin{aligned}
R_{14}^{(1)} &= \left(1+\cos\frac{1}{2}\alpha\right)^2,\\
R_{14}^{(2)} &= 2\cos\frac{1}{2}\alpha-1,\\
R_{14}^{(3)} &= \left(2\cos^3\frac{1}{2}\alpha-2\cos^2\frac{1}{2}\alpha+\cos\frac{1}{2}\alpha-\frac{1}{2}\right)\sqrt{3}\sin\frac{1}{2}\alpha\\
&\quad +6\cos^4\frac{1}{2}\alpha-2\cos^3\frac{1}{2}\alpha-8\cos^2\frac{1}{2}\alpha+5\cos\frac{1}{2}\alpha-\frac{1}{4}.
\end{aligned}$$

The first factor has no zero in the range $(0,\frac{3}{4}\pi)$ for α. The zero of the second factor within the range is $\alpha=\frac{2}{3}\pi$, which is also forbidden. To get the zero of the third factor, we solve $R_{14}^{(3)}=0$ for $\sin\frac{1}{2}\alpha$ and substitute into $\cos^2\frac{1}{2}\alpha+\sin^2\frac{1}{2}\alpha=1$. What we get is the product of two factors

$$\begin{aligned}
F_1 &= 24\cos^3\frac{1}{2}\alpha-24\cos^2\frac{1}{2}\alpha+2\cos\frac{1}{2}\alpha+1,\\
F_2 &= 16\cos^4\frac{1}{2}\alpha+8\cos^3\frac{1}{2}\alpha-24\cos^2\frac{1}{2}\alpha-8\cos\frac{1}{2}\alpha+11.
\end{aligned}$$

The zeros of the two factors within the range $(0, \frac{3}{4}\pi)$ are

$$\alpha = 0.7961\pi,\ 0.4742\pi.$$

Substituting the two α into the other resultants, we find that there is no ϵ satisfying $R_{13} = R_{23} = R_{35} = 0$ at the same time. We conclude that the spherical pentagon does not exist.

Similar argument shows that the spherical pentagon does not exist in Cases 4.1, 4.3, 4.4, for all the arrangements. For Cases 4.1a and 4.1c, the argument can actually be carried out with $\cos\alpha, \sin\alpha$ instead of $\cos\frac{1}{2}\alpha, \sin\frac{1}{2}\alpha$.

3.2 Case 4.2a

The problem here is that β, γ, δ cannot be expressed in terms of α only, so that there is no "ϵ-free" resultant. We simply need to treat all resultants equally and consider the common zero of four resultants.

Consider the arrangement $B1$

$$\{[\alpha, \beta, \gamma, \delta, \epsilon] \colon \alpha\beta\gamma, \delta^3, \alpha\epsilon^3\}.$$

The equations $\alpha + \beta + \gamma = 3\delta = \alpha + 3\epsilon = 2\pi$ imply that

$$\gamma = 2\pi - \alpha - \beta, \quad \delta = \frac{2}{3}\pi, \quad \epsilon = \frac{2}{3}\pi - \frac{1}{3}\alpha.$$

The condition $f \geq 16$ implies

$$\alpha + \beta + \gamma + \delta + \epsilon - 3\pi = \frac{1}{3}(\pi - \alpha) = \frac{4}{f}\pi \leq \frac{1}{4}\pi.$$

Therefore

$$\frac{1}{4}\pi \leq \alpha < \pi, \quad \alpha + \beta = 2\pi - \gamma < 2\pi.$$

In Figure 4, we plot the four resultant curves $R_{12} = R_{13} = R_{14} = R_{25} = 0$. The green lines correspond to $\alpha = \frac{1}{4}\pi$, $\alpha = \pi$ and $\alpha + \beta = 2\pi$. We need to look for solutions between the two vertical lines and below the scant line. We find three possible intersections of the four curves within the range. We may zoom in to get a more accurate values of the solutions

$$(\alpha, \beta) = (0.2584\pi, 1.5603\pi),\ (0.5187\pi, 0.7018\pi).$$

The two solutions have $f = 16.18,\ 24.93$, contradicting to the requirement that f is an even integer.

We observe that there is another solution that appears to be $(\alpha, \beta) = (\pi, \frac{1}{3}\pi)$. We may use the symbolic computation to confirm that the exact value is indeed a common zero of the four resultants. Therefore this solution violates the requirement that $\alpha < \pi$ ($\alpha = \pi$ corresponds to $f = \infty$).

So we conclude that the arrangement $B1$ of Case 4.2a does not admit spherical pentagon suitable for tiling. Similar argument shows that all arrangements of Cases 4.2a do not admit suitable spherical pentagon suitable for tiling.

The argument for the case is typical. After getting the approximate (sometimes exact) values of all the angles from the common zero of four resultants, we calculate the approximate value of f. If the approximate value implies that f is not an even integer ≥ 16, then the solution can be dismissed. This is exactly what happens to all the arrangements of Cases 4.2a.

In case we see a "borderline solution", we always have an exact value of the solution. Then we can use symbolic computation to confirm the exact value, so that the solution can be dismissed due to the violation of some strict inequality.

Finally, we remark that Cases 4.1, 4.3, 4.4 can also be treated by the method for Cases 4.2a.

3.3 Case 4.2b

Up to the permutation of α, β, γ, we only need to consider the arrangements $B1$ and $B3$. The equations $\alpha + \beta + \gamma = 3\delta = \delta + 3\epsilon = 2\pi$ imply that

$$\gamma = 2\pi - \alpha - \beta, \quad \delta = \frac{2}{3}\pi, \quad \epsilon = \frac{4}{9}\pi, \quad f = 36.$$

The range for (α, β) is

$$\alpha > 0, \quad \beta > 0, \quad 0 < \alpha + \beta < 2\pi,$$

For the arrangement $B1$, we look for the common zero of four resultant curves $R_{12} = R_{23} = R_{24} = R_{25} = 0$, and find three solutions within the range

$$(\alpha, \beta) = (0.29539\pi, 1.62453\pi),\ (0.47\pi, 0.71\pi),\ (0.8757\pi, 0.4299\pi).$$

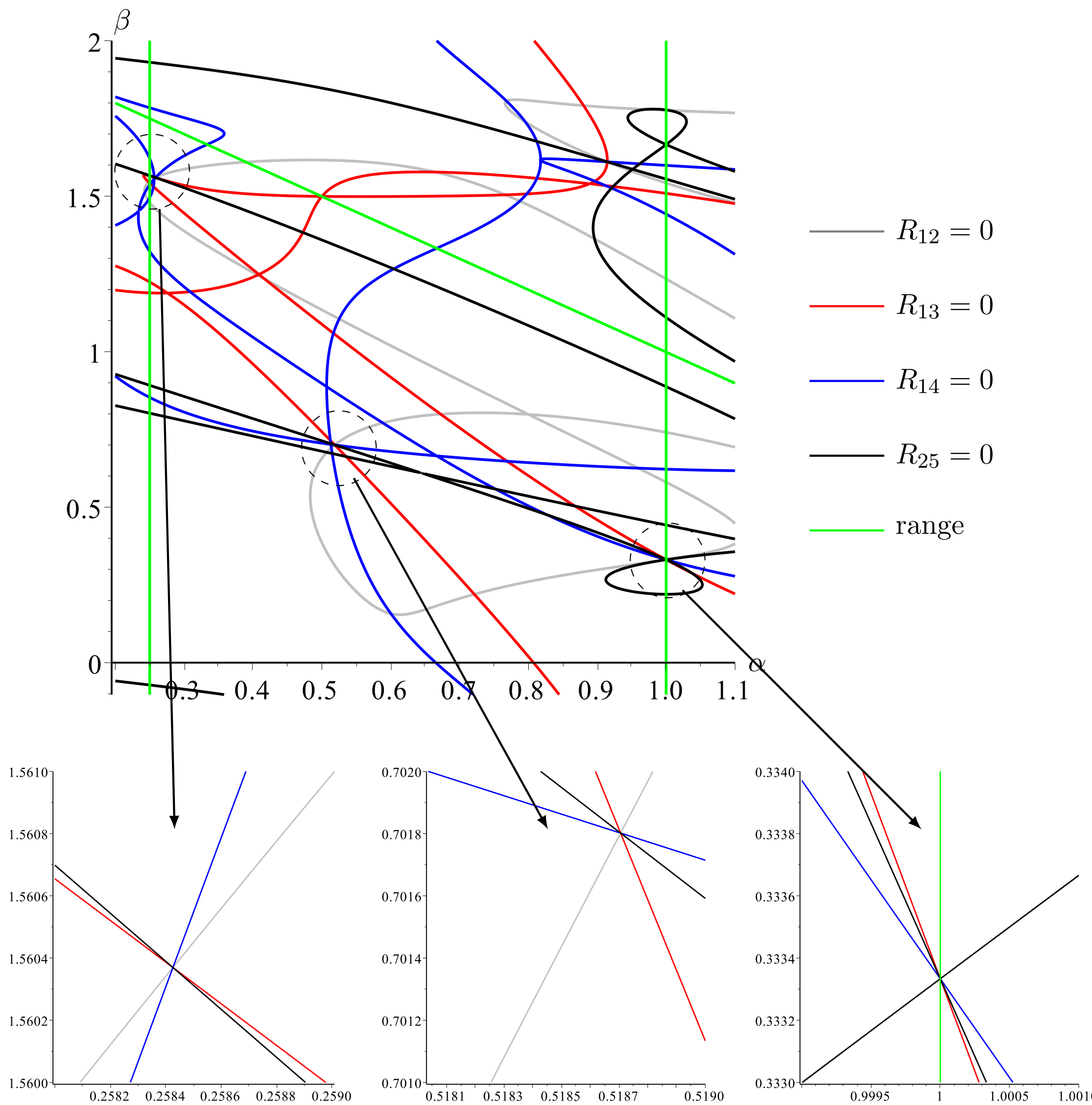

Figure 4: Case 4.2a, arrangement $B1$, $R_{12} = R_{13} = R_{14} = R_{25} = 0$.

However, the second solution violates [2, Lemma 3]. For the arrangement $B3$, we look for the intersection of four curves $R_{13} = R_{14} = R_{23} = R_{35} = 0$ and find two solutions within the range

$$(\alpha, \beta) = (0.77\pi, 0.86\pi),\ (0.855\pi, 0.455\pi).$$

However, the first solution violates [2, Lemma 3].

For the solution $(\alpha, \beta) = (0.29539\pi, 1.62453\pi)$, we have $\gamma = 2\pi - \alpha - \beta = 0.08008\pi$ (accurate up to -0.00002π, i.e., $\gamma \in [0.08006\pi, 0.08008\pi]$). Then we try to find all the possible angle combinations $\alpha^i\beta^j\gamma^k\delta^l\epsilon^m$ at vertices by solving

$$\alpha i + \beta j + \gamma k + \delta l + \epsilon m = 2\pi.$$

Unfortunately, we cannot solve the exact equation because we do not have the exact values for all the angles. Still, since all five terms on the left are positive, the approximate values of the five angles imply

$$i \le 6,\ j \le 1,\ k \le 25,\ l \le 3,\ m \le 4.$$

Therefore any solution to the exact equation also satisfies

$$\left| 0.29539i + 1.62453j + 0.08008k + \frac{2}{3}l + \frac{4}{9}m - 2 \right| < 0.0006.$$

The choice of the right side is due to

$$(6 + 1) \cdot 0.00001 + 25 \cdot 0.00002 < 0.0006.$$

We substitute all combinations of indices i, j, k, l, m within the bounds to the inequality above and found exactly three combinations $\alpha\beta\gamma$, δ^3, $\delta\epsilon^3$ satisfying the inequality.

Similar argument shows that for the solutions $(\alpha, \beta) = (0.8757\pi, 0.4299\pi)$ and $(0.855\pi, 0.455\pi)$, there are also exactly three combinations $\alpha\beta\gamma$, δ^3, $\delta\epsilon^3$. We need 5 digit approximation for the first solution because γ is very small, which means k can be as big as 25. The bounds for the two solutions here are much smaller, and 4 digit and 3 digit approximations are sufficient.

It remains to find the tiling for AVC $= \{\alpha\beta\gamma\delta\epsilon\colon \alpha\beta\gamma, \delta^3, \delta\epsilon^3\}$. By the proof of [4, Theorem 6] for the full AVC $\{36\alpha\beta\gamma\delta\epsilon\colon 36\alpha\beta\gamma, 8\delta^3, 12\delta\epsilon^3\}$, there is no tiling with $\alpha\beta\gamma$, δ^3, $\delta\epsilon^3$ as the only vertices.

3.4 Cases 4.2c and 4.2d

For Case 4.2c, we have

$$\alpha+\beta+\gamma=2\pi, \quad \delta=\frac{2}{3}\pi, \quad \epsilon=\frac{1}{2}\pi, \quad f=24.$$

The common zero of four resultants similar to the case 4.2b gives five solutions

$$\begin{aligned}(\alpha,\beta)=&(0.27849\pi,1.59984\pi),(0.52\pi,0.70\pi),(0.820\pi,0.484\pi); && (\text{for } B1)\\ &(0.73\pi,0.81\pi),(0.801\pi,0.511\pi). && (\text{for } B3)\end{aligned}$$

For Case 4.2d, we have

$$\alpha+\beta+\gamma=2\pi, \quad \delta=\frac{2}{3}\pi, \quad \epsilon=\frac{2}{5}\pi, \quad f=60.$$

The common zero of four resultants gives five solutions

$$\begin{aligned}(\alpha,\beta)=&(0.31031\pi,1.64260\pi),(0.44\pi,0.72\pi),(0.9229\pi,0.3890\pi); && (\text{for } B1)\\ &(0.81\pi,0.90\pi),(0.9059\pi,0.4093\pi). && (\text{for } B3)\end{aligned}$$

For both cases, the second and fourth solutions violate [2, Lemma 3]. For each of the remaining six solutions that do not violate [2, Lemma 3], we may calculate all the angle combinations at vertices similar to Case 4.2b. We find that $\alpha\beta\gamma, \delta^3, \epsilon^4$ are the only vertices for Case 4.2c, and $\alpha\beta\gamma, \delta^3, \epsilon^5$ are the only vertices for Case 4.2d. Then by the proof of [4, Theorem 6] for the full AVCs $\{24\alpha\beta\gamma\delta\epsilon\colon 24\alpha\beta\gamma, 8\delta^3, 6\epsilon^4\}$ and $\{60\alpha\beta\gamma\delta\epsilon\colon 60\alpha\beta\gamma, 20\delta^3, 12\epsilon^5\}$, and the argument for [4, Theorem 7], only the fifth solutions of the two cases admit tilings, and the tilings are the pentagonal subdivisions of platonic solids.

It remains to verify that the fifth solutions of the two cases can be realized by actual spherical pentagons. By finding the common solution of the five resultants, which are quadratic equations of $\cos a$, we can find the approximate values of edge length a for the two solutions. For example, we find $a = 0.17\pi$ for the fifth solution of Case 4.2c. This mean that we expect the exact value $a \in [a_-, a_+] = [0.17\pi, 0.18\pi]$. We will see that $[a_-, a_+] \subset (0, \frac{1}{2}\pi)$ for two solutions here as well as all the later solutions.

Figure 5 shows the possible pentagons in arrangement $B3$, with A, B, C, D, E being respectively the vertices where the angles α, β, γ, δ, ϵ are located. We consider the pentagon as obtained by glueing the isosceles triangles $\triangle ACE$, $\triangle BCD$ and the middle triangle $\triangle ABC$ together. This is indeed the case for the left and middle situations, where the triangle $\triangle ABC$ lies inside the pentagon. Our subsequent discussion will also be based on this assumption. In the right situation, the triangle $\triangle ABC$ is not inside the pentagon, and we will explain why this and the similar situation do not happen for our solutions.

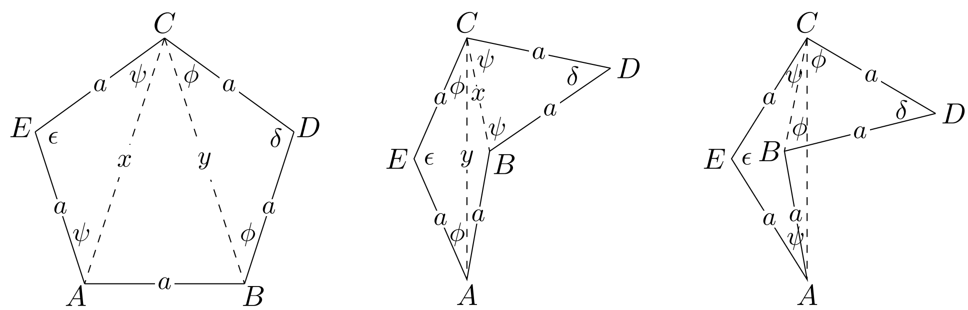

Figure 5: Various possible shapes of the pentagon for $B3$.

The known precise values of δ and ϵ determine the triangles $\triangle ACE$ and $\triangle BCD$ as functions of a. In particular,

$$\begin{aligned}
x &= BC = \arccos(\cos^2 a + \sin^2 a \cos\delta),\\
y &= AC = \arccos(\cos^2 a + \sin^2 a \cos\epsilon),\\
\phi &= \angle CBD = \arctan\left(\sec a \cot\frac{\delta}{2}\right),\\
\psi &= \angle CAE = \arctan\left(\sec a \cot\frac{\epsilon}{2}\right).
\end{aligned}$$

For the range $[a_-, a_+]$ of a, we find the corresponding ranges $[x_-, x_+]$, $[y_-, y_+]$, $[\phi_-, \phi_+]$, $[\psi_-, \psi_+]$ for x, y, ϕ, ψ. The existence of the middle triangle $\triangle ABC$ can be verified by showing that the ranges satisfy

$$a + x + y < 2\pi, \quad a < x + y, \quad x < a + y, \quad y < a + x.$$

This shows the existence of the pentagon with the given precise values of δ, ϵ and a range $[a_-, a_+]$ for a. Yet this does not prevent the shape of the pentagon to be the right of Figure 5. We may verify that the triangle $\triangle ABC$ indeed lies inside the pentagon, by further showing that the ranges of ϕ and

ψ and the initial approximate values of α, β from the two solutions satisfy the following inequalities

$$\alpha > \psi, \quad \beta > \phi, \quad \gamma = 2\pi - \alpha - \beta > \phi + \psi.$$

To verify the original definition of Case 4.2c, it remains to show that the equality $\alpha + \beta + \gamma = 2\pi$ can be achieved for some $a \in [a_-, a_+]$. For this purpose, we have the angles of the triangle $\triangle ABC$

$$\angle CAB = \arccos\left(\frac{\cos x - \cos a \cos y}{\sin a \sin y}\right),$$
$$\angle CBA = \arccos\left(\frac{\cos y - \cos a \cos x}{\sin a \sin x}\right),$$
$$\angle ACB = \arccos\left(\frac{\cos a - \cos x \cos y}{\sin x \sin y}\right),$$

and further the angles α, β, γ as functions of a, x, y

$$\alpha = \psi + \angle CAB, \quad \beta = \phi + \angle CBA, \quad \gamma = \phi + \psi + \angle ACB.$$

Substituting the formulae of x, y, ϕ, ψ as functions of a, the angles α, β, γ, $\angle CAB$, $\angle CBA$, $\angle ACB$ may be expressed as functions of the single variable a. Then we need to achieve the vanishing of the following function of a

$$f(a) = \alpha + \beta + \gamma - 2\pi = \angle CAB + \angle CBA + \angle ACB + 2\phi + 2\psi - 2\pi.$$

We get the existence of a by showing that $f(a_-)$ and $f(a_+)$ have opposite signs and then applying the mean value theorem.

For the fifth solution of Case 4.2c, we find the approximate edge length $a = 0.17\pi$ from the common solution of two resultants. By

$$f(0.17\pi) = -0.028, \quad f(0.18\pi) = 0.035,$$

and the intermediate value theorem, there is $a \in [0.17\pi, 0.18\pi]$ satisfying $f(a) = 0$. We may further apply the intermediate value theorem to $f(a) = 0$ to get more and more digits for a. In fact, we get

$$a = 0.17452731854247459669847381026\pi$$

because for this value of a, we have

$$f(a) = -3.8 \times 10^{-29} < 0, \quad f(a + 10^{-29}\pi) = 2.6 \times 10^{-29} > 0.$$

Using this value of a, we get

$$\alpha = 0.80106832905992046260731242 2969\pi,$$
$$\beta = 0.51139177170631338496460382209\pi,$$
$$\gamma = 0.68753989923376615242808375493\pi,$$

and

$$\phi = 0.189\pi, \quad \psi = 0.275\pi, \quad x = 0.298\pi, \quad y = 0.240\pi.$$

We verify that the inequalities between a, x, y and between $\alpha, \beta, \phi, \psi$ are satisfied, so that the pentagon indeed exists and is shaped like the left of Figure 5. The numerical data for the pentagon is depicted on the left of Figure 6.

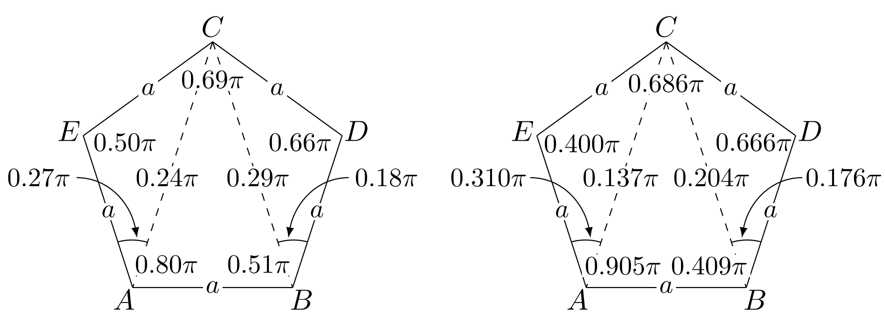

Figure 6: Pentagon for $\{[\alpha, \beta, \delta, \gamma, \epsilon] \colon \alpha\beta\gamma, \delta^3, \epsilon^4 \text{ or } \epsilon^5\}$.

Similarly, for the fifth solution of Case 4.2d, we have

$$a = 0.11864733486550189358293111 8986\pi,$$
$$\alpha = 0.90594259357454383276918243 9026\pi,$$
$$\beta = 0.40930345489814618068554640 2290\pi,$$
$$\gamma = 0.68475395152730998654527115868\pi,$$

and

$$\phi = 0.176\pi, \quad \psi = 0.310\pi, \quad x = 0.204\pi, \quad y = 0.137\pi.$$

This also implies the existence of the pentagon, which is also shaped like the left of Figure 5, with the numerical data depicted on the right of Figure 6.

4 Five Angles at Degree 3 Vertices

Suppose the pentagon has five distinct angles $\alpha, \beta, \gamma, \delta, \epsilon$. By [4, Lemma 3], no angle can appear at all the degree 3 vertices. For the first combination $\{\alpha\beta\gamma, \alpha\delta\epsilon\}$ in the five angle part of Table 1, therefore, some optional vertex not involving α must appear. Up to the symmetry of symbols, we may assume that either $\gamma\epsilon^2$ or γ^3 appears. If we further include the optional vertex α^3 for the combination $\{\alpha\beta\gamma, \delta\epsilon^2\}$, then we get the following list of possible triples of angle combinations that must appear at vertices from the five angle part of Table 1. The total number of cases is 102.

5.1a $\{\alpha\beta\gamma\delta\epsilon\colon \alpha\beta\gamma, \alpha\delta\epsilon, \gamma\epsilon^2\}$. 12 arrangements B.

5.1b $\{\alpha\beta\gamma\delta\epsilon\colon \alpha\beta\gamma, \alpha\delta\epsilon, \gamma^3\}$. 6 arrangements ($\delta, \epsilon$ exchange).

5.2 $\{\alpha\beta\gamma\delta\epsilon\colon \alpha\beta\gamma, \alpha\delta^2, \alpha^2\epsilon\}$. 6 arrangements ($\beta, \gamma$ exchange).

5.3 $\{\alpha\beta\gamma\delta\epsilon\colon \alpha\beta\gamma, \alpha\delta^2, \beta\epsilon^2\}$. 6 arrangements ($(\alpha, \delta), (\beta, \epsilon)$ exchange).

5.4 $\{\alpha\beta\gamma\delta\epsilon\colon \alpha\beta\gamma, \alpha\delta^2, \beta^2\epsilon\}$. 12 arrangements B.

5.5 $\{\alpha\beta\gamma\delta\epsilon\colon \alpha\beta\gamma, \alpha\delta^2, \delta\epsilon^2\}$. 6 arrangements ($\beta, \gamma$ exchange).

5.6 $\{\alpha\beta\gamma\delta\epsilon\colon \alpha\beta\gamma, \alpha\delta^2, \epsilon^3\}$. 6 arrangements ($\beta, \gamma$ exchange).

5.7 $\{\alpha\beta\gamma\delta\epsilon\colon \alpha\beta\gamma, \alpha^2\delta, \beta^2\epsilon\}$. 6 arrangements ($(\alpha, \delta), (\beta, \epsilon)$ exchange).

5.8 $\{\alpha\beta\gamma\delta\epsilon\colon \alpha\beta\gamma, \alpha^2\delta, \delta^2\epsilon\}$. 12 arrangements B.

5.9 $\{\alpha\beta\gamma\delta\epsilon\colon \alpha\beta\gamma, \alpha^2\delta, \epsilon^3\}$. 6 arrangements ($\beta, \gamma$ exchange).

5.10 $\{\alpha\beta\gamma\delta\epsilon\colon \alpha\beta\gamma, \delta\epsilon^2, \alpha^3\}$. 6 arrangements ($\beta, \gamma$ exchange).

5.11 $\{\alpha\beta\gamma\delta\epsilon\colon \alpha\beta^2, \gamma\delta^2, \alpha^2\epsilon\}$. 12 arrangements B.

5.12 $\{\alpha\beta\gamma\delta\epsilon\colon \alpha\beta^2, \gamma\delta^2, \epsilon^3\}$. 6 arrangements ($(\alpha, \beta), (\gamma, \delta)$ exchange).

It remains to consider the combination $\{\alpha\beta\gamma, \delta\epsilon^2\}$ in Table 1, with the additional assumption that there are no other degree 3 vertices. If there are degree 4 vertices, then we consider all the possible combinations at a degree 4 vertex and get the following complete list. Here the angle combinations at the degree 4 vertex are ordered by the types $****$, $***^2$, $*^2*^2$, $**^3$, $*^4$. We also note that the combination $\{\alpha\beta\gamma, \delta\epsilon^2, \epsilon^4\}$ is dismissed because it implies $f = 8$. The total number of cases is 124.

1.1 $\{\alpha\beta\gamma\delta\epsilon\colon \alpha\beta\gamma, \delta\epsilon^2, \alpha\beta\delta\epsilon\}$. 6 arrangements.

1.2a $\{\alpha\beta\gamma\delta\epsilon\colon \alpha\beta\gamma, \delta\epsilon^2, \alpha\beta^2\delta\}$. 12 arrangements B.

1.2b $\{\alpha\beta\gamma\delta\epsilon\colon \alpha\beta\gamma, \delta\epsilon^2, \alpha\beta^2\epsilon\}$. 12 arrangements B.

1.2c $\{\alpha\beta\gamma\delta\epsilon\colon \alpha\beta\gamma, \delta\epsilon^2, \alpha\beta\delta^2\}$. 6 arrangements.

1.2d $\{\alpha\beta\gamma\delta\epsilon\colon \alpha\beta\gamma, \delta\epsilon^2, \alpha\beta\epsilon^2\}$. 6 arrangements.

1.2e $\{\alpha\beta\gamma\delta\epsilon\colon \alpha\beta\gamma, \delta\epsilon^2, \alpha\delta^2\epsilon\}$. 6 arrangements.

1.3a $\{\alpha\beta\gamma\delta\epsilon\colon \alpha\beta\gamma, \delta\epsilon^2, \alpha^2\beta^2\}$. 6 arrangements.

1.3b $\{\alpha\beta\gamma\delta\epsilon\colon \alpha\beta\gamma, \delta\epsilon^2, \alpha^2\delta^2\}$. 12 arrangements B.

1.3c $\{\alpha\beta\gamma\delta\epsilon\colon \alpha\beta\gamma, \delta\epsilon^2, \alpha^2\epsilon^2\}$. 12 arrangements B.

1.4a $\{\alpha\beta\gamma\delta\epsilon\colon \alpha\beta\gamma, \delta\epsilon^2, \alpha\beta^3\}$. 12 arrangements B.

1.4b $\{\alpha\beta\gamma\delta\epsilon\colon \alpha\beta\gamma, \delta\epsilon^2, \alpha\delta^3\}$. 6 arrangements.

1.4c $\{\alpha\beta\gamma\delta\epsilon\colon \alpha\beta\gamma, \delta\epsilon^2, \alpha\epsilon^3\}$. 6 arrangements.

1.4d $\{\alpha\beta\gamma\delta\epsilon\colon \alpha\beta\gamma, \delta\epsilon^2, \alpha^3\delta\}$. 6 arrangements.

1.4e $\{\alpha\beta\gamma\delta\epsilon\colon \alpha\beta\gamma, \delta\epsilon^2, \alpha^3\epsilon\}$. 6 arrangements.

1.4f $\{\alpha\beta\gamma\delta\epsilon\colon \alpha\beta\gamma, \delta\epsilon^2, \delta^3\epsilon\}$. 2 arrangements.

1.5a $\{\alpha\beta\gamma\delta\epsilon\colon \alpha\beta\gamma, \delta\epsilon^2, \alpha^4\}$. 6 arrangements.

1.5b $\{\alpha\beta\gamma\delta\epsilon\colon \alpha\beta\gamma, \delta\epsilon^2, \delta^4\}$. 2 arrangements.

If there are degree 5 vertices, then we consider all the possible combinations at a degree 5 vertex and get the following complete list. Here the angle combinations at the degree 5 vertex are ordered by the types $***{*}^2$, $**^2*^2$, $**{*}^3$, $**^4$, $*^5$. We also note that the combination $\{\alpha\beta\gamma, \delta\epsilon^2, \epsilon^5\}$ is dismissed because it implies $f = \frac{20}{3}$. The total number of cases is 190.

2.1a $\{\alpha\beta\gamma\delta\epsilon\colon \alpha\beta\gamma, \delta\epsilon^2, \alpha\beta^2\delta\epsilon\}$. 12 arrangements B.

2.1b $\{\alpha\beta\gamma\delta\epsilon\colon \alpha\beta\gamma, \delta\epsilon^2, \alpha\beta\delta^2\epsilon\}$. 6 arrangements.

2.2a $\{\alpha\beta\gamma\delta\epsilon\colon \alpha\beta\gamma, \delta\epsilon^2, \alpha^2\beta^2\delta\}$. 6 arrangements.

2.2b $\{\alpha\beta\gamma\delta\epsilon\colon \alpha\beta\gamma, \delta\epsilon^2, \alpha^2\beta^2\epsilon\}$. 6 arrangements.

2.2c $\{\alpha\beta\gamma\delta\epsilon\colon \alpha\beta\gamma, \delta\epsilon^2, \alpha^2\delta^2\epsilon\}$. 6 arrangements.

2.2d $\{\alpha\beta\gamma\delta\epsilon\colon \alpha\beta\gamma, \delta\epsilon^2, \alpha\beta^2\delta^2\}$. 12 arrangements B.

2.2e $\{\alpha\beta\gamma\delta\epsilon\colon \alpha\beta\gamma, \delta\epsilon^2, \alpha\beta^2\epsilon^2\}$. 12 arrangements B.

2.3a $\{\alpha\beta\gamma\delta\epsilon\colon \alpha\beta\gamma, \delta\epsilon^2, \alpha^3\delta\epsilon\}$. 6 arrangements.

2.3b $\{\alpha\beta\gamma\delta\epsilon\colon \alpha\beta\gamma, \delta\epsilon^2, \alpha\beta^3\delta\}$. 12 arrangements B.

2.3c $\{\alpha\beta\gamma\delta\epsilon\colon \alpha\beta\gamma, \delta\epsilon^2, \alpha\beta^3\epsilon\}$. 12 arrangements B.

2.3d $\{\alpha\beta\gamma\delta\epsilon\colon \alpha\beta\gamma, \delta\epsilon^2, \alpha\delta^3\epsilon\}$. 6 arrangements.

2.4a $\{\alpha\beta\gamma\delta\epsilon\colon \alpha\beta\gamma, \delta\epsilon^2, \alpha^2\beta^3\}$. 12 arrangements B.

2.4b $\{\alpha\beta\gamma\delta\epsilon\colon \alpha\beta\gamma, \delta\epsilon^2, \alpha^2\delta^3\}$. 12 arrangements B.

2.4c $\{\alpha\beta\gamma\delta\epsilon\colon \alpha\beta\gamma, \delta\epsilon^2, \alpha^2\epsilon^3\}$. 12 arrangements B.

2.4d $\{\alpha\beta\gamma\delta\epsilon\colon \alpha\beta\gamma, \delta\epsilon^2, \alpha^3\delta^2\}$. 6 arrangements.

2.4e $\{\alpha\beta\gamma\delta\epsilon\colon \alpha\beta\gamma, \delta\epsilon^2, \alpha^3\epsilon^2\}$. 6 arrangements.

2.5a $\{\alpha\beta\gamma\delta\epsilon\colon \alpha\beta\gamma, \delta\epsilon^2, \alpha\beta^4\}$. 12 arrangements B.

2.5b $\{\alpha\beta\gamma\delta\epsilon\colon \alpha\beta\gamma, \delta\epsilon^2, \alpha\delta^4\}$. 6 arrangements.

2.5c $\{\alpha\beta\gamma\delta\epsilon\colon \alpha\beta\gamma, \delta\epsilon^2, \alpha\epsilon^4\}$. 6 arrangements.

2.5d $\{\alpha\beta\gamma\delta\epsilon\colon \alpha\beta\gamma, \delta\epsilon^2, \alpha^4\delta\}$. 6 arrangements.

2.5e $\{\alpha\beta\gamma\delta\epsilon\colon \alpha\beta\gamma, \delta\epsilon^2, \alpha^4\epsilon\}$. 6 arrangements.

2.5f $\{\alpha\beta\gamma\delta\epsilon\colon \alpha\beta\gamma, \delta\epsilon^2, \delta^4\epsilon\}$. 2 arrangements.

2.6a $\{\alpha\beta\gamma\delta\epsilon\colon \alpha\beta\gamma, \delta\epsilon^2, \alpha^5\}$. 6 arrangements.

2.6b $\{\alpha\beta\gamma\delta\epsilon\colon \alpha\beta\gamma, \delta\epsilon^2, \delta^5\}$. 2 arrangements.

Finally, we need to consider the case that $\alpha\beta\gamma$ and $\delta\epsilon^2$ are the only degree 3 vertices, and there are no vertices of degree 4 or 5. In Section 4.1, we will show that the only possibility is that δ^6 is a vertex, and δ, ϵ are not adjacent in the pentagon.

For each case, we carry out the calculation similar to what is outlined for Case 4.2a in Section 3.2. In most cases, we find no common zero for the resultants within the natural range for the angles. If there are solutions within the natural range, then we further calculate the approximate value of the number f of tiles. In many cases, we find that the approximate value implies that f cannot be an even integer ≥ 16, so the cases can also be dismissed. After eliminating all these "trivial" cases, the remaining cases are 1.2e, 1.4f, 1.5a, 1.5b, 2.4b, 2.5f, 2.6b, 5.5, and the exceptional case that $\alpha\beta\gamma$ and $\delta\epsilon^2$ are the only degree 3 vertices, and there are no vertices of degree 4 or 5. We will study these cases in the subsequent sections.

4.1 $\{\alpha\beta\gamma\delta\epsilon\colon \alpha\beta\gamma, \delta\epsilon^2\}$, $v_4 = v_5 = 0$

We first study the exceptional case that $\alpha\beta\gamma$ and $\delta\epsilon^2$ are the only degree 3 vertices, and there are no vertices of degree 4 or 5.

By [2, Proposition 1], any pentagonal spherical tiling must have a tile with four vertices having degree 3, and the fifth vertex having degree 3, 4, or 5. Since $v_4 = v_5 = 0$ in our exceptional case, there is a tile with all vertices having degree 3. We call such a tile 3^5-*tile*.

The neighborhood of the 3^5-tile is given in Figure 7. We denote the tiles by P_i, denote the vertex shared by P_i, P_j, P_k by V_{ijk}, and denote the angle of P_i at V_{ijk} by $A_{i,jk}$.

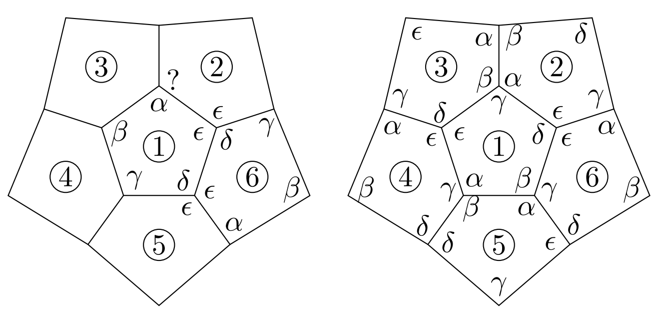

Figure 7: Neighborhood tiling for $\text{AVC}_3 = \{\alpha\beta\gamma, \delta\epsilon^2\}$.

Up to the symmetry of $\text{AVC}_3 = \{\alpha\beta\gamma, \delta\epsilon^2\}$, we only need to consider the arrangements $B1$ and $B3$. The center tile of the left of Figure 7 has arrangement $B1$. By AVC$_3$, we get $V_{156} = \delta\epsilon^2$, so that $V_{5,16} = V_{6,15} = \epsilon$. Since $\alpha\epsilon\cdots$ is not in AVC$_3$, the angle α of P_6 adjacent to ϵ must be located as indicated. Then the angles α and ϵ of P_6 determine the locations of all the angles of P_6. By AVC$_3$, we further get $A_{2,16} = \epsilon$. Then the angle of P_2 labeled ? is adjacent to ϵ and therefore must be α or δ. This means either $\alpha^2\cdots$ or $\alpha\delta\cdots$ belongs to AVC$_3$, a contradiction.

So we only need to consider the arrangement $B3$, as indicated by the center tile of the right of Figure 7. Using the similar argument, we get the unique locations of all the angles in the neighborhood.

Next we will argue that the number of tiles $f \le 24$. Since f is even, it is sufficient to show that $f < 26$. We note that AVC$_3$ implies

$$\alpha + \beta + \gamma + \delta + \epsilon - 3\pi = \frac{1}{2}\delta = \frac{4}{f}\pi, \quad \delta = \frac{8}{f}\pi.$$

Since $f \ge 16$, we have $\delta \le \frac{1}{2}\pi$. We will have two inequality restrictions on f.

We consider pentagon in Figure 5. We have $a < \pi$ because otherwise any two adjacent edges would intersect at two points. We may determine arcs x and y by the cosine laws

$$\begin{aligned}\cos x &= \cos^2 a + \sin^2 a \cos\delta,\\ \cos y &= \cos^2 a + \sin^2 a \cos\epsilon = \cos^2 a - \sin^2 a \cos\frac{\delta}{2}.\end{aligned}$$

The inequality $y - x \le a$ then defines a region on the rectangle $(a, \delta) \in (0, \pi) \times (0, \frac{1}{2}\pi]$.

For $\frac{1}{2}\pi < a < \pi$, another inequality may be obtained by estimating the area of the pentagon. Since $\delta \le \frac{1}{2}\pi$, the triangle BCD lies outside the quadrilateral $ABCE$. Therefore

$$\frac{4}{f}\pi = \text{Area(pentagon } ABDCE) \ge \text{Area(quadrilateral } ABCE).$$

The area of the quadrilateral can be further estimated

$$\text{Area(quadrilateral } ABCE) \ge \text{Area(triangle } ACE) - \text{Area(triangle } ABC).$$

By the assumption $\frac{1}{2}\pi < a < \pi$, we have

$$\text{Area(triangle } ACE) \ge \epsilon = \pi - \frac{\delta}{2} = \pi - \frac{4}{f}\pi.$$

Moreover, Area(triangle ABC) $+ \pi$ is the sum $\sum$ of the three angles of the triangle ABC. Combining all the inequalities together, we get

$$\sum \geq 2\left(\pi - \frac{4}{f}\pi\right).$$

The triangle ABC has sides x, y, a, and its three angles can be calculated by the cosine law. Then $\sum$ may be explicitly expressed as a function of (a, δ).

To show that $f \geq 26$ leads to contradiction, we note that $f \geq 26$ implies $\sum \geq \frac{22}{13}\pi$ by the estimation above. In Figure 8, the solid curve separates the regions $y - x < a$ and $y - x > a$, and the dashed curve separates the regions $\sum > \frac{22}{13}\pi$ and $\sum < \frac{22}{13}\pi$. Moreover, the horizontal dotted line corresponds to $f = 26$, and the vertical dotted line corresponds to $a = \frac{1}{2}\pi$. We see that, for $f \geq 26$, the condition $y - x < a$ is not satisfied for $a \in (0, \frac{1}{2}\pi]$, and the condition $\sum \geq \frac{22}{13}\pi$ is not satisfied for $a \in [\frac{1}{2}\pi, \pi)$. Thus we conclude that $f \leq 24$.

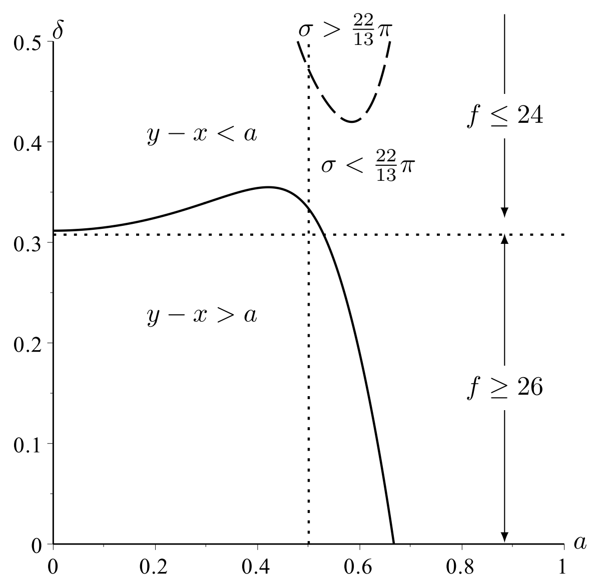

Figure 8: $f \leq 24$ for $\text{AVC}_3 = \{\alpha\beta\gamma, \delta\epsilon^2\}$.

By the vertex counting equation (see [3, page 750], for example)

$$\frac{f}{2} - 6 = \sum_{k \geq 4}(k-3)v_k = v_4 + 2v_5 + 3v_6 + \cdots, \tag{1}$$

and [7, Theorem 6], $f \le 24$ implies that either the tiling has vertices of degree 4 or 5, or $f = 24$ and the tiling is the earth map tiling with exactly two vertices of degree 6. The former case is covered by the calculation of the cases 1.$*$ and 2.$*$ and will be discussed in Sections 4.3 and 4.4. So we will only study the earth map tiling.

There are five families of earth map tilings, corresponding to distances $5, 4, 3, 2, 1$ between the two vertices of degree > 3, called "poles". They are obtained by glueing copies of the "timezones" in Figure 9 (three timezones are shown for distance 5) along the "meridians". The vertical edges at the top meet at the north pole, and the vertical edges at the bottom meet at the south pole. For $f = 24$, the tiling consists of two time zones for distances $4, 3, 2, 1$ and six timezones for distance 5.

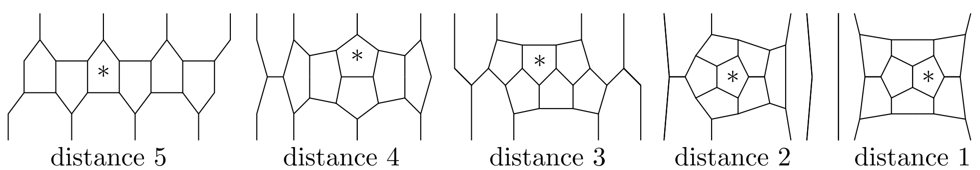

Figure 9: Timezones for earth map tilings.

Next we carry out the propagation argument in [2, Section 2] leading to the proof of Proposition 4 of that paper. The neighborhood of a 3^5-tile is given by the right of Figure 7. If a nearby tile is still a 3^5-tile, then its neighborhood is again given by the right of Figure 7. To see whether this is possible, we simplify the presentation of the neighborhood tiling on the right of Figure 7 by keeping only γ and the orientations of the angle arrangement. This gives the left picture in Figure 10. The middle picture is the mirror flipping of the left picture.

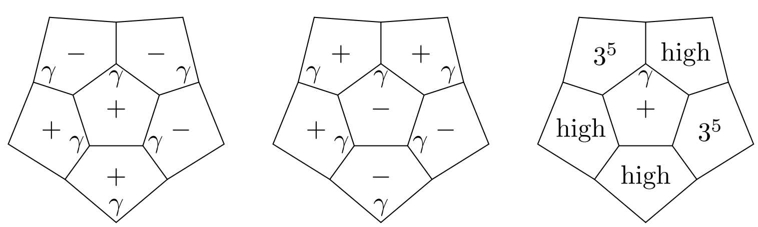

Figure 10: Propagation of the neighborhood tiling.

Now each nearby tile is adjacent to three tiles in the neighborhood tiling.

We may compare the location of γ and the orientations of the three tiles with the left or the middle picture (depending on whether the nearby tile is positively or negatively oriented). If everything matches, then the tile can be (but is not necessarily) a 3^5-tile, and we indicate the tile by 3^5 on the right of Figure 10. If there is a mismatch, then the tile must have a vertex of degree > 3 (i.e., *high degree*), and we indicate the tile by "high".

We apply the propagation to the $*$-labeled 3^5-tiles in Figure 9. For distances $4, 3, 2, 1$, all $*$-labeled tiles have at least three nearby 3^5-tiles. Since the right of Figure 10 has only two nearby 3^5-tiles, it cannot be the neighborhoods of the $*$-labeled tiles. For distance 5, we note that only the two tiles on the left and right of the $*$-labeled tile are 3^5-tiles. These two must be the two nearby 3^5-tiles on the right of Figure 10. Guided by this observation, it is easy to derive the unique earth map tiling of distance 5 in Figure 11 (only three of the six timezones are shown). In particular, we find that δ^6 must be a vertex.

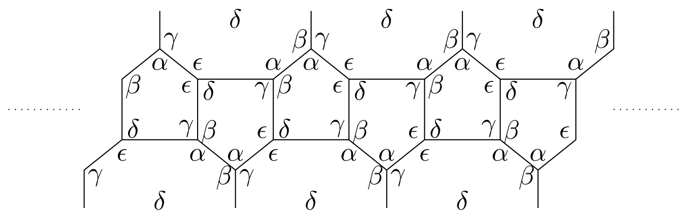

Figure 11: Tiling for $\text{AVC}_3 = \{\alpha\beta\gamma, \delta\epsilon^2\}$, $v_4 = v_5 = 0$.

So we calculate the equilateral pentagon with the angle arrangement $B3$ and three known vertices $\alpha\beta\gamma, \delta\epsilon^2, \delta^6$. The common zero of four resultants gives three solutions

$$\begin{aligned}(\alpha, \beta) &= (0.1440\pi, 1.3333\pi), (0.1192\pi, 1.3807\pi), (0.88\pi, 0.61\pi),\\ (\delta, \epsilon) &= (\tfrac{1}{3}\pi, \tfrac{5}{6}\pi).\end{aligned}$$

The third solution violates [2, Lemma 3]. Using the exact values of δ and ϵ, we may further get the following data for the first and second solutions

$$\begin{aligned}&a = 0.2501\pi,\quad x = 0.2301\pi,\quad y = 0.4788\pi,\quad \phi = 0.3766\pi,\quad \psi = 0.1153\pi;\\ &a = 0.2614\pi,\quad x = 0.2385\pi,\quad y = 0.5000\pi,\quad \phi = 0.3807\pi,\quad \psi = 0.1192\pi.\end{aligned}$$

Similar to Section 3.4, we can verify that the inequalities between a, x, y and between $\alpha, \beta, \phi, \psi$ are satisfied for the first solution. The data for the second solution suggests $y - x = a$, $\alpha = \psi$ and $\beta = \pi + \phi$. Since the existence of the pentagon depends on these exact equalities, the approximate numerical computation is not enough to verify the existence. We will use symbolic computations to exactly verify the equalities. The details will be given in Section 4.3. In fact, the data also suggests $\beta = \frac{4}{3}\pi$ for the first solution. We will also verify this by symbolic computation. The details will be given in Section 4.2.

4.2 Case 5.5

Only the arrangement $B5$ gives non-trivial solution. To make the arrangement consistent with the 4.$*$ cases and the exceptional case, we exchange α and β to translate the arrangement $B5$ to the arrangement $B3$. So we consider

$$\text{AVC}_3 = \{[\alpha, \beta, \delta, \gamma, \epsilon] \colon \alpha\beta\gamma, \beta\delta^2, \delta\epsilon^2\}.$$

The common zero of four resultants gives

$$\alpha = 0.1440\pi, \quad \beta = \frac{4}{3}\pi, \quad \gamma = 0.5226\pi, \quad \delta = \frac{1}{3}\pi, \quad \epsilon = \frac{5}{6}\pi, \quad f = 24.$$

Note that at the moment, all the values are only approximate, and the exact value of β, δ, γ will be justified by symbolic computation.

By the method in Section 3.3, we get all the possible angle combinations at vertices

$$\text{AVC} = \{[\alpha, \beta, \delta, \gamma, \epsilon] \colon \alpha\beta\gamma, \beta\delta^2, \delta\epsilon^2, \alpha^3\gamma^3, \alpha^2\gamma^2\delta^2, \alpha\gamma\delta^4, \delta^6\}.$$

Of course "possible" does not mean necessarily appearing. So the actual AVC is contained in the right side.

The AVC implies that the degrees of the vertices are either 3 or 6. By $f = 24$, the vertex counting equation (1) and [7, Theorem 6], the tiling is the earth map tiling with exactly two vertices of degree 6.

In Section 4.1, we explained that every earth map tiling has 3^5-tiles. So we study the possible ways of assigning the angles in the neighborhood of the 3^5-tile subject to our AVC. We will keep using the notations P_i, V_{ijk}, $A_{i,jk}$ as before.

By AVC, we have $V_{134} = \delta\epsilon^2$. This implies either $A_{3,14} = \delta, A_{4,13} = \epsilon$, or $A_{3,14} = \epsilon, A_{4,13} = \delta$. The left of Figure 12 describes the former case. By AVC, we know $A_{3,12} \neq \gamma$, therefore the angle γ of P_3 adjacent to $A_{3,14} = \delta$

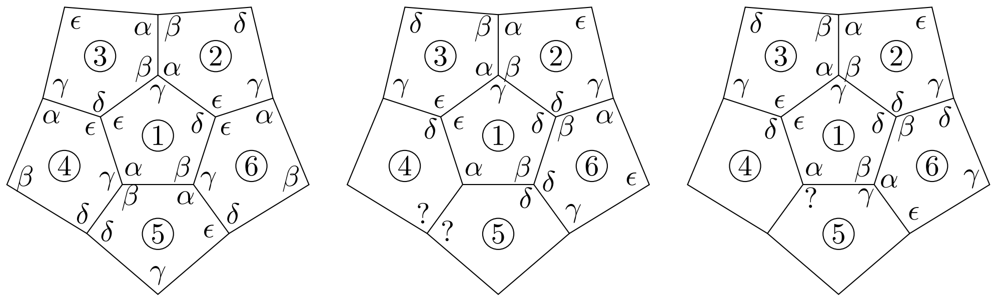

Figure 12: Neighborhood tiling for
$\{[\alpha,\beta,\delta,\gamma,\epsilon]\colon \alpha\beta\gamma, \beta\delta^2, \delta\epsilon^2, \alpha^3\gamma^3, \alpha^2\gamma^2\delta^2, \alpha\gamma\delta^4, \delta^6\}$.

must be located as indicated. This determines all the angles of P_3. By the same reason, we may determine all the angles of P_4. By AVC, we further get $A_{2,13} = \alpha$ and $A_{5,14} = \beta$. Since $\alpha\epsilon\cdots$ is not a vertex, the angle ϵ of P_2 adjacent to $A_{2,13} = \alpha$ must be located as indicated. This determines all the angles of P_2. Then by AVC, we get $A_{6,12} = \epsilon$. If the angle α of P_6 adjacent to $A_{6,12} = \epsilon$ is $A_{6,15}$, then we get $A_{5,16} = \gamma$ by AVC, so that β and γ are adjacent in P_5. The contradiction implies that the angle α of P_6 must be located as indicated. This determines all the angles of P_6. Then we get all the angles of P_5. The tiling is the same as the right of Figure 7.

The middle and right of Figure 12 describe the case $A_{3,14} = \epsilon, A_{4,13} = \delta$. We can successively determine all the angles of P_3, P_2 as before, and get $A_{4,13} = \delta, A_{6,12} = \beta$. Then the middle and the right describe two ways the angles of P_6 may be arranged. In the middle, we find that the two ?-labeled angles are α and ϵ. On the right, the ?-labeled angle is δ or ϵ. Since $\alpha\epsilon\cdots$ is not a vertex, and $\alpha\delta\cdots, \alpha\epsilon\cdots$ are not degree 3 vertices, we always get contradictions.

So we conclude that the right of Figure 7 (which is the same as the left of Figure 12) is the only neighborhood tiling fitting the AVC. Then the propagation argument in Section 4.1 (which no longer uses the AVC) shows that the tiling is the earth map tiling in Figure 11. We find that the actual AVC at the end is $\{\alpha\beta\gamma, \delta\epsilon^2, \delta^6\}$. Since the AVC does not include $\beta\delta^2$, the pentagon and the tiling is for the exceptional case in Section 4.1, and is not for Case 5.5. Furthermore, the approximate value of α shows that the tiling is for the first solution of the exceptional case.

In Section 4.1, we already verified the existence of the pentagon, and get the approximate value $a = 0.2501\pi$. It is given by Figure 13, with the left being the scheme and the right being the actual shape. However, the numerical computation cannot show that $f(a) = 0$ exactly matches $\beta = \frac{4}{3}\pi$.

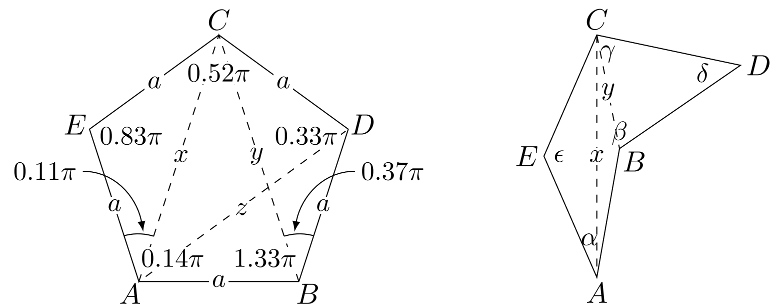

Figure 13: Pentagon for $\{[\alpha, \beta, \delta, \gamma, \epsilon]\colon \alpha\beta\gamma, \beta\delta^2, \delta\epsilon^2\}$.

To prove the exact value of β, δ, γ, we reconstruct the pentagon by starting with these exact values. This is possible because an equilateral pentagon allows three free variables. The goal is to verify that $\alpha + \beta + \gamma = 2\pi$ is exactly satisfied for the pentagon, so that the original assumption on the appearance of the vertices $\alpha\beta\gamma, \beta\delta^2, \delta\epsilon^2$ is satisfied.

The two ways of calculating $\cos x$ by using the triangle ACE and the quadrilateral $ABDC$ gives the quadratic equation for the cosine of the edge length

$$\begin{aligned} 0 = &\left(1 - \cos\frac{4}{3}\pi\right)\left(1 - \cos\frac{1}{3}\pi\right)\cos^2 a \\ &+ \left[\cos\frac{5}{6}\pi + \cos\left(\frac{4}{3} + \frac{1}{3}\right)\pi - \cos\frac{4}{3}\pi - \cos\frac{1}{3}\pi\right]\cos a \\ &+ \left[\cos\frac{5}{6}\pi - \sin\frac{4}{3}\pi \sin\frac{1}{3}\pi\right]. \end{aligned}$$

By $a = 0.2501\pi \in (0, \frac{1}{2}\pi)$, the exact value of $\cos a$ is

$$\cos a = \frac{1}{3}\left(-1 + \sqrt{3} + \sqrt{-5 + 4\sqrt{3}}\right).$$

We may construct the triangle ACE using this a and $\epsilon = \frac{5}{6}\pi$, and construct the quadrilateral $ABDC$ using this a and $\beta = \frac{4}{3}\pi$ and $\delta = \frac{1}{3}\pi$. The validity of the quadratic equation above means that the triangle and the quadrilateral have matching $AC = x$ edge. Therefore they can be glued together to form a pentagon.

We can use a, δ, ϵ to calculate the triangles AEC, BCD and then further use β to calculate the triangle ABC. Then we can confirm the approximate values $\alpha = 0.1440\pi$ and $\gamma = 0.5226\pi$. In the subsequent calculation of the exact values of α and γ, we will only choose the exact values consistent with the approximate values.

The two ways of calculating $\cos y$ by using the triangle BCD and the quadrilateral $ABCE$ gives another quadratic equation for $\cos a$

$$\begin{aligned}0 = &\left(1 - \cos\frac{5}{6}\pi\right)(1 - \cos\alpha)\cos^2 a \\ &+ \left[\cos\frac{1}{3}\pi + \cos\left(\frac{5}{6}\pi + \alpha\right) - \cos\frac{5}{6}\pi - \cos\alpha\right]\cos a \\ &+ \left[\cos\frac{1}{3}\pi - \sin\frac{5}{6}\pi\sin\alpha\right].\end{aligned}$$

Substituting the value of $\cos a$ into the equation, we get a linear equation relating $\cos\alpha$ and $\sin\alpha$

$$\begin{aligned}&\left(7 + 6\sqrt{3} + 8\sqrt{-5 + 4\sqrt{3}} + 5\sqrt{3}\sqrt{-5 + 4\sqrt{3}}\right)\cos\alpha \\ &+ 3\left(2 + \sqrt{3} + \sqrt{-5 + 4\sqrt{3}}\right)\sin\alpha \\ &= 19 + 3\sqrt{3} + 5\sqrt{-5 + 4\sqrt{3}} + 5\sqrt{3}\sqrt{-5 + 4\sqrt{3}}.\end{aligned}$$

Then we get two possible α. The one consistent with the approximate value $\alpha = 0.1440\pi$ is

$$\begin{aligned}\alpha &= \arctan\frac{1}{33}\left(4 + 3\sqrt{3} - 2\sqrt{-5 + 4\sqrt{3}} + 4\sqrt{3}\sqrt{-5 + 4\sqrt{3}}\right) \\ &= 0.14400988468593670938539230388\pi.\end{aligned}$$

Similarly, the two ways of calculating $\cos z$ by using the triangle ABD

and the quadrilateral $ADCE$ gives a linear equation relating $\cos\gamma$ and $\sin\gamma$

$$\begin{aligned}&\left(7+6\sqrt{3}+8\sqrt{-5+4\sqrt{3}}+5\sqrt{3}\sqrt{-5+4\sqrt{3}}\right)\cos\gamma\\&+3\left(2+\sqrt{3}+\sqrt{-5+4\sqrt{3}}\right)\sin\gamma\\=\,&7-3\sqrt{3}-\sqrt{-5+4\sqrt{3}}+5\sqrt{3}\sqrt{-5+4\sqrt{3}}.\end{aligned}$$

The solution consistent with the approximate value $\gamma = 0.5226\pi$ is

$$\begin{aligned}\gamma&=\pi-\arctan\frac{1}{3}\left(12+7\sqrt{3}+6\sqrt{-5+4\sqrt{3}}+4\sqrt{3}\sqrt{-5+4\sqrt{3}}\right)\\&=0.52265678198072995728127436277\pi.\end{aligned}$$

Then we may symbolically verify

$$\tan(\pi-\alpha-\gamma)=\frac{\tan(\pi-\gamma)-\tan\alpha}{1+\tan(\pi-\gamma)\tan\alpha}=\sqrt{3}.$$

The only exact value of $\pi-\alpha-\gamma$ consistent with the approximate value is $\pi-\alpha-\gamma=\frac{1}{3}\pi=\beta-\pi$.

4.3 Cases 1.2e, 1.5a and 2.4b

For the three cases, only the arrangement $B11=[\alpha,\delta,\beta,\gamma,\epsilon]$ admits meaningful solutions. So the cases can be summarized as

$$\{[\alpha,\delta,\beta,\gamma,\epsilon]\colon \alpha\beta\gamma, \delta\epsilon^2, \alpha\delta^2\epsilon \text{ or } \alpha^4 \text{ or } \alpha^2\delta^3\}.$$

Again we translate into the arrangement $B3$ by exchanging α and γ

$$\{[\alpha,\beta,\delta,\gamma,\epsilon]\colon \alpha\beta\gamma, \delta\epsilon^2, \gamma\delta^2\epsilon \text{ or } \gamma^4 \text{ or } \gamma^2\delta^3\}.$$

In all three cases, the common zero of four resultants gives

$$\gamma=\frac{1}{2}\pi,\quad \delta=\frac{1}{3}\pi,\quad \epsilon=\frac{5}{6}\pi,\quad f=24,$$

and two combinations of α,β

$$\alpha+\beta=\frac{3}{2}\pi,\quad \alpha=\psi \text{ or } \pi-\psi,\quad \psi=0.1192\pi.$$

The exact values of γ, δ, ϵ will be justified by symbolic computation.

The second solution $\alpha = \pi - \psi$ violates [2, Lemma 3]. From the solution $\alpha = \psi$, we get all the possible angle combinations at vertices by the method in Section 3.3,

$$\text{AVC} = \{[\alpha, \beta, \delta, \gamma, \epsilon] \colon \alpha\beta\gamma, \delta\epsilon^2, \gamma\delta^2\epsilon, \gamma^4, \gamma^2\delta^3, \delta^6\}.$$

The actual AVC should be contained in the right side.

Since β appears only at $\alpha\beta\gamma$, and the total number of times β appears in the tiling is $f = 24$, we find that $\alpha\beta\gamma$ appears 24 times. This implies that γ already appears 24 times at $\alpha\beta\gamma$, and therefore cannot appear at any other vertex. This implies that $\gamma\delta^2\epsilon, \gamma^4, \gamma^2\delta^3$ actually cannot appear, and the actual AVC is contained in

$$\{[\alpha, \beta, \delta, \gamma, \epsilon] \colon \alpha\beta\gamma, \delta\epsilon^2, \delta^6\}.$$

Since this is contained in the maximal possible AVC studied in Section 4.2, the tiling is given by Figure 11. In particular, the actual AVC is $\{\alpha\beta\gamma, \delta\epsilon^2, \delta^6\}$, which includes none of $\gamma\delta^2\epsilon, \gamma^4, \gamma^2\delta^3$. Therefore the pentagon and the tiling is for the exceptional case in Section 4.1, and is not for Cases 1.2e, 1.5a and 2.4b. Furthermore, the approximate value of α shows that the tiling is for the second solution of the exceptional case.

It remains to verify the existence of the pentagon. The approximate values in Section 4.1 suggests $\alpha = \psi$, which means that the pentagon is obtained by glueing two triangles ACE and BCD together, and the third triangle ABC is reduced to an arc. The situation is described in Figure 14.

To prove the configuration in Figure 14, we reconstruct the pentagon by starting with the exact values of γ, δ, ϵ, and the assumption $AE = AC = BD = CD = a$. The goal is to verify that $AC = BC + a$, so that glueing the two isosceles triangles ACE and BCD gives a pentagon with equal sides.

We have

$$\tan\phi = \sec a \cot\frac{\delta}{2}, \quad \tan\psi = \sec a \cot\frac{\epsilon}{2}.$$

Then $\phi + \psi = \gamma = \frac{1}{2}\pi$ implies that

$$\sec a \cot\frac{\delta}{2} \cdot \sec a \cot\frac{\epsilon}{2} = 1.$$

Therefore

$$\cos a = \sqrt{-3 + 2\sqrt{3}}, \quad \sin a = \sqrt{1 - (-3 + 2\sqrt{3})} = -1 + \sqrt{3}.$$

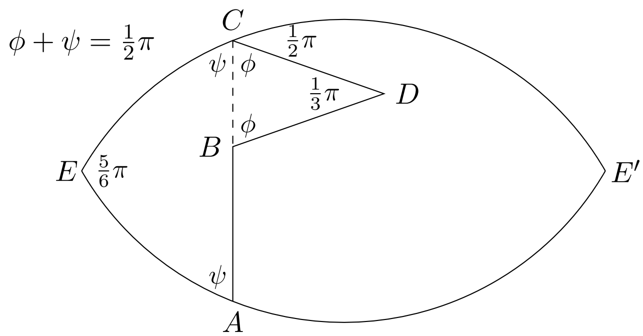

Figure 14: Pentagon for $\{[\alpha, \beta, \delta, \gamma, \epsilon]\colon \alpha\beta\gamma, \delta\epsilon^2, \gamma\delta^2\epsilon$ or γ^4 or $\gamma^2\delta^3\}$.

Here we choose $\cos a$ to be positive because the approximate value $\psi = \alpha = 0.1192\pi$ implies $\tan\psi > 0$. Combined with $\cot\frac{\epsilon}{2} > 0$, we get $\sec a > 0$. The approximate value of a is

$$a = \arccos\sqrt{-3 + 2\sqrt{3}} = 0.26143665075066716501668366 30\pi.$$

Now we have

$$\begin{aligned}
\cos AC &= \cos^2 a + \sin^2 a \cos\frac{5}{6}\pi \\
&= (-3 + 2\sqrt{3}) - (4 - 2\sqrt{3})\frac{\sqrt{3}}{2} = 0, \\
\cos BC &= \cos^2 a + \sin^2 a \cos\frac{1}{3}\pi \\
&= (-3 + 2\sqrt{3}) + (4 - 2\sqrt{3})\frac{1}{2} = -1 + \sqrt{3} = \sin a.
\end{aligned}$$

The first equality implies $AC = \frac{1}{2}\pi$. The second equality implies $\cos BC > 0$, so that $0 < BC < \frac{1}{2}\pi$. Since we also have $0 < a < \frac{1}{2}\pi$, the second equality above implies $BC + a = \frac{1}{2}\pi = AC$.

The shape of the pentagon and the exact value of a imply the exact values of α and β

$$\begin{aligned}
\alpha &= \psi = \frac{1}{2}\pi - \phi = \frac{1}{2}\pi - \arctan\sqrt{3 + 2\sqrt{3}}, \\
\beta &= \pi + \phi = \pi + \arctan\sqrt{3 + 2\sqrt{3}}.
\end{aligned}$$

Finally, we remark that the second solution with $\alpha = \pi - \psi$ (from the common zero of our resultants) actually gives the complementary pentagon $ABDCE'$ in the ϵ-part of the sphere (the 2-gon of angle $\epsilon = \frac{5}{6}\pi$).

4.4 Cases 1.4f, 1.5b, 2.5f and 2.6b

For Cases 1.4f and 2.6b, we have the following solutions from the common zeros of four resultants

$$\begin{aligned} (\alpha,\beta) &= (0.93\pi, 0.72\pi), (0.6055\pi, 0.5024\pi), && \text{(for } B1) \\ & \quad (0.84\pi, 0.62\pi), (0.3095\pi, 1.0615\pi), && \text{(for } B3) \\ (\delta,\epsilon) &= \left(\tfrac{2}{5}\pi, \tfrac{4}{5}\pi\right), \quad f = 20. \end{aligned}$$

For Case 1.5b, we have

$$\begin{aligned} (\alpha,\beta) &= (0.84\pi, 0.69\pi), (0.6338\pi, 0.5642\pi), (0.10133\pi, 1.56723\pi), && \text{(for } B1) \\ & \quad (0.78\pi, 0.64\pi), (0.4536\pi, 0.8823\pi), && \text{(for } B3) \\ (\delta,\epsilon) &= \left(\tfrac{1}{2}\pi, \tfrac{3}{4}\pi\right), \quad f = 16. \end{aligned}$$

For Case 2.5f, we have

$$\begin{aligned} (\alpha,\beta) &= (0.99\pi, 0.78\pi), (0.5588\pi, 0.4371\pi), && \text{(for } B1) \\ & \quad (0.90\pi, 0.61\pi), && \text{(for } B3) \\ (\delta,\epsilon) &= \left(\tfrac{2}{7}\pi, \tfrac{6}{7}\pi\right), \quad f = 28. \end{aligned}$$

In all cases, the first and third solutions violate [2, Lemma 3]. For the remaining solutions, we use the method of Section 3.3 to find all the possible angle combinations at all the vertices

$$\begin{aligned} \text{AVC} &= \{\alpha\beta\gamma, \delta\epsilon^2, \delta^3\epsilon, \delta^5\}, && \text{(for Cases 1.4f and 2.6b)} \\ \text{AVC} &= \{\alpha\beta\gamma, \delta\epsilon^2, \delta^4\}, && \text{(for Case 1.5b)} \\ \text{AVC} &= \{\alpha\beta\gamma, \delta\epsilon^2, \delta^4\epsilon, \delta^7\}. && \text{(for Case 2.5f)} \end{aligned}$$

We first prove that the AVCs above do not admit tilings with arrangement $B1$. The left of Figure 15 shows what happens at a vertex of degree >3, which means $\delta^3\epsilon, \delta^4\epsilon, \delta^4, \delta^5$ or δ^7 in our AVCs. We always have three consecutive δ at the vertex, and we may assume that the angles of P_1 are arranged as indicated. By AVC, $\gamma\epsilon\cdots$ is not a vertex, so that the angle γ of P_2 adjacent to δ must be located as indicated. This determines all the angles of P_2. Then by AVC, the vertex $\epsilon^2\cdots$ shared by P_1, P_2 is $\delta\epsilon^2$. So we get a tile P_3 outside P_1, P_2, together with the location of the angle δ of P_3. Since the angle ϵ in P_3 is adjacent to δ, it is located at one of the ? marks. This gives a vertex $\alpha\epsilon\cdots$, contradicting to the AVC. This proves that the arrangement $B1$ does not admit a tiling.

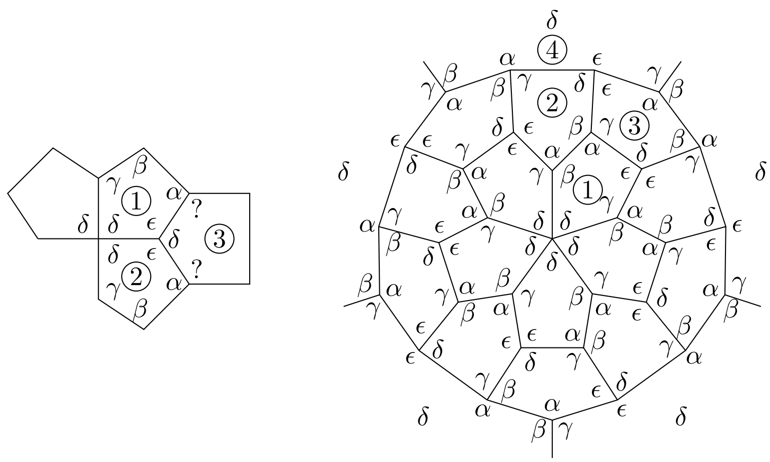

Figure 15: Tiling for $\{[\alpha,\beta,\delta,\gamma,\epsilon]\colon \alpha\beta\gamma, \delta\epsilon^2, \delta^3\epsilon, \delta^5\}$, actually Case 2.6b.

Next we consider the arrangement $B3$, which means the fourth solutions of Cases 1.4f, 2.6b and 1.5b. First assume δ^5 is a vertex, for the Cases 1.4f and 2.6b. On the right of Figure 15, we start from such a vertex at the center. We may assume that the angles in P_1 are arranged as indicated. By AVC, we may determine all the angles in the other tiles at the vertex δ^5. Then we use the AVC to further determine the five tiles similar to P_2 and all their angles. Next we determine the five tiles similar to P_3 and all their angles. Finally we determine the five tiles similar to P_4 and all their angles. The result is the earth map tiling of distance 5. Since $\delta^3\epsilon$ is not a vertex, the tiling is really for Case 2.6b.

The same argument shows that the tiling for the fourth solution of Case 1.5b is also the earth map tiling of distance 5.

It remains to consider the arrangement $B3$ with the additional assumption that δ^5 is not a vertex, and $\delta^3\epsilon$ is a vertex. This means Case 1.4f. We will show that the tiling is given by Figure 16. The tiling has two tiles with two degree 4 vertices, which are drawn as the north and south "regions" (as opposed to the two poles in the earth map tiling) P_1, P_{10}. The tiling is obtained by glueing the left and right together.

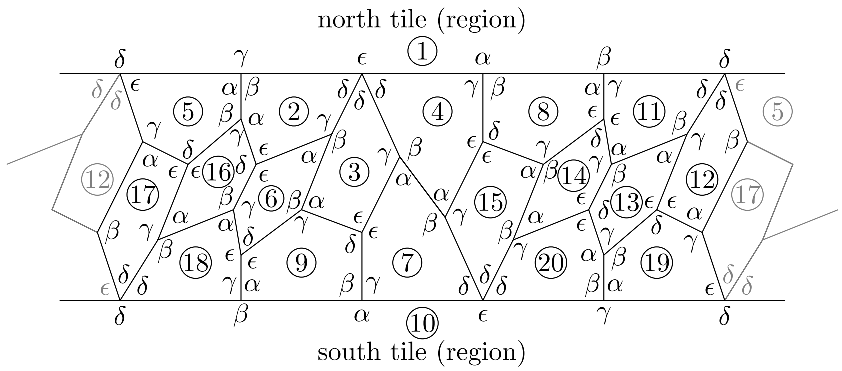

Figure 16: Tiling for $\{[\alpha, \beta, \delta, \gamma, \epsilon] \colon \alpha\beta\gamma, \delta\epsilon^2, \delta^3\epsilon\}$, actually Case 1.4f.

To argue for the tiling in Figure 16, we start with a vertex $\delta^3\epsilon$ at the ϵ-angle of a tile P_1. At this moment, we do not yet know the configurations at the other four vertices of P_1. By AVC, we may determine all the angles in the other three tiles P_2, P_3, P_4 at the vertex. Then we may further determine the tiles P_5, P_6, P_7, P_8 and all their angles. By AVC, the vertex $\epsilon^2 \cdots$ shared by P_3, P_7 is $\delta\epsilon^2$. Then we may determine tiles P_9, P_{10} and all their angles.

The vertex $\delta\epsilon \cdots$ shared by P_4, P_8 can be either $\delta\epsilon^2$ or $\delta^3\epsilon$. Suppose the vertex is $\delta\epsilon^2$, then we may successively determine $P_{15}, P_{14}, P_{20}, P_{13}, P_{11}$ and find that the δ-vertex of P_1 is $\delta^2\epsilon \cdots$, which by AVC is $\delta^3\epsilon$. Note that we used $\delta^2\epsilon \cdots = \delta^3\epsilon$ in determining P_{20} and used $\epsilon^2 \cdots = \delta\epsilon^2$ in determining P_{13}.

What we have proved can be interpreted as follows. If P_4 has only one degree 4 vertex (which implies that the vertex $\delta\epsilon \cdots$ shared by P_4, P_8 is $\delta\epsilon^2$), then P_1 has two degree 4 vertices (which must be the δ-vertex and the ϵ-vertex). Therefore we have proved that there must be at least one tile with two degree 4 vertices.

So without loss of generality, we may assume that the tile P_1 we started with has two degree 4 vertices. Then we repeat the argument and determine the tiles $P_1, \dots, P_{10}$ and all their angles. With the knowledge that the vertex $\delta\epsilon \cdots$ shared by P_1, P_5 is $\delta^3\epsilon$, we may further determine the tiles $P_{11}, \dots, P_{20}$

and all their angles as indicated by Figure 16.

Finally, we need to verify the existence of the pentagon. We may compute the edge lengths and angles as in Section 3.4 and verify the inequalities needed for the existence. For the fourth solution of Cases 1.4f and 2.6b, we have

$$\begin{aligned} a &= 0.21683706135091000335136566 1654\pi, \\ \alpha &= 0.30959211826772392541573224 7869\pi, \\ \beta &= 1.06152432808957675934745630 289\pi, \\ \gamma &= 0.62888355364269931523681144 9235\pi, \end{aligned}$$

and

$$\phi = 0.336\pi, \quad \psi = 0.126\pi, \quad x = 0.241\pi, \quad y = 0.408\pi.$$

The inequalities for the existence can be verified and we get the pentagon on the left of Figure 17. For the fifth solution of Case 1.5b, we have

$$\begin{aligned} a &= 0.21550569507830775211792346 1726\pi, \\ \alpha &= 0.45368481897671186294479110 5935\pi, \\ \beta &= 0.88238808379725439682846672 428\pi, \\ \gamma &= 0.66392709722603374022674216 977\pi, \end{aligned}$$

and

$$\phi = 0.289\pi, \quad \psi = 0.155\pi, \quad x = 0.292\pi, \quad y = 0.392\pi.$$

Again the pentagon exists and is depicted on the right of Figure 17.

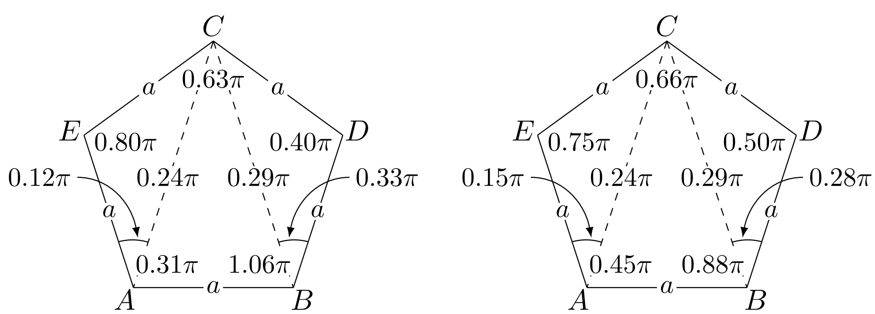

Figure 17: Pentagon for $\{[\alpha,\beta,\delta,\gamma,\epsilon]\colon \alpha\beta\gamma, \delta\epsilon^2, \delta^3\epsilon \text{ or } \delta^5 \text{ or } \delta^4\}$.

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