NOTES

Rearranging Terms in Alternating Series

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It is well known that if you change the order of the terms in a conditionally convergent series, the result may have a different sum. We will demonstrate a method for calculating the sum of a broad class of such reordered series. The result is not new; it originally appeared in a 19th century German paper [**8**], and similar work appeared later in [**2**], [**4**], and [**6**]. The purpose of this note is to bring attention to the result, which could appropriately be mentioned in introductory calculus classes right after Riemann's theorem [**1**]; the main result uses nothing more complicated than the integral test, and the discussion uses nothing more complicated than the concept of $o(\frac{1}{x})$, which could be explained (see [**5**]) or circumnavigated in lecture. The results from [**3**] on the alternating harmonic series could well be included in the lecture.

Let $S=\sum_{j=1}^{\infty}(-1)^{j+1}a_j$, with $a_j>0$ and $\{a_j\}$ eventually monotonically decreasing to zero. Define S^{mn} as the rearrangement (no signs changed) obtained from S by taking groups of m positive terms followed by groups of n negative terms (thus $S^{11}=S$ and $S^{21}=a_1+a_3-a_2+a_5+a_7-a_4+\cdots$).

THEOREM. *Let f be a continuous and eventually monotone function such that $f(2k-1)=a_{2k-1}$ for positive integral k, and let $m\geqslant n>0$. Then*

$$S^{mn}=S+\frac{1}{2}\lim_{k\to\infty}\int_{nk}^{mk}f(x)\,dx,$$

with the understanding that both sides may be infinite or may diverge by oscillation.

Proof. Since $a_k\to 0$, S^{mn} is the limit of its $(m+n)k$th partial sum. Identically,

$$S^{mn}_{(m+n)k}=S_{2nk}+\sum_{j=nk+1}^{mk}a_{2j-1}. \tag{1}$$

Since f is eventually monotone decreasing to zero, the limit

$$\lim_{t\to\infty}\left(\sum_{j=1}^{t}f(2j-1)-\int_1^t f(2x-1)\,dx\right)$$

exists; this follows from the standard proof of the integral test for convergence. It follows that the tail expression goes to 0; therefore,

$$\lim_{k\to\infty}\left(\sum_{j=nk+1}^{mk}a_{2j-1}-\int_{nk}^{mk}f(2x-1)\,dx\right)=0.$$

Therefore,

$$\lim_{k\to\infty}\sum_{j=nk+1}^{mk}a_{2j-1}=\lim_{k\to\infty}\int_{nk}^{mk}f(2x-1)\,dx$$

$$= \lim_{k\to\infty} \frac{1}{2}\int_{2nk-1}^{2mk-1} f(x)\,dx$$

$$= \frac{1}{2}\lim_{k\to\infty}\int_{nk}^{mk} f(x)\,dx, \tag{2}$$

again with the understanding that both limits may be infinite or may diverge by oscillation. Let $k \to \infty$ in (1) and use (2) to finish the proof.

REMARK. If $\{a_j\}$ is absolutely convergent, then we get the expected result that $S^{mn} = S$. If $m = n$ we also find, as expected, that $S^{mn} = S$. (Several less obvious sum-preserving rearrangements of infinite series are discussed in [7].)

Several examples will illustrate the diverse results that can be obtained by applying our Theorem to rearrangements S^{mn} of particular alternating series.

EXAMPLE 1. If H^{mn} is obtained from the alternating harmonic series $(1 - 1/2 + 1/3 - 1/4 + \cdots)$ by taking groups of m positive terms and n negative terms, then

$$H^{mn} = \log 2 + \frac{1}{2}\lim_{k\to\infty}\int_{nk}^{mk}\frac{1}{x}\,dx = \log 2 + \frac{1}{2}\log\frac{m}{n} = \frac{1}{2}\log\frac{4m}{n}.$$

EXAMPLE 2. If P^{mn} is obtained similarly from the alternating series $1 - 1/2^p + 1/3^p - 1/4^p + \cdots$, with $0 < p < 1$ and $m > n$, then P^{mn} diverges to positive infinity.

EXAMPLE 3. Let L be Leibniz's sum for $\pi/4$: $L = 1 - 1/3 + 1/5 - 1/7 + \cdots$. Then

$$L^{mn} = \frac{\pi}{4} + \frac{1}{2}\lim_{k\to\infty}\int_{nk}^{mk}(1/(2x-1))\,dx = \frac{\pi}{4} + \frac{1}{4}\log\frac{m}{n}.$$

EXAMPLE 4. Let $S = \sum_{j=1}^{\infty}(-1)^{j+1}/j\log j$. Then

$$S^{mn} = S + \frac{1}{2}\lim_{k\to\infty}(\log\log mk - \log\log nk) = S.$$

EXAMPLE 5. Let $S = \sum_{j=1}^{\infty}(-1)^{j+1}f(j)$, where $f(x) = (\cos(\log x) + 2)/x$. It is easy to check that f is monotone.

$$S^{mn} = S + \tfrac{1}{2}\lim_{k\to\infty}(\sin(\log mk) + 2\log mk - \sin(\log nk) - 2\log nk),$$

which diverges by oscillation unless $m = n$.

The curious reader may be wondering if we can find an example for which S^{mn} converges to a value differing from S by something other than a logarithmic term. Barring "artificial" functions for which $\lim_{x\to\infty} xf(x)$ diverges by oscillation, there is none.

Proof. If $f(x) = o(1/x)$, then

$$\lim_{k\to\infty}\int_{nk}^{mk} f(x)\,dx \leqslant \lim_{k\to\infty}(mk - nk)f(nk), \text{ since } f \text{ is monotone decreasing,}$$

$$= \lim_{k\to\infty}\frac{m-n}{n}nkf(nk),$$

$$= 0.$$

Therefore $S^{mn} = S$. If $f(x) = c/x + o(1/x)$, then

$$\lim_{k\to\infty}\int_{nk}^{mk} f(x)\,dx = \lim_{k\to\infty}\int_{nk}^{mk}\frac{c}{x} + o\left(\frac{1}{x}\right)dx$$

$$= \lim_{k\to\infty}\int_{nk}^{mk}\frac{c}{x}\,dx, \text{ by the preceding result,}$$

$$= c\log\frac{m}{n},$$

so $S^{mn} = S + \frac{1}{2}c\log(m/n)$. The reader may check that S^{mn} diverges to positive infinity if $f(x) > O(1/x)$.

The remaining possibility, that $xf(x)$ diverges by oscillation, seems to merit attention. Suppose we modify Example 5 slightly by letting $f(x) = (\cos(\alpha\log x) + 2)/x$, where $\alpha = 2\pi/\log(m/n)$. Then

$$\alpha\log mk - \alpha\log nk = \alpha\log\frac{m}{n} = 2\pi, \text{ so } \sin(\alpha\log mk) - \sin(\alpha\log nk) = 0.$$

Hence

$$S^{mn} = S + \frac{1}{2}\lim_{k\to\infty}\left(\sin(\alpha\log mk) + 2\log mk - \sin(\alpha\log nk) - 2\log nk\right) = S + \log\frac{m}{n}.$$

Once again the difference is proportional to $\log(m/n)$. Also, notice that S^{pq} converges if and only if $\alpha\log(p/q)$ is an integral multiple of 2π, or equivalently if and only if p/q is an integral power of m/n. It would be interesting to find answers to the following two questions:

Is there a series S for which $S^{mn} - S$ converges for some values of m and n, but is not proportional to $\log(m/n)$?

Is there a series $S = \sum_{j=1}^{\infty}(-1)^{j+1}f(j)$ such that $xf(x)$ diverges by oscillation but $S^{mn} - S$ converges for all values of m and n?

EXAMPLE 6. Suppose $S = \sum_{j=1}^{\infty}(-1)^{j+1}\sin(1/j)$. Naturally we take $f(x)$ to be $\sin(1/x)$, which is $1/x + o(1/x)$. Thus we find that

$$S^{mn} = S + \frac{1}{2}\log\frac{m}{n}\lim_{x\to\infty} x\sin\frac{1}{x} = S + \frac{1}{2}\log\frac{m}{n}.$$

Getting back to our theorem, we should note that the assumption that $m \geq n$ is unnecessary—to see this, just identify $\sum_{j=nk+1}^{mk} a_{2j-1}$ with $-\sum_{j=mk+1}^{nk} a_{2j-1}$. In fact, a slight modification (left to the reader) of our proof yields a similar result for any rearrangement of S that leaves the order of its positive and negative subsequences intact; if S' is such a rearrangement of S with $\phi(k)$ negative terms in S'_k, and f satisfies our old hypothesis, then

$$S' = S + \frac{1}{2}\lim_{k\to\infty}\int_{\phi(k)}^{k-\phi(k)} f(x)\,dx.$$

Applying this result to the series $S = 1 - 2^{-1/2} + 3^{-1/2} - 4^{-1/2} + \cdots$, and taking $\phi(k)$ to be the integral part of $((l - \sqrt{2k - l^2})/2)^2$ for $k > \frac{1}{2}l^2$ (ϕ can be arbitrary elsewhere), a simple calculation shows that the rearranged series sums to $S + l$.

For another application of this result, choose the reordering, H', of the alternating harmonic series for which $\phi(k)$ equals the integral part of $k/(\frac{1}{4}e^{2\pi} + 1)$. Then $H' = \pi$.

I wish to express my thanks to all those who read over and commented on this manuscript, especially Alan Siegel.

References

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