Tilings of the Sphere by Geometrically Congruent Pentagons I

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Abstract

There are no edge-to-edge tilings of the sphere by more than 12 congruent pentagons, such that there is a tile with all vertices having degree 3 and the edge length combinations are a^2b^2c, a^3bc, or a^3b^2, with a,b,c distinct.

1 Introduction

Mathematicians have studied tilings for more than 100 years. A lot is known about tilings of the plane or the Euclidean space. However, results about tilings of the sphere are relatively rare. A major achievement in this regard is the complete classification of edge-to-edge tilings of the sphere by congruent triangles [5, 6]. For tilings of the sphere by congruent pentagons, we completely classified the minimal case of 12 tiles [1, 2]. This paper is the first of a series that classifies beyond the minimal case.

The spherical tilings should be easier to study than the planar tilings, simply because the former involves only finitely many tiles. The classifications in [2, 6] not only give the complete list of tiles, but also the ways the tiles are fit together. It is not surprising that such kind of classifications for the planer tilings are only possible under various symmetry conditions, because the quotients of the plane by the symmetries often become compact.

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Like the earlier works, we restrict ourselves to edge-to-edge tilings of the sphere by congruent polygons, such that all vertices have degree ≥ 3. These are mild and natural assumptions that simplify the discussion. The polygon in such a tiling must be triangle, quadrilateral, or pentagon [7]. We believe that the pentagonal tilings should be relatively easier to study than the quadrilateral ones because 5 is an "extreme" among 3, 4, 5.

Our classification program starts with Proposition 1, which says that a spherical pentagonal tiling must have a tile, such that four vertices have degree 3, and the fifth vertex has degree 3, 4 or 5. In case all five vertices have degree 3, [2, Proposition 8] shows that there are five possible ways the edge lengths of the pentagon can be arranged: $a^5, a^4b, a^2b^2c, a^3bc, a^3b^2$. This paper shows that the last three cases are impossible beyond the minimal case of 12 tiles classified by [2].

Theorem. *If a spherical tiling by more than* 12 *geometrically congruent pentagons has edge length combination a^2b^2c, a^3bc, or a^3b^2, with a, b, c distinct, then every tile has at least one vertex of degree > 3.*

The proof for the edge length combinations a^2b^2c and a^3bc takes only one page and can be found after Proposition 1. The rest of the paper deals with the case of a^3b^2, which turns out to be much more complicated. In Proposition 2, we find that there are four possible tilings of the neighborhood of the tile with all vertices having degree 3. The proposition also provides some information about the angles, from which we can extract all the possible angle combinations at vertices (which we call anglewise vertex combination, or AVC). Then we study the vertices at the boundary of the neighborhood. We always arrive at contradictions at the end, either due to the wrong configuration of edges at the vertices, or due to the contradiction of the pentagonal tiles with the spherical geometry. Therefore the case of a^3b^2 is also impossible.

Beyond the cases discussed in this paper, the following cases are yet to be classified:

1. Every tile has at least one vertex of degree ≥ 4, and there is a tile such that four vertices have degree 3, and the fifth vertex has degree 4 or 5.

2. Most edges have the same length, such as the edge length combinations a^5 or a^4b, even in case there is a tile such that all vertices have degree 3.

We expect that the technique in this paper can be applied as long as there is enough variation in edge lengths. For the extreme case that all edges have the same length (i.e., edge length combination a^5), however, completely new idea is needed. It turns out that in this case, the pentagon has 3 degrees of freedom. On the other hand, our work on the angle relations in pentagonal tilings [3, 4] shows that in almost all cases, we have three independent linear relations among angles. Therefore the possible pentagons can be completely determined, and then finding tilings for the specific pentagons should not be difficult. For other cases such as a^4b, we expect to combine this idea with the technique of this paper. At the end, we are fairly optimistic that the complete classification of the pentagonal tilings of the sphere can be achieved.

2 Neighborhood Tiling

We review some known combinatorial results about edge-to-edge pentagonal tilings of the sphere, such that all vertices have degree ≥ 3.

Let v_k be the number of vertices of degree k. Let f be the number of tiles. It is easy to show the following *vertex counting equation* (see [2, page 750], for example):

$$\frac{f}{2} - 6 = \sum_{k\geq 4}(k-3)v_k = v_4 + 2v_5 + 3v_6 + \cdots . \tag{2.1}$$

This implies that f is an even integer ≥ 12. Since the tilings for the case $f = 12$ are completely classified by [2], we will assume $f > 12$.

Proposition 1. *In any pentagonal spherical tiling, there must be a tile with four vertices of degree* 3 *and a fifth vertex of degree* 3, 4 *or* 5.

Proof. If a tiling does not have the said property, then any tile either has at least one vertex of degree ≥ 6, or has at least two vertices of degree 4 or 5. Since a degree k vertex is shared by at most k tiles, the number of tiles of first kind is $\leq 6v_6 + 7v_7 + 8v_8 + \cdots$, and the number of tiles of the second kind is $\leq \frac{1}{2}(4v_4 + 5v_5)$. Then we get

$$f \leq \frac{1}{2}(4v_4 + 5v_5) + 6v_6 + 7v_7 + 8v_8 + \cdots .$$

This contradicts the vertex counting equation (2.1). □

In this paper, we restrict to the case that there is a tile with all vertices having degree 3. By [2, Proposition 8], the tile must have the edge lengths arranged in one of the five ways in Figure 1. We further restrict to the last three ways a^3bc, a^2b^2c, a^3b^2 in this paper.

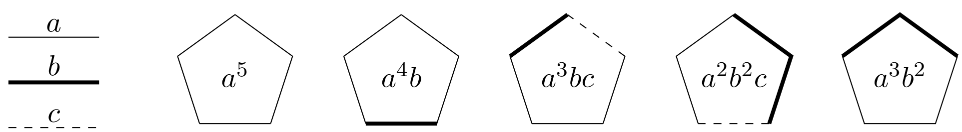

Figure 1: Edges in a tile with all vertices having degree 3.

For the angles, we note that, if all tiles are geometrically congruent, then the area of each tile is $\frac{4\pi}{f}$, so that the sum of five angles in the pentagon is $3\pi + \frac{4\pi}{f}$. This is the *angle sum equation for the pentagon.* In fact, by [4, Lemma 2], the angle sum equation for the pentagon remains true as long as the tiles are angle congruent.

We also know that the sum of all angles at a vertex is 2π. This is the *angle sum equation for the vertex.*

The preliminary knowledge we have so far about the spherical pentagonal tilings is enough for proving the first two cases of the main theorem.

Proof of the main theorem for the edge length combination a^2b^2c. If there is a tile with all vertices having degree 3, then its neighborhood is combinatorially given by the left of Figure 2. We denote the tile by P_1 and its five neighboring tiles by $P_2, \dots, P_6$.

Up to the combinatorial symmetry of the neighborhood, we may assume that the edges of P_1 are given by the left and the middle of Figure 2. The left describes the case that the edge shared by P_2, P_3 is a. We may successively determine all the edges of P_2, P_6, P_5, P_4, and then find three a-edges in P_3, a contradiction. Similar argument shows that P_2, P_3 cannot share a b-edge. Therefore the edge shared by P_2, P_3 must be c, as described in the middle of Figure 2. Then we may determine all the edges of P_2, P_3. Denote the unique a^2-angle, b^2-angle, ab-angle, ac-angle, bc-angle in the tile by $\alpha, \beta, \gamma, \delta, \epsilon$. Then we get all the indicated angles in the middle of Figure 2. The angle sums at three vertices give

$$3\alpha = 3\beta = \gamma + \delta + \epsilon = 2\pi.$$

This implies

$$\alpha + \beta + \gamma + \delta + \epsilon = \frac{2\pi}{3} + \frac{2\pi}{3} + 2\pi = 3\pi + \frac{4\pi}{12}.$$

By the angle sum equation for the pentagon, we conclude $f = 12$. $\square$

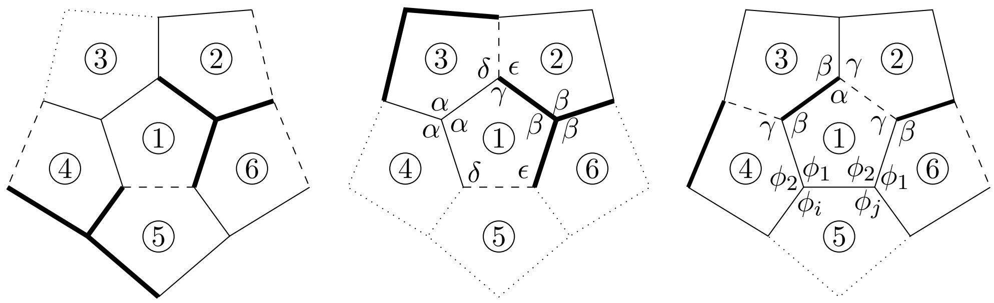

Figure 2: Geometrically congruent neighborhood tilings for a^2b^2c and a^3bc.

Proof of the main theorem for the edge length combination a^3bc. Up to symmetry, we may assume that the edges of P_1 are given by the right of Figure 2. Since each tile has only one b-edge and one c-edge, the edge shared by P_2, P_3 must be a. This determines all the edges of P_2, P_3. Then we may further determine all the edges of P_4, P_6. Denote the unique bc-angle, ab-angle, ac-angle by α, β, γ. Moreover, denote the a^2-angle adjacent to β by ϕ_1, and denote the a^2-angle adjacent to γ by ϕ_2. Then we get all the indicated angles on the right of Figure 2. The angle sums at three vertices give

$$\alpha + \beta + \gamma = \phi_1 + \phi_2 + \phi_i = \phi_1 + \phi_2 + \phi_j = 2\pi.$$

This implies $\phi_i = \phi_j$, which actually means $\phi_1 = \phi_2$. Then $\phi_1 + \phi_2 + \phi_i = 2\pi$ further implies $\phi_1 = \phi_2 = \frac{2\pi}{3}$, so that

$$\alpha + \beta + \gamma + \phi_1 + \phi_2 = 2\pi + \frac{2\pi}{3} + \frac{2\pi}{3} = 3\pi + \frac{4\pi}{12}.$$

By the angle sum equation for the pentagon, we conclude $f = 12$. $\square$

Proposition 2. *If a spherical tiling by more than* 12 *geometrically congruent pentagons has edge length combination a^3b^2, with a, b distinct, then up to symmetry, the neighborhood of a tile with all vertices having degree* 3 *has four possible geometrically congruent tilings given in Figure 3.*

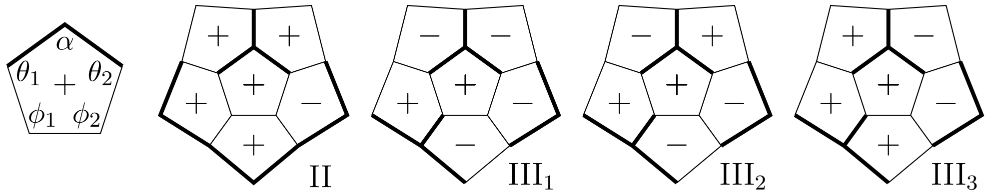

Figure 3: Geometrically congruent neighborhood tilings for a^3b^2.

Here is the way to read the tilings in Figure 3. The five angles of the pentagon are indicated on the left. In each tile, the two thick b-edges determine the location of the five angles up to the horizontal flipping. The tiles labeled + (considered as positively oriented) have the angles arranged the same as the pentagon on the left, and the tiles labeled − (considered as negatively oriented) have the angles arranged as the horizontal flipping of the pentagon on the left.

The four tilings in Figure 3 are the middle tiling in Figure 5, two tilings in Figure 6, and the left tiling in Figure 7 (with θ_i, ϕ_i abbreviated as i). We also note that the proof of the proposition computes the values of the angles, give in Table 1.

tiling	α	θ_1	θ_2	ϕ_1	ϕ_2
II	$\frac{2}{3}\pi$	$\theta_1+\theta_2=\left(\frac{2}{3}+\frac{8}{f}\right)\pi$		$\left(\frac{1}{3}+\frac{4}{f}\right)\pi$	$\left(\frac{4}{3}-\frac{8}{f}\right)\pi$
III_1	$\frac{2}{3}\pi$	$\left(\frac{1}{3}+\frac{4}{f}\right)\pi$	$\left(\frac{4}{3}-\frac{8}{f}\right)\pi$	$\left(\frac{4}{3}-\frac{8}{f}\right)\pi$	$\left(-\frac{2}{3}+\frac{16}{f}\right)\pi$
III_2	$\frac{2}{3}\pi$	$\left(\frac{5}{6}-\frac{2}{f}\right)\pi$	$\left(-\frac{1}{6}+\frac{10}{f}\right)\pi$	$\left(\frac{1}{3}+\frac{4}{f}\right)\pi$	$\left(\frac{4}{3}-\frac{8}{f}\right)\pi$
III_3	$\frac{2}{3}\pi$	$\left(-\frac{1}{6}+\frac{10}{f}\right)\pi$	$\left(\frac{5}{6}-\frac{2}{f}\right)\pi$	$\left(\frac{4}{3}-\frac{8}{f}\right)\pi$	$\left(\frac{1}{3}+\frac{4}{f}\right)\pi$

Table 1: Angles in geometrically congruent neighborhood tilings for a^3b^2.

The proof of Proposition 2 makes use of the following constraint from the spherical geometry. The constraint is a stronger version of [2, Lemma 21] and will be proved in Section 3. We note that the boundary of a spherical pentagon is always assumed to be a simple closed curve, as the pentagon in a geometrically congruent tiling should be.

Lemma 3. *If the spherical pentagon in Figure 4 has two pairs of equal edges a and b, then $\beta > \gamma$ is equivalent to $\delta < \epsilon$.*

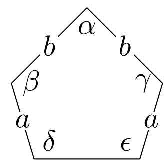

Figure 4: Geometrical constraint for pentagon.

Proof of Proposition 2. Up to the combinatorial symmetry of the neighborhood, we may assume that the edges of P_1 are given by Figure 5. We ignore the marks for the angles for the moment, and concentrate on the edge lengths. If the edge shared by P_2, P_3 is a, then we may successively determine all the edges of P_2, P_3, P_4, P_6, P_5 and get the type I tiling on the left of Figure 5. If the edge shared by P_2, P_3 is b, then we may determine all the edges of P_2, P_3. Then up to the horizontal flipping, there are two ways of arranging the edges of P_5. Each way then further determines all the edges of P_4, P_6. The two ways give the type II tiling in the middle and the type III tiling on the right.

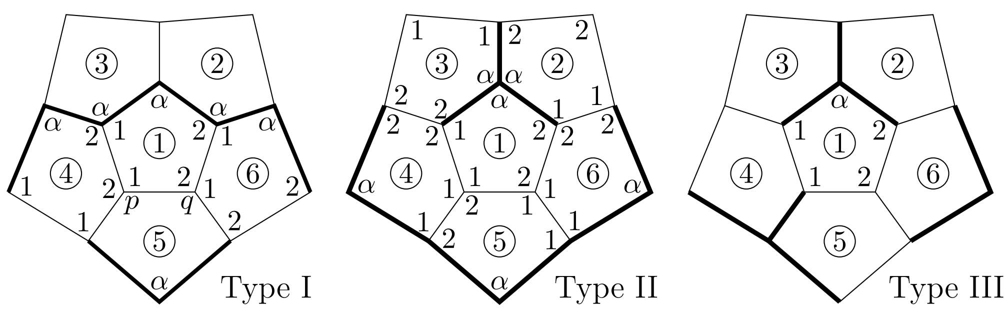

Figure 5: Three types of edge congruent neighborhood tilings for a^3b^2.

Now we consider the angles. We first prove that $\theta_1 \neq \theta_2$ and $\phi_1 \neq \phi_2$. By Lemma 3 (or [2, Lemma 21]), we know that $\theta_1 \neq \theta_2$ is equivalent to $\phi_1 \neq \phi_2$. Therefore we only need to prove that $\theta_1 = \theta_2$ and $\phi_1 = \phi_2$ imply $f = 12$. The type I neighborhood tiling has a vertex with three a-edges and a vertex with one a-edge and two b-edges. The angle sums at the vertices give

$$\phi_* + \phi_* + \phi_* = \alpha + \theta_* + \theta_* = 2\pi.$$

If $\theta_1 = \theta_2$ and $\phi_1 = \phi_2$, then $\phi_1 = \phi_2 = \frac{2\pi}{3}$ and $\alpha + \theta_1 + \theta_2 = 2\pi$. By the angle sum equation for the pentagon, this implies $f = 12$. Similarly, the angles in type II and III neighborhood tilings satisfy

$$3\alpha = \phi_* + \phi_* + \phi_* = \theta_* + \theta_* + \phi_* = 2\pi.$$

If $\theta_1 = \theta_2$ and $\phi_1 = \phi_2$, then all angles are $\frac{2\pi}{3}$, and the angle sum equation for the pentagon also implies $f = 12$.

We denote by V_{ijk} the vertex shared by P_i, P_j, P_k, and denote by $A_{i,jk}$ the angle of P_i at V_{ijk}. Up to the symmetry of exchanging the subscripts 1 and 2, we may assume that the angles of P_1 are arranged as in Figure 5. This is the starting point of further argument about angles.

To make the pictures more legible, we will also indicate θ_1, ϕ_1 by [1] and indicate θ_2, ϕ_2 by [2]. Since $\theta_1 \neq \theta_2$ and $\phi_1 \neq \phi_2$, this will not cause ambiguities.

Type I Neighborhood

The neighborhood is the left of Figure 5, with all the edges of the tiling and all the angles of P_1 given. We have $A_{4,13} = \theta_i$ and $A_{6,12} = \theta_j$. The angle sums at the vertices V_{134} and V_{126} give $\alpha + \theta_1 + \theta_i = \alpha + \theta_2 + \theta_j$. By $\theta_1 \neq \theta_2$, we must have $i = 2$ and $j = 1$. This determines all the angles of P_4, P_6. The angle sums at V_{145}, V_{156} give $\phi_1 + \phi_2 + \phi_p = \phi_1 + \phi_2 + \phi_q$. This implies $\phi_p = \phi_q$, which is $\phi_1 = \phi_2$, a contradiction. We conclude that there is no geometrically congruent tiling of type I.

Type II Neighborhood

The neighborhood is the middle of Figure 5, with all the edges of the tiling and all the angles of P_1 given. We either have $A_{5,14} = \phi_1, A_{5,16} = \phi_2$, or have $A_{5,14} = \phi_2, A_{5,16} = \phi_1$.

If $A_{5,14} = \phi_1, A_{5,16} = \phi_2$, then the angle sums at V_{145}, V_{156} give $2\phi_1 + \phi_* = 2\phi_2 + \phi_* = 2\pi$. This implies that $\phi_1 = \phi_2$ no matter what the two ϕ_* are, a contradiction.

So we must have $A_{5,14} = \phi_2, A_{5,16} = \phi_1$. By comparing the angle sums at V_{145}, V_{156}, we get $A_{4,15} = A_{6,15}$. Up to symmetry (exchanging the subscripts 1 and 2, followed by the horizontal flipping), we may assume that $A_{4,15} = A_{6,15} = \phi_1$. This determines all the angles of P_4, P_5, P_6. By $\theta_1 \neq \theta_2$ and the angle sums at V_{126}, V_{134}, we get $A_{2,16} = \theta_1, A_{3,14} = \theta_2$ and subsequently all the angles of P_2, P_3. At the end, we get the geometrically congruent tiling of type II in Figure 3.

The angle sums at $V_{123}, V_{134}, V_{145}$ give

$$3\alpha = \theta_1 + \theta_2 + \phi_2 = 2\phi_1 + \phi_2 = 2\pi.$$

Together with the angle sums equation for the pentagon

$$\alpha + \theta_1 + \theta_2 + \phi_1 + \phi_2 = 3\pi + \frac{4\pi}{f},$$

we get the angles in Table 1.

Type III Neighborhood

The neighborhood is the right of Figure 5, with all the edges of the tiling and all the angles of P_1 given. To determine all the other angles in the tiling, we consider the various orientations of P_5 and P_6.

Case 1 P_5 and P_6 are negatively oriented.

The case is described by Figure 6, in which we know all the angles of P_1, P_5, P_6.

Let $A_{2,16} = \theta_i, A_{3,14} = \theta_j, A_{4,15} = \theta_k$. Then $A_{4,13} = \phi_k$, and the angle sums at $V_{126}, V_{134}, V_{145}$ give $\theta_2 + \theta_i + \phi_2 = \theta_1 + \theta_j + \phi_k = \theta_1 + \theta_k + \phi_1 = 2\pi$. If $k = 2$, then the first equality says $\theta_2 + \theta_i = \theta_1 + \theta_j$. By $\theta_1 \neq \theta_2$, we get $i = 1$ and $j = 2$. Then the second equality becomes $\theta_1 + \theta_2 + \phi_2 = \theta_1 + \theta_2 + \phi_1$, which implies $\phi_2 = \phi_1$, a contradiction. Therefore we must have $k = 1$, and the angle sums at the three vertices imply $j = 1$. These determine all the angles of P_3, P_4. Then the two choices of the orientation of P_2 give two tilings in Figure 6. They are the type III_1 and III_2 tilings in Figure 3.

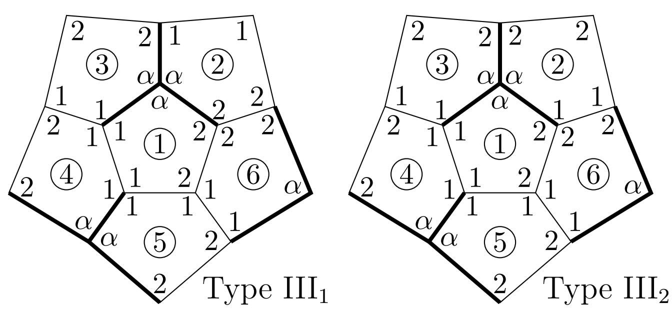

Figure 6: P_5, P_6 negatively oriented: III_1 and III_2.

For the left tiling, the angle sums at $V_{123}, V_{134}, V_{126}, V_{156}$ give

$$3\alpha = 2\theta_1 + \phi_1 = 2\theta_2 + \phi_2 = 2\phi_1 + \phi_2 = 2\pi.$$

For the right tiling, the angle sums at the same vertices give

$$3\alpha = 2\theta_1 + \phi_1 = \theta_1 + \theta_2 + \phi_2 = 2\phi_1 + \phi_2 = 2\pi.$$

Together with the angle sum equation for the pentagon, we get the angles in the type III_1 and III_2 tilings in Table 1.

<u>*Case 2*</u> P_5 and P_6 have different orientations.

The left and the middle of Figure 7 describe the case that P_5 is positively oriented and P_6 is negatively oriented. By $\phi_1 \neq \phi_2$ and comparing the angle sums at V_{126}, V_{145}, we get $A_{2,16} \neq A_{4,15}$. So we either have $A_{2,16} = \theta_2, A_{4,15} = \theta_1$, or have $A_{2,16} = \theta_1, A_{4,15} = \theta_2$. The two cases determine all the angles of P_2, P_4 in the left and the middle of Figure 7. Then we compare the angle sums at V_{134}, V_{145} in the first case, and compare V_{126}, V_{134} in the second case. In both cases, we get $A_{3,14} = \theta_2$ and then determine all the angles of P_3.

The right of Figure 7 describes the case that P_5 is negatively oriented and P_6 is positively oriented. By $\theta_1 \neq \theta_2$ and comparing the angle sums at V_{126}, V_{145}, we get $A_{2,16} = \theta_1$, $A_{4,15} = \theta_2$. This determines all the angles of P_2, P_4. By comparing the angle sums at V_{126}, V_{134}, together with $\phi_1 \neq \phi_2$, we get $A_{3,14} = \theta_1$. This determines all the angles of P_3.

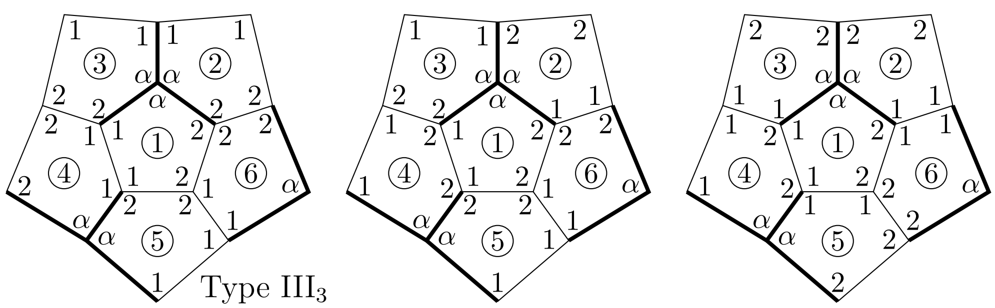

Figure 7: P_5, P_6 differently oriented: III_3 and other impossible tilings.

For each tiling in Figure 7, we have angle sum equations at four vertices. Solving the equations together with the angle sum equation for the pentagon, we get the following angles

$$
\begin{aligned}
\text{left}\colon \alpha &= \frac{2}{3}\pi,\ \theta_1 = \left(-\frac{1}{6}+\frac{10}{f}\right)\pi,\ \theta_2 = \left(\frac{5}{6}-\frac{2}{f}\right)\pi,\\
\phi_1 &= \left(\frac{4}{3}-\frac{8}{f}\right)\pi,\ \phi_2 = \left(\frac{1}{3}+\frac{4}{f}\right)\pi;\\
\text{middle}\colon \alpha &= \frac{2}{3}\pi,\ \theta_1 = \left(\frac{1}{3}+\frac{4}{f}\right)\pi,\ \theta_2 = \left(\frac{5}{6}-\frac{2}{f}\right)\pi,\\
\phi_1 &= \left(\frac{1}{3}+\frac{4}{f}\right)\pi,\ \phi_2 = \left(\frac{5}{6}-\frac{2}{f}\right)\pi;\\
\text{right}\colon \alpha &= \frac{2}{3}\pi,\ \theta_1 = \left(\frac{5}{6}-\frac{2}{f}\right)\pi,\ \theta_2 = \left(-\frac{1}{6}+\frac{10}{f}\right)\pi,\\
\phi_1 &= \left(\frac{4}{3}-\frac{8}{f}\right)\pi,\ \phi_2 = \left(\frac{1}{3}+\frac{4}{f}\right)\pi.
\end{aligned}
$$

The left tiling is the type III_3 tiling in Figure 3 and Table 1. By $f > 12$, we have $\theta_1 < \theta_2$ and $\phi_1 < \phi_2$ for the middle tiling and $\theta_1 > \theta_2$ and $\phi_1 > \phi_2$ for the right tiling. Both contradict Lemma 3.

Case 3 P_5 and P_6 are positively oriented.

The case is described by Figure 8. The angle sums at V_{123}, V_{156} give $\alpha = \phi_2 = \frac{2\pi}{3}$. By the angle sum equation for the pentagon and $f \neq 12$, we get $\phi_1 + \theta_1 + \theta_2 \neq 2\pi$. By comparing the inequality with the angle sums at V_{126}, V_{145}, we get $A_{2,16} = A_{4,15} = \theta_2$. This determines all the angles of P_2, P_4. Then the two choices of the orientation of P_3 give two tilings in Figure 8. Similar to the earlier cases, we may calculate the angles in the two tilings and find that both satisfy $\theta_1 < \theta_2$ and $\phi_1 < \phi_2$, contradicting to Lemma 3. □

Given a neighborhood tiling, we ask whether a nearby tile can still have all its vertices having degree 3. Moreover, if this is the case, we would like to know what type of neighborhood tiling around this nearby tile must be. This is the *propagation* problem.

Take type III_3 as an example. We ask whether P_2 can be the center of one of the four tilings in Figure 3. By comparing the angles in Table 1, the neighborhood tiling around P_2 cannot be of type III_1. Then we compare the orientations of tiles in the quadruple P_2, P_3, P_1, P_6 in the type III_3 tiling with the orientations of tiles in the quadruple P_1, P_2, P_3, P_4 or the quadruple P_1, P_3, P_2, P_6 in the type II, III_2 or III_3 tilings. Since we cannot find any

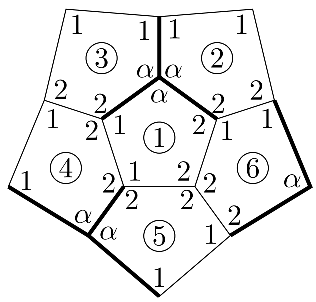

Figure 8: P_5, P_6 positively oriented: impossible tilings.

matches, the tile P_2 of the type III_3 tiling must have vertices of degree > 3. We indicate this by assigning $\times$ to P_2. By similar method, we get all the possible propagations in Figure 9.

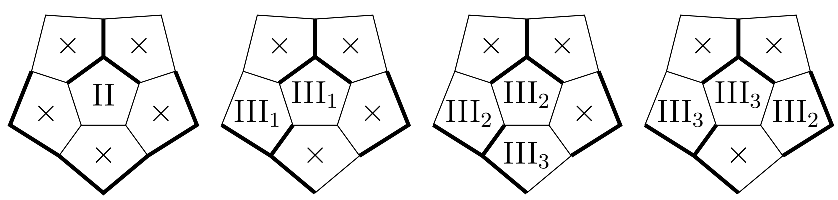

Figure 9: Propagation of geometrically congruent neighborhoods for a^3b^2.

The propagation can be used to show the following.

Proposition 4. *There are no geometrically congruent earth map tilings with edge length combinations a^2b^2c, a^3bc, or a^3b^2, with a, b, c distinct.*

Proof. According to [8], there are five families of earth map tilings, corresponding to the distances $1, 2, 3, 4, 5$ between the only two vertices of equal degree > 3 (called *poles*). For each distance, the tilings are obtained by repeating a timezone connecting the two poles. The cases of distances 4 and 5 are given on the left and the right of Figure 10. We mark the tiles with vertices of degree > 3 by $\times$.

Since earth map tilings always have tiles with all vertices having degree 3, by the first two cases of the main theorem, the combinations a^2b^2c and a^3bc can be dismissed. It remains to consider the combination a^3b^2. The earth map tilings of distances $1, 2, 3, 4$ always have a tile (such as the tiles marked P) with all vertices having degree 3, such that three consecutive nearby tiles (such as the tile marked Q) also have this property. By the propagations in Figure 9, this does not happen for geometrically congruent tilings with edge length combination a^3b^2.

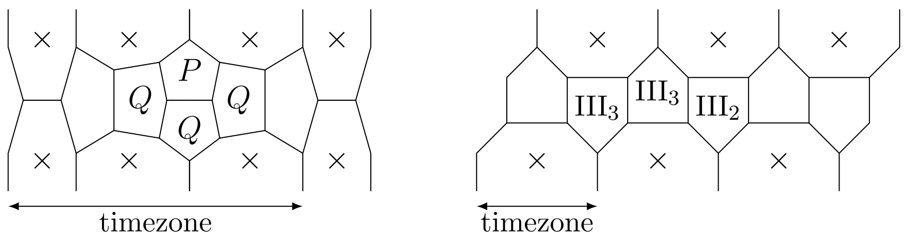

Figure 10: Earth map tiling and core tile.

It remains to consider the earth map tiling of distance 5 on the right of Figure 10 for the combination a^3b^2. By looking at the distribution of tiles with all vertices having degree 3 in this earth map tiling, we find that only the propagation from type III_3 neighborhood fits the tiling. However, this propagation also leads to the appearance of type III_2 neighborhood in the tiling, which does not fit the earth map tiling of distance 5. □

By the classification in [2], we only need to consider $f > 12$. Proposition 4 and [8, Theorems 1 and 6] further imply that we cannot have $v_4+v_5+v_6+\cdots =$ 1 or 2. Then by the vertex counting equation (2.1), we only need to consider even $f \geq 18$ in the rest of the paper. We will also find the following two simple observations very useful.

Proposition 5. *In a spherical tiling by geometrically congruent pentagons with edge length combination a^3b^2, a, b distinct, the number of ab-angles at any vertex is even.*

Proof. It is easy to show that the number of b-edges at a vertex is the number of b^2-angles plus half of the number of ab-angles. The proposition then follows. □

Proposition 6. *Suppose in a spherical tiling by pentagons geometrically congruent to the left of Figure 3, the angles $\theta_1, \theta_2, \phi_1, \phi_2$ are distinct. If ϕ_2 appears at most once in a vertex, then the angles cannot be configured as $\cdots\theta_2\phi_1^k\theta_2\cdots$ at a vertex. In case $k = 0$, this means that two θ_2 cannot share an a-edge at a vertex.*

Proof. An angle configuration $\cdots\theta_2\phi_1^k\theta_2\cdots$ at a vertex is given by Figure 11. Since θ_2 is an ab-angle and ϕ_1 is an a^2-angle, the lengths of the edges at the vertex must be as indicated. Note that for $k=0$, the picture shows the case that two θ_2 share an a-edge. Then we get the full information about the tiles P_1, P_5. Since $\phi_2^2\cdots$ is not a vertex, the angle ϕ_2 adjacent to ϕ_1 in P_2 must be located as indicated. This determines the full information about P_2. Then by the similar argument, we get the full information about P_3. Keep going, we can determine the full information about P_4. Then we find a vertex $\phi_2^2\cdots$ shared by P_4, P_5, a contradiction. □

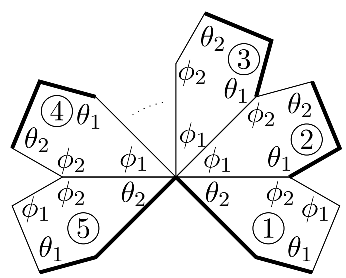

Figure 11: Impossible configuration in case $\phi_2^2\cdots$ is not a vertex.

3 Geometry of Spherical Pentagon

This section is devoted to the geometric constraints on spherical pentagons. First we prove Lemma 3. Then we establish a constraint for pentagons of edge length combination a^3b^2.

It is easy to see that Lemma 3 can also be reformulated in terms of the outside pentagon, and the two formulations are equivalent. We will always use strict inequalities in our argument, because the special case of equalities can be easily analyzed. We will also use the sine law and the following well known result in the spherical trigonometry: If a spherical triangle has the angles $\alpha, \beta, \gamma < \pi$ and has the edges a, b, c opposite to the angles, then $\alpha > \beta$ if and only if $a > b$.

Let A, B, C, D, E be the vertices of the pentagon at the angles $\alpha, \beta, \gamma, \delta, \epsilon$. Among the two great arcs connecting B and C, take the one with length $< \pi$ and form the edge BC. Since AB and AC intersect only at one point A and have the same length b, we get $b < \pi$. Since all three edges AB, AC, BC are

$< \pi$, by the sine law, among the two triangles bounded by the three edges, one has all three angles $< \pi$ and the other has all three angles $> \pi$. We denote the first triangle by $\triangle ABC$. The angle $\angle BAC$ of $\triangle ABC$ is either α of the pentagon, or its complement $2\pi - \alpha$. In the second case, we may replace the pentagon by its outside. Then we may always assume $\alpha = \angle BAC < \pi$.

The pentagon is obtained by choosing D, E, and then connecting B to D, C to E, and D to E by great arcs. Since $BC < \pi$, we find that BC does not intersect BD and CE, and BC and DE intersect at at most one point.

If BC and DE intersect at one point F, then one of D, E is outside $\triangle ABC$ and one is inside (we omit the "equality case" of D or E is on BC). The left of Figure 12 shows the case D is outside and E is inside. Since $BC < \pi$, the interiors of BD and CE do not intersect BC. This implies that

$$\beta > \angle ABC = \angle ACB > \gamma.$$

On the other hand, since $AC = b < \pi$ and $BC < \pi$, the prolongation of CE intersects the boundary of $\triangle ABC$ at a point G on AB. Using $AB < \pi$, $\alpha < \pi$, $\gamma < \angle ACB < \pi$ and applying the sine law to $\triangle ACG$, we get $CG < \pi$, so that $a = CE < CG < \pi$. Using $a < \pi$, $BF < BC < \pi$, $CF < BC < \pi$, $\angle BFD = \angle CFE < \pi$ and applying the sine law to $\triangle BDF$ and $\triangle CEF$, we find $\angle BDF < \pi$ and $\angle CEF < \pi$. Therefore

$$\delta = \angle BDF < \pi < 2\pi - \angle CEF = \epsilon.$$

So in case D is outside and E is inside, we have $\beta > \gamma$ and $\delta < \epsilon$. Similarly, in case D is inside and E is outside, we have $\beta < \gamma$ and $\delta > \epsilon$.

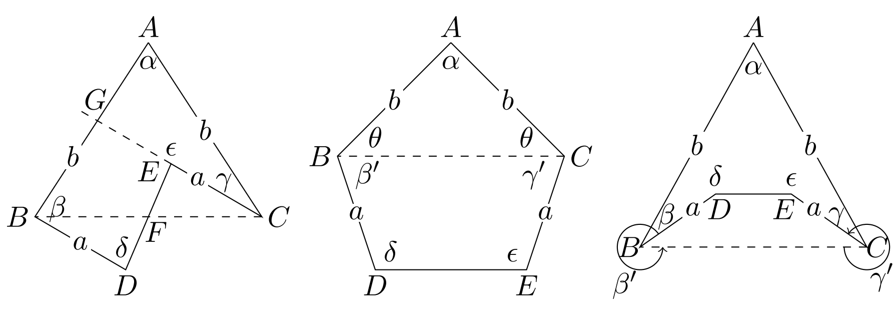

Figure 12: Geometrical constraint for pentagon.

If BC and DE are disjoint, then we have a quadrilateral $\square BDEC$ with δ, ϵ as two interior angles. Moreover, let $\theta = \angle ABC = \angle ACB$. Then the angles β', γ' of the quadrilateral at B and C are related to β, γ either by $\beta' = \beta - \theta, \gamma' = \gamma - \theta$ (in case both D and E are outside $\triangle ABC$, see the middle of Figure 12), or by $\beta' = \beta - \theta + 2\pi, \gamma' = \gamma - \theta + 2\pi$ (in case both D and E are inside $\triangle ABC$, see the right of Figure 12). Therefore $\beta > \gamma$ is equivalent to $\beta' > \gamma'$. Then the proof of Lemma 3 is reduced to the proof of the following similar lemma for quadrilaterals.

Lemma 7. *If the spherical quadrilateral on the left of Figure 13 has a pair of equal edges a, then $\beta > \gamma$ is equivalent to $\delta < \epsilon$.*

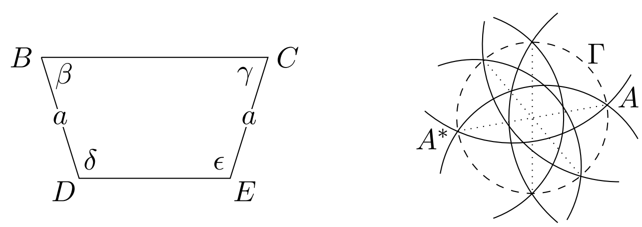

Figure 13: Geometrical constraint for quadrilateral, and great circles.

We note that the boundary of a spherical quadrilateral is assumed to be a simple closed curve, similar to the pentagon in Lemma 3. The argument before Lemma 7 reduced the problem to such quadrilaterals.

To prove Lemma 7, we use the conformally accurate way of drawing great circles on the sphere, described on the right of Figure 13. Let the circle Γ be the stereographic projection (from the north pole to the tangent space of the south pole) of the equator. The antipodal points on the equator are then projected to the antipodal points on Γ. We denote the antipodal point of A by A^*. Since the intersection of any great arc with the equator is a pair of antipodal points on the equator, the great circles of the sphere are in one-to-one correspondence with the circles (and straight lines) on the plane that intersect Γ at a pair of antipodal points.

We note that the lemma can also be reformulated in terms of the outside quadrilateral, and the two formulations are equivalent.

Proof. Suppose $a > \pi$. In Figure 14, we draw the great circles containing the two a-edges. They intersect at a pair of antipodal points and divide the sphere into four 2-gons. Since $a > \pi$ and the boundary of the quadrilateral is simple, each antipodal point lies in one b-edge. Up to symmetry, therefore, there are two ways the four vertices B, C, D, E of the two a-edges can be located, described in the two pictures in Figure 14. Moreover, since $a > \pi$, the antipodal point B^* of B lies on BD.

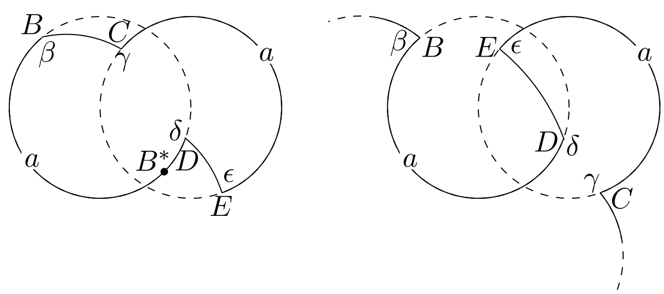

Figure 14: The case $a > \pi$.

On the left of Figure 14, we have one great arc BC completely contained in the indicated 2-gon. The other great arc connecting B and C intersects BD at B^* and therefore cannot be an edge of the quadrilateral. By the same reason, the great arc DE is also completely contained in the indicated 2-gon. This implies

$$\beta < \pi < \gamma, \quad \delta > \pi > \epsilon.$$

Similar argument gives the quadrilateral on the right, and we get the same inequalities above.

So in all the subsequent argument, we may always assume $a < \pi$. We argue that, if $\delta, \epsilon < \pi$, then we may further assume $DE < \pi$. If $DE > \pi$, then we draw the great circle $\bigcirc DE$ containing DE on the left of Figure 15, such that the disk bounded by $\bigcirc DE$ is the hemisphere containing the angles $\delta, \epsilon < \pi$. The assumption $DE > \pi$ implies that the antipodal D^* of D lies in the interior of the DE edge. Moreover, the edge BD must lie inside the hemisphere because the other great arc connecting B and D intersects DE at D^*. By the same reason, the edge CE also lies inside the hemisphere. Moreover, the edge BC also lies inside the hemisphere because by $DE > \pi$, the other great arc connecting B and C must intersect DE. Now we may consider a new quadrilateral $\square' BCED$ obtained by replacing DE by the other great arc (of length $< \pi$) connecting D and E. This new quadrilateral (i.e., the hemisphere subtracting the original quadrilateral) satisfies $\delta' = \pi - \delta, \epsilon' = \pi - \epsilon < \pi$ and $DE < \pi$. Moreover, it is easy to see that the lemma for the new quadrilateral is equivalent to the lemma for the original quadrilateral.

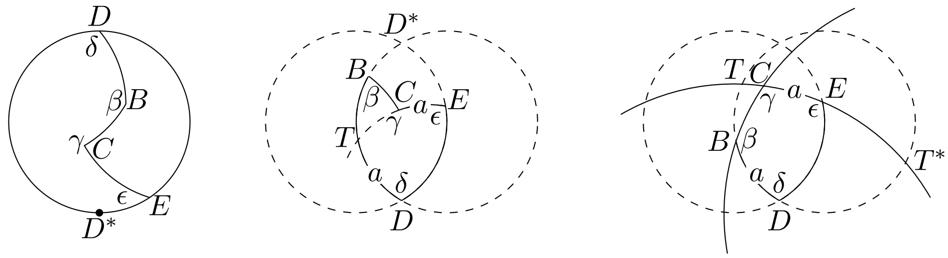

Figure 15: The case $a < \pi$ and three angles $< \pi$.

Now we consider the case that the quadrilateral has at least three angles $< \pi$. Up to symmetry, we may assume $\beta, \delta, \epsilon < \pi$. By the discussion above, we may further assume that $a < \pi$ and $DE < \pi$. We draw the great circles $\bigcirc BD$ and $\bigcirc DE$ containing BD and DE as in the middle and the right of Figure 15. The two great circles divide the sphere into four 2-gons, and by $\delta < \pi$, we may assume that δ is an angle of the middle 2-gon, and BD and DE are contained in the edges of the middle 2-gon. By $\beta, \epsilon < \pi$, we find that BC and EC are inside the middle 2-gon. The middle of Figure 15 describes the case $\gamma > \pi$, and the right describes the case $\gamma < \pi$. In the middle, the prolongation of EC intersects BD at T. Then $DT < a < ET$. Since all angles of $\triangle DET$ are $< \pi$, this implies that $\angle DET < \angle EDT$. We conclude that

$$\beta < \pi < \gamma, \quad \delta > \epsilon.$$

On the right of Figure 15, the great circles $\bigcirc BD$ and $\bigcirc CE$ containing the two a-edges intersect at antipodal points T and T^*. The topology of the picture shows that T and T^* do not lie in the two a-edges, so that we have $BT + a + DT^* = \pi = CT + a + ET^*$. This implies

$$BT > CT \iff DT^* < ET^*.$$

Since the angles in $\triangle BCT$ and $\triangle DET^*$ are $< \pi$, we also have

$$BT > CT \iff \pi - \beta = \angle CBT < \angle BCT = \pi - \gamma,$$

and

$$DT^* < ET^* \iff \pi - \delta = \angle EDT^* > \angle DET^* = \pi - \epsilon.$$

Combining all the equivalences together, we get

$$\beta > \gamma \iff \delta < \epsilon.$$

So the lemma is proved for the case of at least three angles $< \pi$. By considering the outside quadrilateral, we also know the lemma holds for the case of at least three angles $> \pi$. It remains to consider the case that two angles $< \pi$ and the other two angles $> \pi$. Up to symmetry, this means the following three cases

1. $\beta, \epsilon > \pi$ and $\gamma, \delta < \pi$.

2. $\beta, \gamma > \pi$ and $\delta, \epsilon < \pi$.

3. $\gamma, \epsilon > \pi$ and $\beta, \delta < \pi$.

Again we may always additionally assume $a < \pi$.

The first case is consistent with the conclusion of the lemma.

As we argued before, in the second case, we may additionally assume $DE < \pi$. Let λ be the other great arc connecting B and C, as on the left of Figure 16. Since $a < \pi$, λ does not intersect BD and CE. By the topology of the picture, if λ intersects DE, then they intersect at two points. By $DE < \pi$, this cannot happen. Therefore by replacing the original edge BC by λ, we get a new quadrilateral, in which all four angles $\beta-\pi, \gamma-\pi, \delta, \epsilon < \pi$. We have proved the lemma for the new quadrilateral, which gives $\beta - \pi > \gamma - \pi \iff \delta < \epsilon$. This is the same as $\beta > \gamma \iff \delta < \epsilon$.

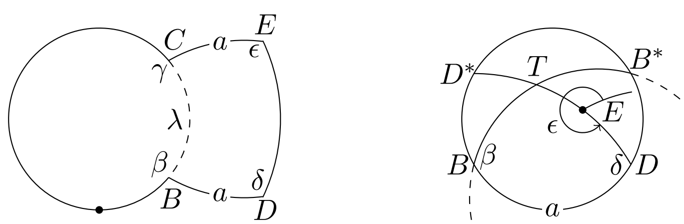

Figure 16: The case $a < \pi$ and two angles $< \pi$, two angles $> \pi$.

It remains to show that the third case is actually impossible. We draw the great circle $\bigcirc BD$ containing BD on the right of Figure 16, such that the disk bounded by $\bigcirc BD$ is the hemisphere containing the angles $\beta, \delta < \pi$. The great circles $\bigcirc BC$ and $\bigcirc DE$ containing BC and DE intersect $\bigcirc BD$ at the antipodal points B^*, D^*. Since $BD = a < \pi$, the antipodal points lie outside the BD edge, and we get the configuration as in Figure 16. In particular, the great circles $\bigcirc BC$ and $\bigcirc DE$ intersect at a point T inside the hemisphere. Since BC and DE do not intersect, either C lies in BT, or E lies in DT.

Without loss of generality, we may assume E lying in DT, as described by the picture. Then we find that the edges BD, DE are contained in the same hemisphere bounded by $\bigcirc BC$. Moreover, since $EC = a < \pi$, EC cannot intersect $\bigcirc BD$ at two points. This implies that EC is also contained in the same hemisphere bounded by $\bigcirc BC$, and further implies that $\gamma < \pi$. $\square$

Now we turn to another geometrical constraint, which we will use at the very end of this paper. A general spherical pentagon allows 7 free parameters. The spherical pentagon of edge length combination a^3b^2 on the left of Figure 17 satisfies 3 independent edge length equalities. Therefore the pentagon allows $7 - 3 = 4$ free parameters. This means that four angles ($\beta, \gamma, \delta, \epsilon$, for example) completely determine the pentagon.

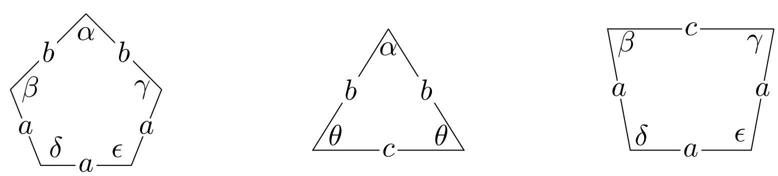

Figure 17: One equality for the angles.

We start with the simpler case of the isosceles triangle in the middle of Figure 17. A general triangle allows 3 free parameters. The isosceles triangle introduces 1 equality and allows $3 - 1 = 2$ free parameters. Therefore α and θ completely determine the isosceles triangle. The edge length c is given by

$$\cos c = \frac{\cos\alpha + \cos^2\theta}{\sin^2\theta} = \cos\alpha + (1 + \cos\alpha)\cot^2\theta. \tag{3.1}$$

The edge length b is further given by

$$\sin c \cos\theta = \sin b \cos b(1 - \cos\alpha), \quad \sin c \sin\theta = \sin b \sin\alpha. \tag{3.2}$$

Next consider the quadrilateral of edge length combination a^3c on the right of Figure 17. A general quadrilateral allows 5 free parameters. The edge length combination a^3c introduces 2 equalities and leaves $5 - 2 = 3$ free parameters. Therefore its four angles satisfy one equality, and the edge lengths a and c can be expressed in terms of the four angles.

We add an arc $y = DC$ to the quadrilateral as in Figure 18. By the sine law, we have

$$\sin a \sin\beta = \sin y \sin(\gamma - \phi).$$

Then by using the equality (3.2) for the isosceles triangle $\triangle CDE$, we have

$$\begin{aligned}\sin a \sin \beta &= \sin y \sin \gamma \cos \phi - \sin y \cos \gamma \sin \phi \\ &= \sin a \cos a (1 - \cos \epsilon) \sin \gamma - \sin a \sin \epsilon \cos \gamma.\end{aligned}$$

Canceling $\sin a$, we get

$$\sin \beta + \cos \gamma \sin \epsilon = \cos a \sin \gamma (1 - \cos \epsilon). \tag{3.3}$$

By the symmetry of exchanging β and γ, and exchanging δ and ϵ, we get

$$\sin \gamma + \cos \beta \sin \delta = \cos a \sin \beta (1 - \cos \delta). \tag{3.4}$$

If we cancel $\cos a$ from (3.3) and (3.4), then we get an equality for the four angles of the quadrilateral

$$(\sin \beta + \cos \gamma \sin \epsilon) \sin \beta (1 - \cos \delta) = (\sin \gamma + \cos \beta \sin \delta) \sin \gamma (1 - \cos \epsilon). \tag{3.5}$$

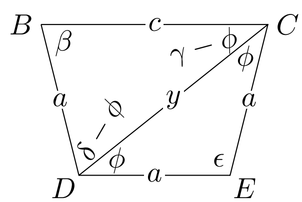

Figure 18: Geometrical constraint for quadrilateral.

The equalities (3.3) and (3.4) give expressions of a in terms of the four angles. The following further gives an expression for c

$$\begin{aligned}\cos c &= \cos a \cos y + \sin a \sin y \cos(\delta - \phi) \\ &= \cos a \cos y + \sin a \cos \delta \sin y \cos \phi + \sin a \sin \delta \sin y \sin \phi \\ &= \cos a (\cos^2 a + \sin^2 a \cos \epsilon) \\ &\quad + \sin a \cos \delta \sin a \cos a (1 - \cos \epsilon) + \sin a \sin \delta \sin a \sin \epsilon \\ &= \cos a - \sin^2 a \cos a (1 - \cos \delta)(1 - \cos \epsilon) + \sin^2 a \sin \delta \sin \epsilon \\ &= \cos^3 a (1 - \cos \delta)(1 - \cos \epsilon) - \cos^2 a \sin \delta \sin \epsilon \\ &\quad + \cos a (\cos \delta + \cos \epsilon - \cos \delta \cos \epsilon) + \sin \delta \sin \epsilon. \end{aligned} \tag{3.6}$$

The general formula without assuming the two edges are equal to a can be found in [1].

Now consider the pentagon of edge length combination a^3b^2 on the left of Figure 17. We try to find out how the angles $\beta, \gamma, \delta, \epsilon$ determine the pentagon.

We divide the pentagon into an isosceles triangle of edge length combination b^2c and a quadrilateral of edge length combination a^3c as in Figure 12. Suppose we are in the middle situation, then the quadrilateral has angles $\beta - \theta, \gamma - \theta, \delta, \epsilon$. Substituting these angles in place of $\beta, \gamma, \delta, \epsilon$ in (3.3) and (3.4) and dividing $\sin\theta$, we get

$$\begin{aligned}
&(\sin\beta + \cos\gamma\sin\epsilon)\cot\theta - \cos\beta + \sin\gamma\sin\epsilon \\
&= \cos a(\sin\gamma\cot\theta - \cos\gamma)(1 - \cos\epsilon), \qquad (3.7)\\
&(\sin\gamma + \cos\beta\sin\delta)\cot\theta - \cos\gamma + \sin\beta\sin\delta \\
&= \cos a(\sin\beta\cot\theta - \cos\beta)(1 - \cos\delta). \qquad (3.8)
\end{aligned}$$

Eliminating $\cot\theta$ from (3.7) and (3.8) gives a quadratic equation for $\cos a$

$$L\cos^2 a + M\cos a + N = 0, \qquad (3.9)$$

where

$$\begin{aligned}
L &= \sin(\beta - \gamma)(1 - \cos\delta)(1 - \cos\epsilon), \\
M &= \cos(\beta - \gamma)(\sin\epsilon - \sin\delta + \sin(\delta - \epsilon)), \\
N &= \sin\delta - \sin\epsilon - \sin(\beta - \gamma)(1 - \sin\delta\sin\epsilon).
\end{aligned}$$

Alternatively, eliminating $\cos a$ from (3.7) and (3.8) gives a quadratic equation for $\cot\theta$

$$P\cot^2\theta + Q\cot\theta + R = 0, \qquad (3.10)$$

where

$$\begin{aligned}
P &= \sin\beta(\sin\beta + \cos\gamma\sin\epsilon)(1 - \cos\delta) \\
&\quad - \sin\gamma(\sin\gamma + \cos\beta\sin\delta)(1 - \cos\epsilon), \\
Q &= -(\sin 2\beta + \cos(\beta + \gamma)\sin\epsilon)(1 - \cos\delta) \\
&\quad + (\sin 2\gamma + \cos(\beta + \gamma)\sin\delta)(1 - \cos\epsilon), \\
R &= \cos\beta(\cos\beta - \sin\gamma\sin\epsilon)(1 - \cos\delta) \\
&\quad - \cos\gamma(\cos\gamma - \sin\beta\sin\delta)(1 - \cos\epsilon).
\end{aligned}$$

After finding a and θ in terms of $\beta, \gamma, \delta, \epsilon$, we may substitute $\cot\theta$ and $\cos a$ into the following combination of (3.1) and (3.6) to determine $\cos\alpha$

$$\begin{aligned}&\cos\alpha + (1+\cos\alpha)\cot^2\theta \\ &= \cos a - \sin^2 a \cos a(1-\cos\delta)(1-\cos\epsilon) + \sin^2 a \sin\delta \sin\epsilon.\end{aligned}$$

Then α, θ, c further determine b by (3.2).

We may also consider the other two situations in Figure 12. It is not difficult to see that we still get the same equalities showing how the angles $\beta, \gamma, \delta, \epsilon$ determine the pentagon.

4 Geometrically Congruent Tilings

4.1 Type III_1

For all the angles to be positive, we must have $-\frac{2}{3} + \frac{16}{f} > 0$, or $f < 24$. So we only need to consider $f = 18, 20, 22$.

For $f = 18$, the angles are

$$\alpha = \frac{2}{3}\pi, \quad \theta_1 = \frac{5}{9}\pi, \quad \theta_2 = \frac{8}{9}\pi, \quad \phi_1 = \frac{8}{9}\pi, \quad \phi_2 = \frac{2}{9}\pi.$$

At the vertex $\theta_1\theta_2\cdots$ shared by P_2, P_3 on the left of Figure 6, the remaining angle $2\pi - \theta_1 - \theta_2 = \theta_1$ is a combination of five angles above. Since the only such combination is the single θ_1, the vertex must be $\theta_1^2\theta_2$, and we get a tile P outside P_2, P_3 with angle θ_1 at the vertex. On the other hand, the angle of P at the vertex is an a^2-angle, which must be either ϕ_1 or ϕ_2. Since neither ϕ_1 nor ϕ_2 is equal to θ_1, we get a contradiction. For $f = 22$, the same argument leads to the same contradiction.

For $f = 20$, the vertex $\theta_1\theta_2\cdots$ can be $\theta_1^2\theta_2$ or $\theta_1\theta_2\phi_2^4$. The first case leads to the same contradiction as above. The second case implies that there is a vertex of degree 6. By the vertex counting equation (2.1), however, we find $v_4 = v_6 = 1$ and $v_5 = v_7 = v_8 = \cdots = 0$. This is combinatorially impossible by [8, Theorems 6].

We conclude that there are no type III_1 geometrically congruent tilings.

4.2 Types III_2 and III_3

For type III_2, the angle sum at a vertex $\alpha^a\theta_1^{b_1}\theta_2^{b_2}\phi_1^{c_1}\phi_2^{c_2}$ is

$$\frac{2}{3}a + \left(\frac{5}{6} - \frac{2}{f}\right) b_1 + \left(-\frac{1}{6} + \frac{10}{f}\right) b_2 + \left(\frac{1}{3} + \frac{4}{f}\right) c_1 + \left(\frac{4}{3} - \frac{8}{f}\right) c_2 = 2.$$

Since $f \geq 18$ and all angles are positive, we get

$$\frac{2}{3}a + \frac{13}{18}b_1 + \frac{1}{3}c_1 + \frac{8}{9}c_2 \leq 2.$$

There are finitely many possible choices of (a, b_1, c_1, c_2) satisfying the inequality above. We rewrite the angle sum equation as an expression for b_2

$$b_2 = -(5a+6b_1+3c_1+9c_2-15)F + (4a+5b_1+2c_1+8c_2-12),\ F = \frac{48}{60-f},$$

and then substitute the finitely many choices of (a, b_1, c_1, c_2). Those that yield non-negative integer b_2 and have degree ≥ 3 are listed in Table 2.

a	b_1	b_2	c_1	c_2
3	0	0	0	0
0	1	1	0	1
0	2	0	1	0
0	0	0	2	1
1	0	F	0	1
1	1	$F-1$	1	0
1	0	$F-2$	3	0
2	0	$2F-2$	1	0
0	2	$3F-2$	0	0
0	0	$3F-2$	1	1
0	1	$3F-3$	2	0
0	0	$3F-4$	4	0

a	b_1	b_2	c_1	c_2
1	1	$4F-3$	0	0
1	0	$4F-4$	2	0
2	0	$5F-4$	0	0
0	0	$6F-4$	0	1
0	1	$6F-5$	1	0
0	0	$6F-6$	3	0
1	0	$7F-6$	1	0
0	1	$9F-7$	0	0
0	0	$9F-8$	2	0
1	0	$10F-8$	0	0
0	0	$12F-10$	1	0
0	0	$15F-12$	0	0

Table 2: All the vertices $\alpha^a\theta_1^{b_1}\theta_2^{b_2}\phi_1^{c_1}\phi_2^{c_2}$ for type III_2.

It is easy to verify that the five angles are distinct. Moreover, since Table 2 shows that $\phi_2^2\cdots$ is not a vertex, Proposition 6 can be applied. By the proposition and considering the possible configurations of edges and angles at vertices, the following are not vertices (a or c are allowed to be 0)

$$\alpha^{\geq 1}\phi_1^{\geq 1},\ \alpha^a\theta_1\theta_2^{\geq 2}\phi_1^c,\ \alpha^a\theta_2^{\geq 1}\phi_1^c,\ \alpha^a\theta_2^{\geq 3}\phi_2,\ \theta_1^2\theta_2^{\geq 3},\ \theta_2^{\geq 3}\phi_1\phi_2.$$

This eliminates many vertices from Table 2. Combined with Proposition 5, the only remaining possible angle combinations at vertices are

AVC for $f = 24$: $\alpha^3,\ \theta_1\theta_2\phi_2,\ \theta_1^2\phi_1,\ \phi_1^2\phi_2,\ \theta_1^2\theta_2^2,\ \theta_1\theta_2\phi_1^2,\ \theta_2^2\phi_1\phi_2,\ \phi_1^4$;
AVC for $f = 36$: $\alpha^3, \theta_1\theta_2\phi_2,\ \theta_1^2\phi_1,\ \phi_1^2\phi_2,\ \alpha\theta_1\theta_2\phi_1,\ \alpha\theta_2^2\phi_2$.

For $f = 36$, the AVC implies that the vertex $\theta_2^2\cdots$ shared by P_2, P_3 on the right of Figure 6 is $\alpha\theta_2^2\phi_2$. Since α is a b^2-angle, θ_2 is an ab-angle, and ϕ_2 is an a^2-angle, the vertex $\alpha\theta_2^2\phi_2$ can be configured in unique way, and two θ_2 are not adjacent in such configuration. Since two θ_2 are adjacent on the right of Figure 6, we conclude that there are no type III_2 geometrically congruent tilings for $f = 36$.

For $f = 24$, we have

$$\alpha = \frac{2}{3}\pi,\quad \theta_1 = \frac{3}{4}\pi,\quad \theta_2 = \frac{1}{4}\pi,\quad \phi_1 = \frac{1}{2}\pi,\quad \phi_2 = \pi.$$

Substituting the angles into (3.9), we get $L = 2, M = 0, N = 0$, so that $\cos a = 0$. Since $0 < a < \pi$ (because BD and DE intersect only at D), we get $a = \frac{1}{2}\pi$.

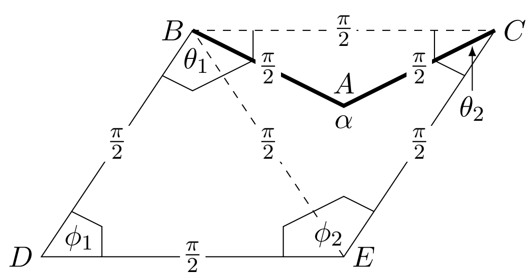

Figure 19: Impossible spherical pentagon.

In Figure 19, by $BD = DE = a = \frac{1}{2}\pi$ and $\angle BDE = \frac{1}{2}\pi$, we know $\triangle BDE$ is an equilateral triangle with side length $\frac{1}{2}\pi$ and inner angle $\frac{1}{2}\pi$. Then by $BE = CE = \frac{1}{2}\pi$ and $\angle BEC = \angle CED - \angle BED = \pi - \frac{1}{2}\pi = \frac{1}{2}\pi$, we know $\triangle BCE$ is also an equilateral triangle with side length $\frac{1}{2}\pi$ and inner angle $\frac{1}{2}\pi$. Furthermore, by $\angle ABE = \angle ABD - \angle DBE = \frac{3}{4}\pi - \frac{1}{2}\pi = \frac{1}{4}\pi$ and $\angle ACE = \frac{1}{4}\pi$, the edges AB and AC evenly divide the angles $\angle CBE$ and $\angle BCE$. Therefore A is the center of the equilateral triangle $\triangle BCE$, so that $\alpha = \frac{4}{3}\pi$, contradicting to $\alpha = \frac{2}{3}\pi$. Another way to see the contradiction is that the areaa of $\triangle BCE$ and $\triangle BDE$ are both $\frac{1}{8}$ of the area of the sphere, so that the area of the pentagon is $\left(1+\frac{2}{3}\right)\frac{1}{8} = \frac{5}{24}$ of the area of the sphere. In other words, the area of the sphere is not an integer multiple of the area of the pentagon.

We conclude that there are no type III_2 geometrically congruent tilings. Moreover, the tile in type III_3 tilings is the same as the one for type III_2 tilings, except θ_1 and θ_2 are exchanged, and ϕ_1 and ϕ_2 are exchanged. The pentagonal tile leads to the same contradiction, so that there are no type III_3 geometrically congruent tilings.

4.3 Type II, $\theta_1 = \alpha$

For type II tilings, we need to first determine the specific values of θ_1 and θ_2. By $f \geq 18$, we know $\phi_1 < \alpha$, $\phi_2 > \alpha$ and $\theta_1 + \theta_2 < 2\alpha$. By Lemma 3, this further implies $\theta_1 > \theta_2$, so that

$$\theta_1 > \frac{1}{2}(\theta_1 + \theta_2) = \left(\frac{1}{3} + \frac{4}{f}\right)\pi, \quad \theta_2 < \left(\frac{1}{3} + \frac{4}{f}\right)\pi < \alpha.$$

By $\theta_1 + \theta_2 = 2\phi_1$ and $\theta_1 \neq \theta_2$, we get $\theta_1 \neq \phi_1$ and $\theta_2 \neq \phi_1$. By $\theta_1 + \theta_2 < \phi_2$, we get $\theta_1 \neq \phi_2$ and $\theta_2 \neq \phi_2$. Therefore $\theta_1, \theta_2, \phi_1, \phi_2$ are distinct. The discussion also shows that if $\alpha, \theta_1, \theta_2, \phi_1, \phi_2$ are not distinct, then $\theta_1 = \alpha$.

If $\theta_1 = \alpha$, then we know all the angles

$$\theta_1 = \alpha = \frac{2}{3}\pi, \quad \theta_2 = \frac{8}{f}\pi, \quad \phi_1 = \left(\frac{1}{3} + \frac{4}{f}\right)\pi, \quad \phi_2 = \left(\frac{4}{3} - \frac{8}{f}\right)\pi.$$

As far as the values of the angles are concerned, we use α to denote both α and θ_1. By $f > 12$, we see that $2\pi - 2\phi_2 = \left(-\frac{2}{3} + \frac{16}{f}\right)\pi$ is strictly smaller than all the angles. Therefore $\phi_2^2\cdots$ is not a vertex, so that Proposition 6 holds.

The angle sum at a vertex $\alpha^a\theta_2^b\phi_1^c\phi_2^d$ (a is the total number of α and θ_1 at the vertex) is

$$\frac{2}{3}a + \frac{8}{f}b + \left(\frac{1}{3} + \frac{4}{f}\right)c + \left(\frac{4}{3} - \frac{8}{f}\right)d = 2.$$

By $f \geq 18$, we get

$$\frac{2}{3}a + \frac{1}{3}c + \frac{8}{9}d < 2.$$

Substituting those (a, c, d) satisfying the inequality into the solution of the angle sum equation

$$b = (6 - 2a - c - 4d)\frac{f}{24} + d - \frac{1}{2}c,$$

we get all the possible vertices. Those with $a \geq 1$ are listed in Table 3. Using Proposition 6, we may carry out the argument similar to type III_2 tilings and find that $f = 24, 36, 60$, with the corresponding vertices with $a \geq 1$ listed in the table. We also find all the configurations of the vertices in the table, given in Figure 20, where we note that the ab-angle α is really θ_1.

a	b	c	d	$f = 24$	$f = 36$	$f = 60$
3	0	0	0	α^3	α^3	α^3
1	1	0	1	$\alpha\theta_2\phi_2$	$\alpha\theta_2\phi_2$	$\alpha\theta_2\phi_2$
2	$\frac{f-12}{24}$	1	0		$\alpha^2\theta_2\phi_1$	$\alpha^2\theta_2^2\phi_1$
1	$\frac{f-36}{24}$	3	0			$\alpha\theta_2\phi_1^3$
2	$\frac{f}{12}$	0	0	$\alpha^2\theta_2^2$		
1	$\frac{f}{12} - 1$	2	0	$\alpha\theta_2\phi_1^2$		
1	$\frac{f-4}{8}$	1	0			
1	$\frac{f}{6}$	0	0			

Table 3: All the vertices $\alpha^a\theta_2^b\phi_1^c\phi_2^d$ with $a \geq 1$ for type II, $\theta_1 = \alpha$.

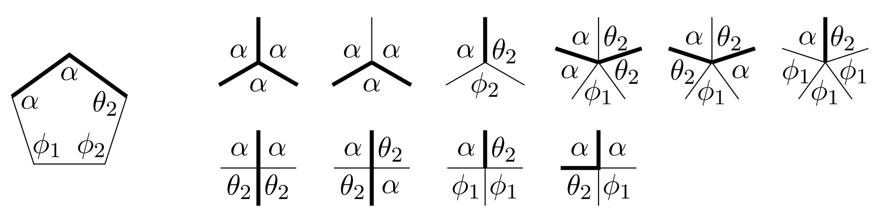

Figure 20: Vertex configurations for type II, $\theta_1 = \alpha$.

By Figure 20, the vertex $\theta_1^2 \cdots = \alpha^2 \cdots$ shared by P_5, P_6 in the middle of Figure 5 is α^3 with one a-edge and two b-edges. So we get a tile P_7 outside P_5, P_6. Then P_7 share a vertex $\alpha\theta_2 \cdots$ with either P_5 or P_6, such that α is a b^2-angle. By Figure 20, the vertex must be $\alpha^2\theta_2\phi_1$, which implies $f = 36$. Now for $f = 36$, the vertex $\alpha\theta_2\cdots$ shared by P_4, P_5 in the middle of Figure 5 is either $\alpha\theta_2\phi_2$ or $\alpha^2\theta_2\phi_1$. However, the edge length arrangement at the vertex in the middle of Figure 5 does not match the configuration for $\alpha\theta_2\phi_2$ or $\alpha^2\theta_2\phi_1$ in Figure 20. This concludes the proof that there are no type II geometrically congruent tilings with $\theta_1 = \alpha$.

4.4 Type II, $\theta_1 \neq \alpha$.

The final case is type II tilings with all five angles $\alpha, \theta_1, \theta_2, \phi_1, \phi_2$ distinct. We already have the vertices $\alpha^3, \theta_1\theta_2\phi_2, \phi_1^2\phi_2$ from the middle of Figure 5. We saw the importance of Proposition 6 in the discussion for the tilings of types III_2 and III_3 and for the case $\theta_1 = \alpha$. For the proposition to still hold, we need $\phi_2^2\cdots$ not to be a vertex.

Suppose $\phi_2^2\cdots$ is a vertex. Then the angle sum at the vertex gives

$$2\pi > 2\phi_2 = 2\left(\frac{4}{3} - \frac{8}{f}\right)\pi.$$

This means $f < 24$. Then by the vertex counting equation (2.1), [8, Theorems 1 and 6] and Proposition 4, all vertices must have degree ≤ 6. Moreover, for $f = 18$, the vertex counting equation is $3 = v_4 + 2v_5 + 3v_6$. By [8, Theorems 1], we must have $v_6 = 0$. Furthermore, by the remark after Proposition 4, we also cannot have $v_4 = v_5 = 1$. Therefore the only possibility is $v_4 = 3$ and $v_{\geq 5} = 0$.

For $f \geq 18$, it is easy to see from the angle sum that the vertex $\phi_2^2\cdots$ must be $\theta_2^k\phi_2^2$. By Proposition 5, k must be even. Since all vertices have degree ≤ 6, we conclude that either $\theta_2^4\phi_2^2$ or $\theta_2^2\phi_2^2$ is a vertex. If $\theta_2^4\phi_2^2$ is a vertex, then $v_6 \geq 1$. By $f < 24$ and the remark after Proposition 4, we get $f = 22$, $v_4 = 2$, $v_6 = 1$. Then by $f = 22$ and the angle sum at $\theta_2^4\phi_2^2$, we may calculate all the angles

$$\alpha = \frac{2}{3}\pi, \quad \theta_1 = \frac{67}{66}\pi, \quad \theta_2 = \frac{1}{66}\pi, \quad \phi_1 = \frac{17}{33}\pi, \quad \phi_2 = \frac{32}{33}\pi.$$

Since no four angles from above (repetition allowed) can add up to 2π, we get a contradiction to $v_4 = 2$.

So $\theta_2^2\phi_2^2$ must be a vertex. The angle sum at the vertex gives

$$\theta_1 = \pi, \quad \theta_2 = \left(\frac{8}{f} - \frac{1}{3}\right)\pi.$$

For each $18 \le f < 24$, we know the values of all five angles and can further find all possible angle combinations at vertices (satisfying Proposition 5 and all the constraints above, especially $v_{\ge 5} = 0$ for $f = 18$)

$$\text{AVC for } f = 18, 22\colon \alpha^3,\ \theta_1\theta_2\phi_2,\ \phi_1^2\phi_2,\ \theta_2^2\phi_2^2;$$
$$\text{AVC for } f = 20\colon \alpha^3,\ \theta_1\theta_2\phi_2,\ \phi_1^2\phi_2,\ \theta_2^2\phi_2^2,\ \alpha^2\theta_2^2\phi_1.$$

By the AVC, the vertex $\theta_2\phi_1\cdots$ shared by P_2, P_6 on the right of Figure 5 is $\alpha^2\theta_2^2\phi_1$, and we must have $f = 20$. It is easy to see that $\alpha^2\theta_2^2\phi_1$ must be configured as in Figure 21. We know the full information about P_1 and P_4. Since $\alpha\theta_1\cdots$ is not a vertex by the AVC, we may use the information about P_1 to determine the full information about P_2, and use the information about P_4 to determine the full information about P_3. Then we find a vertex $\theta_1^2\cdots$ shared by P_1, P_2, contradicting to the AVC.

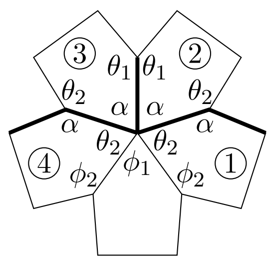

Figure 21: Impossible vertex configuration for type II, $\theta_1 \ne \alpha$.

So we conclude that $\phi_2^2\cdots$ is not a vertex. In particular, Proposition 6 remains valid. Next we find vertices $\alpha^a\theta_1^{b_1}\theta_2^{b_2}\phi_1^c\phi_2$ containing one copy of ϕ_2. Let $b = \min\{b_1, b_2\}$. Then the angle sum at such a vertex gives

$$\begin{aligned} 2 &\ge \frac{2}{3}a + \left(\frac{2}{3} + \frac{8}{f}\right) b + \left(\frac{1}{3} + \frac{4}{f}\right) c + \left(\frac{4}{3} - \frac{8}{f}\right) \\ &= \frac{1}{3}(2a + 2b + c + 4) + \frac{4}{f}(2b + c - 2). \end{aligned}$$

If $2b + c - 2 > 0$, then $2a + 2b + c < 2$, so that $a = b = 0$ and $c = 0$ or 1, which implies $2b + c - 2 \le 0$, contradicting to the assumption. So we must have $2b + c - 2 \le 0$, which implies $(b, c) = (1, 0), (0, 2), (0, 1), (0, 0)$.

If $(b, c) = (1, 0)$ or $(0, 2)$, then the inequality above implies $a = 0$. This gives $\theta_1^k\theta_2\phi_2$, $\theta_1\theta_2^k\phi_2$, with $k \ge 1$, and $\theta_1^k\phi_1^2\phi_2$, $\theta_2^k\phi_1^2\phi_2$, with $k \ge 0$. Since

$\theta_1\theta_2\phi_2$ and $\phi_1^2\phi_2$ are already vertices, the angle sum implies that the four cases must be $\theta_1\theta_2\phi_2$ and $\phi_1^2\phi_2$. The two vertices will be listed in (4.1).

If $(b,c) = (0,1)$, then the inequality above and $f \geq 18$ imply $a = 0$, and we get $[i]^k\phi_1\phi_2$, $i = 1,2$. Since the angle sum of $\theta_1\phi_1\phi_2$ is $> 2\pi$, we only have $\theta_2^k\phi_1\phi_2$. By Propositions 5 and 6, the only possibility is $\theta_2^2\phi_1\phi_2$, which will be listed in (4.1).

If $(b,c) = (0,0)$, then the inequality above and $f \geq 18$ imply $a = 0$ or 1, and we get $\alpha\theta_i^k\phi_2$ and $\theta_i^k\phi_2$, $i = 1,2$. Since the angle sums of $\alpha\theta_1\phi_2$ and $\theta_1^2\phi_2$ are $> 2\pi$, we only have $\alpha\theta_2^k\phi_2$ and $\theta_2^k\phi_2$. By Propositions 5 and 6, the only possibilities are $\alpha\theta_2^2\phi_2$ and $\theta_2^2\phi_2$. The angle sum at $\theta_2^2\phi_2$ implies $\theta_1 = \theta_2$, a contradiction. For the remaining possibility $\alpha\theta_2^2\phi_2$, the angle sum at the vertex gives

$$\theta_1 = \left(\frac{2}{3} + \frac{4}{f}\right)\pi, \quad \theta_2 = \frac{4}{f}\pi.$$

On the other hand, the vertex $\theta_1^2\cdots$ shared by P_5, P_6 in the middle of Figure 5 has the remaining angle $2\pi - 2\theta_1 = \left(\frac{2}{3} - \frac{8}{f}\right)\pi$, which is strictly less than α, θ_1, ϕ_2. Therefore the vertex is $\theta_1^2\theta_2^b\phi_1^c$. Since two θ_1 share an a-edge at the vertex, the vertex $\theta_1^2\theta_2^b\phi_1^c$ contradicts Proposition 6.

It remains to consider a vertex $\alpha^a\theta_1^{b_1}\theta_2^{b_2}\phi_1^c$ without ϕ_2. To avoid contradicting Proposition 6, each θ_2 needs to be combined with one θ_1 into a chain $\theta_1\phi_1\cdots\phi_1\theta_2$ bordered by two b-edges. Therefore we must have $b_1 \geq b_2$. Combined with all the possible vertices we found so far, we get the following complete list of possible angle combinations at vertices

$$\text{AVC: } \alpha^3, \theta_1\theta_2\phi_2,\ \phi_1^2\phi_2,\ \theta_2^2\phi_1\phi_2,\ \alpha^a\theta_1^{b_1}\theta_2^{b_2}\phi_1^c, \quad b_1 \geq b_2. \tag{4.1}$$

Suppose $\theta_2^2\phi_1\phi_2$ is not a vertex. Since the total number of θ_1 and θ_2 in the whole tiling should both be equal to f, to balance the equal total number, we must have $b_1 = b_2$ in every vertex $\alpha^a\theta_1^{b_1}\theta_2^{b_2}\phi_1^c$. The angle sum at $\alpha^a\theta_1^b\theta_2^b\phi_1^c$ is

$$2 = \frac{2}{3}a + \left(\frac{2}{3} + \frac{8}{f}\right)b + \left(\frac{1}{3} + \frac{4}{f}\right)c = \frac{2}{3}a + \left(\frac{1}{3} + \frac{4}{f}\right)(2b + c).$$

This implies $2a + 2b + c \leq 5$. By trying all (a,b,c) satisfying the inequality and using $f \geq 18$, we get all the solutions

$$\begin{aligned}
&a = 0, 2b + c = 4, f = 24\colon \theta_1\theta_2\phi_1^2,\ \theta_1^2\theta_2^2,\ \phi_1^4;\\
&a = 1, 2b + c = 3, f = 36\colon \alpha\theta_1\theta_2\phi_1,\ \alpha\phi_1^3;\\
&a = 0, 2b + c = 5, f = 60\colon \theta_1^2\theta_2^2\phi_1,\ \theta_1\theta_2\phi_1^3,\ \phi_1^5.
\end{aligned}$$

The vertex $\theta_1^2 \cdots$ shared by P_5, P_6 in the middle of Figure 5 does not appear in the AVC for $f = 36$, is $\theta_1^2\theta_2^2$ for $f = 24$, and is $\theta_1^2\theta_2^2\phi_1$ for 60. Since two θ_1 share an a-edge at the vertex, we find that the configuration of the vertex contradicts Proposition 6.

So we conclude that $\theta_2^2\phi_1\phi_2$ must be a vertex. The angle sum at the vertex implies

$$\theta_1 = \left(\frac{1}{2} + \frac{6}{f}\right)\pi,\ \theta_2 = \left(\frac{1}{6} + \frac{2}{f}\right)\pi,\ \phi_1 = \left(\frac{1}{3} + \frac{4}{f}\right)\pi,\ \phi_2 = \left(\frac{4}{3} - \frac{8}{f}\right)\pi.$$

The angle sum at $\alpha^a\theta_1^{b_1}\theta_2^{b_2}\phi_1^c$ is

$$\frac{2}{3}a + \left(\frac{1}{6} + \frac{2}{f}\right)b = 2, \quad b = 3b_1 + b_2 + 2c.$$

If $b = 0$, then the vertex is α^3. If $b > 0$, then this implies $4a + b < 12$. By Proposition 5, we also know that b is even. Substituting those (a, b) satisfying the two conditions into the angle sum equation, we keep those yielding integers $f \geq 18$ and get all the solutions

$$\begin{aligned} f &= 24\colon a = 0, b = 8; \\ f &= 36\colon a = 1, b = 6; \\ f &= 60\colon a = 0, b = 10. \end{aligned}$$

By the AVC (4.1), the vertex $\theta_1^2 \cdots$ shared by P_5, P_6 in the middle of Figure 5 is $\alpha^a\theta_1^{b_1}\theta_2^{b_2}\phi_1^c$, $b_1 \geq 2$. Therefore one of the above three cases must happen. For $f = 24$, solving $b_1 \geq 2$ and $3b_1 + b_2 + 2c = 8$ shows that the vertex is either $\theta_1^2\theta_2^2$ or $\theta_1^2\phi_1$. Since two θ_1 share an a-edge at the vertex, the vertex $\theta_1^2\theta_2^2$ contradicts Proposition 6 and the vertex $\theta_1^2\phi_1$ is impossible.

For $f = 36$, we consider the vertex $\theta_1\theta_2 \cdots$ shared by P_4, P_5 in the middle of Figure 5. By the AVC (4.1), the vertex is $\theta_1\theta_2\phi_2$ or $\alpha^a\theta_1^{b_1}\theta_2^{b_2}\phi_1^c$, $b_1, b_2 \geq 1$. Solving $3b_1 + b_2 + 2c = 6$ for $b_1, b_2 \geq 1$ shows that the later vertex is $\alpha\theta_1\theta_2\phi_1$. Since θ_1 and θ_2 share an a-edge at the vertex, neither $\theta_1\theta_2\phi_2$ nor $\alpha\theta_1\theta_2\phi_1$ are possible.

For $f = 60$, we have

$$\alpha = \frac{2}{3}\pi, \quad \theta_1 = \frac{3}{5}\pi, \quad \theta_2 = \frac{1}{5}\pi, \quad \phi_1 = \frac{2}{5}\pi, \quad \phi_2 = \frac{6}{5}\pi.$$

Substituting the angles into (3.10), we get

$$P = 0, \quad Q = 2\left(\frac{\sqrt{10-2\sqrt{5}}}{4}\right)^3, \quad R = \frac{5-2\sqrt{5}}{4}.$$

Therefore

$$\cot\theta = -\frac{R}{Q} = -\frac{\sqrt{5}-1}{\sqrt{10+2\sqrt{5}}}.$$

By $\theta < \pi$, we get $\theta = \frac{3}{5}\pi = \theta_1$. This means that the pentagon is given by Figure 22, in which D lies in the great arc connecting B and C.

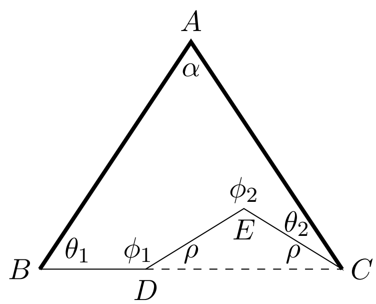

Figure 22: Impossible spherical pentagon.

Since $\triangle CDE$ and $\triangle ABC$ are isosceles triangles, we have $\rho = \angle EDC = \angle ECD = \theta_1 - \theta_2 = \frac{2}{5}\pi$. Then $\phi_1 + \rho = \frac{4}{5}\pi \neq \pi$, contradicting to the fact that D lies on the great arc connecting B and C. We conclude that there are no type II geometrically congruent tilings with $\theta_1 \neq \alpha$.

References

[1] Y. Akama, M. Yan. On deformed dodecahedron tiling. *preprint*, arXiv:1403.6907, 2014.

[2] H. H. Gao, N. Shi, M. Yan. Spherical tiling by 12 congruent pentagons. *J. Combinatorial Theory Ser. A*, 120(4):744–776, 2013.

[3] H. P. Luk. *Angles in spherical pentagonal tilings.* MPhil. Thesis, Hong Kong Univ. of Sci. and Tech., 2012.

[4] H. P. Luk, M. Yan. Angle combinations in spherical tilings by congruent pentagons. *preprint*, arXiv:1308.4207, 2014.

[5] D. M. Y. Sommerville. Division of space by congruent triangles and tetrahedra. *Proc. Royal Soc. Edinburgh*, 43:85–116, 1922-3.

[6] Y. Ueno, Y. Agaoka. Classification of tilings of the 2-dimensional sphere by congruent triangles. *Hiroshima Math. J.*, 32(3):463–540, 2002.

[7] Y. Ueno, Y. Agaoka. Examples of spherical tilings by congruent quadrangles. *Math. Inform. Sci., Fac. Integrated Arts and Sci., Hiroshima Univ., Ser. IV*, 27:135–144, 2001.

[8] M. Yan. Combinatorial tilings of the sphere by pentagons. *Elec. J. of Combi.*, 20(1):#P54, 2013.