Angle Combinations in Spherical Tilings by Congruent Pentagons

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Abstract

We develop a systematic method for computing the angle combinations in spherical tilings by angle congruent pentagons, and study whether such combinations can be realized by actual angle or geometrically congruent tilings. We get major families of angle or geometrically congruent tilings related to the platonic solids.

1 Introduction

Two polygons are *angle congruent* if there is a one-to-one correspondence between the edges, such that the adjacencies of the edges are preserved, and the angles between adjacent edges are also preserved. Similarly, if the edge lengths instead of the angles are preserved, then the two polygons are *edge congruent*. The two polygons are *(geometrically) congruent* if they are angle congruent and edge congruent.

The reason for studying different congruences arises from our attempt at classifying *edge-to-edge* tilings of the sphere by geometrically congruent pentagons. The classification of spherical trianglular tilings was started by Sommerville [6] in 1923 and completed by Ueno and Agaoka [7] in 2002. For pentagonal tilings, we already classified the minimal case of dodecahedron

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tilings [1, 4] (by 12 congruent pentagons). Unlike spherical triangles, for which the angle congruence is equivalent to the edge congruence, we need to separately study angle and edge congruences for pentagons. In [8], we studied the combinatorial aspect of spherical pentagonal tilings and found the next simplest spherical pentagonal tilings beyond the dodecahedron. In [2], we developed the method for classifying edge congruent pentagonal tilings. In [3], we show that certain types of the tiling of the sphere by geometrically congruent pentagons do not exist.

The objective of this paper is to study the angle aspect of spherical pentagonal tilings. We develop a systematic method for obtaining the numerical information about the angle combinations in spherical tilings by angle congruent pentagons. We further study some examples where such numerical information may or may not be realized by actual angle or geometrically congruent tilings. A major finding is Theorems 6 and 7, that naturally generalize the geometrically congruent dodecahedron tiling in [1, 4].

Since we are only concerned with the angles at the vertices, we do not care about the edge lengths and do not even require the edges to be straight (i.e., great arcs on the sphere). The only information we use is the sphere, the pentagon, and that the angles at any vertex should add up to 2π. The edge length information is included only when the results are applied to geometrically congruent tilings.

Since the angle combinations are completely understood for the minimal dodecahedron tiling by Sections 5 and 6 of [4], and the non-minimal pentagonal tilings have at least 16 tiles [8], we will assume that the number of tiles is at least 16 in this paper. Moreover, since we only consider tilings that are naturally given by graphs embedded in the sphere, we will only consider edge-to-edge tilings with all vertices having degree ≥ 3.

Our results are presented as what we call AVCs (*anglewise vertex combinations*, first introduced in [4]). A typical example is

$$\{84\alpha^2\beta\gamma\delta\colon (72-y_1)\alpha\beta\gamma,\ 32\alpha^3,\ (12+y_1)\beta\delta^3 \mid y_1\alpha\gamma^3,\ (12-y_1)\gamma^2\delta^3\},$$
$$\alpha=\tfrac{2}{3}\pi,\ \beta=\tfrac{6}{7}\pi,\ \gamma=\tfrac{10}{21}\pi,\ \delta=\tfrac{8}{21}\pi.$$

Here we have 84 pentagonal tiles, the angles in the pentagon are $\alpha,\alpha,\beta,\gamma,\delta$ with specific values. There are five possible angle combinations $\alpha\beta\gamma$, α^3, $\beta\delta^3$, $\alpha\gamma^3$, $\gamma^2\delta^3$ at the vertices. The coefficients $72-y_1, 32, 12+y_1, y_1, 12-y_1$ are the numbers for the respective combinations. The divider $\mid$ separates the five combinations into two groups. The three combinations before the divider are *necessary* in the sense that they must appear as vertices (i.e., the coefficients are positive). The two combinations after the divider are *optional* in the sense that these are the only vertices that may appear in addition to the necessary three (i.e., the coefficients are non-negative). We may choose y_1 to be any integer, such that all the coefficients in the necessary part are positive and all the coefficients in the optional part are non-negative. The condition means exactly $0 \le y_1 \le 12$, and we have total of 13 choices.

The AVCs are constructed incrementally. We still call such AVC pieces constructed along the way by (partial) AVCs. We also emphasize the final result such as the example above by calling them *full* AVCs. The notions of necessary and optional parts still apply to the AVC pieces, and will still be indicated by the divider.

In Section 2, we classify the angle combinations at degree 3 vertices. As suggested by earlier works [2, 3, 4, 8], degree 3 vertices dominate in spherical pentagonal tilings and should be investigated first. In fact, very little is assumed in Section 2, so that the resulting Theorem 1 can be applied to any angle congruent surface tiling with at most five distinct angles. The example above is an enrichment of the (partial) AVC $\{\alpha\beta\gamma, \alpha^3\}$ in the theorem.

In Section 3, we use the spherical and pentagonal assumptions to further specify the possible angle combinations in the pentagon and at the vertices. In Proposition 5, we list the possibilities for up to three (the full list would involve up to five) distinct angles at degree 3 vertices. We need to consider the case that all angles appear at degree 3 vertices and the case that there is one angle not appearing at degree 3 vertices. The example above is an enrichment of the AVC $\{\alpha^2\beta\gamma\delta\colon \alpha\beta\gamma, \alpha^3, \beta\delta^3\}$ in the second case.

Sections 4, 5 and 6 are devoted to the method for deriving the full AVCs from the AVCs obtained in Section 3. Sections 4 and 5 give two routine processes for computing the full AVC. Section 6 shows four examples not covered by the two routines. In Section 7, we list all the full AVCs with up to three distinct angles at degree 3 vertices. Moreover, we outline how to get all the full AVCs with four distinct angles at degree 3 vertices. We also explore the AVCs with five distinct angles at degree 3 vertices, where the methods in Sections 4, 5, 6 may not be sufficient.

In Section 8, we study the realizations of the AVCs by actual angle congruent or geometrically congruent tilings. We concentrate only on those AVCs allowing free continuous choice of two angles (within some range), because this is what happens to the geometrically congruent tiling in [1, 4], and no AVC allows free continuous choice of three or more angles. Our classification of AVCs shows that, beyond the minimal $f = 12$, there are only three such AVCs, with respective $f = 24, 36, 60$. Then in Theorems 6 and 7, we find that only the AVCs with $f = 24, 60$ can be realized, and the realizations are the *pentagonal subdivisions* of platonic solids. Although the AVC with $f = 36$ allows free continuous choice of two angles, this AVC cannot be realized.

We do not attempt to compute the complete classification of all full AVCs for spherical pentagonal tilings for two reasons. The first is that in applying the knowledge of this paper to specific tiling problems, the information other than the angles can be used to drastically simplify the possibilities. The insight we gain from this paper is sufficient for the partial classification of spherical tilings by geometrically congruent pentagons in [3]. In the subsequent classification work where we assume all edges have equal length, the angle information plays more important role, and the results from this paper is also sufficient. Second, the ideas developed here can certainly be used for angle congruent tilings in other contexts, such as quadrilateral tilings or tilings of other surfaces. Therefore the ideas of this paper are more important than the complete classification. This is also the reason why we do not try to provide theoretical explanation to every observation arising from the routines in Sections 4 and 5.

The work originates and extends the MPhil thesis [5] of the first author.

2 Angle Combinations at Degree 3 Vertices

In this section, we classify the AVCs (anglewise vertex combinations) at degree 3 vertices. A typical answer is

$$\{\alpha\beta\gamma, \alpha\delta^2 \mid \beta^2\delta\}.$$

This means that four *distinct* angles $\alpha, \beta, \gamma, \delta$ appear at degree 3 vertices. We first get the *necessary* part $\{\alpha\beta\gamma, \alpha\delta^2\}$ of the AVC, under the only criterion that all four angles appearing at degree 3 vertices are included. Then we look for the other degree 3 vertices, under the only criterion that distinct angles are not forced to become equal. For example, given the necessary part $\{\alpha\beta\gamma, \alpha\delta^2\}$, we may also allow $\beta^2\delta$ to appear while still keeping all four angles $\alpha, \beta, \gamma, \delta$ distinct. However, $\alpha^2\delta$ cannot be a vertex because the appearance of both $\alpha\delta^2$ and $\alpha^2\delta$ would imply

$$\alpha + 2\delta = 2\pi, \quad 2\alpha + \delta = 2\pi,$$

so that $\alpha = \delta$.

Although $\beta^2\delta$ is allowed to appear, it does not have to appear. Therefore we call the vertex *optional*, and indicate the difference between necessary and optional vertices by a divider $\mid$. We may have several optional vertices, which form the optional part of the AVC. The only assumption for deriving the optional part is that all angles at a vertex add up to 2π, which we call the *angle sum equation (at the vertex).*

Since the very simple criteria used for deriving the AVCs in this section are valid for tilings of any surface, our concluding Theorem 1 is not restricted to pentagonal tilings of the sphere.

In the discussion below, we call a vertex to be of *$\alpha\beta\gamma$-type* if it is $\alpha'\beta'\gamma'$ for some distinct angles α', β', γ'. A degree 3 vertex can also be of $\alpha\beta^2$-type or α^3-type. There are altogether three types of degree 3 vertices.

Case (1)**.** There is only one angle α at degree 3 vertices.

The only AVC is

$$\{\alpha^3\}.$$

Case (2)**.** There are two distinct angles α and β at degree 3 vertices.

The degree 3 vertices must be $\alpha^3, \alpha^2\beta, \alpha\beta^2, \beta^3$. If any two appear simultaneously, then the angle sum equations imply $\alpha = \beta$. In order for both distinct α, β to appear, therefore, either $\alpha^2\beta$ or $\alpha\beta^2$ is a vertex, and is the only degree 3 vertex. Up to the symmetry of exchanging α and β, we get the only AVC

$$\{\alpha\beta^2\}.$$

The AVC does not allow optional degree 3 vertices.

Case (3)**.** There are three distinct angles α, β and γ at degree 3 vertices.

If $\alpha\beta\gamma$ is a vertex, then all three angles already appear and we get the necessary part $\{\alpha\beta\gamma\}$. We cannot have any optional $\alpha\beta^2$-type vertex because the corresponding angle sum equations will force some angles to become equal. On the other hand, it is possible for one of $\alpha^3, \beta^3, \gamma^3$ to appear while still keeping α, β, γ distinct. But simultaneous appearance of two of $\alpha^3, \beta^3, \gamma^3$ will force the corresponding angles to become equal. Up to symmetry, therefore, we get the only AVC with the optional part

$$\{\alpha\beta\gamma \mid \alpha^3\}.$$

Now we assume that there are no $\alpha\beta\gamma$-type vertices, and there are $\alpha\beta^2$-type vertices. Up to symmetry, we may assume that $\alpha\beta^2$ is a vertex. Then γ must appear as $\alpha^2\gamma$, $\beta\gamma^2$ or γ^3, without forcing some angles to become equal. Since $\{\alpha\beta^2, \alpha^2\gamma\}$ can be transformed to $\{\alpha\beta^2, \beta\gamma^2\}$ via $\alpha \to \beta \to \gamma \to \alpha$, up to symmetry, we get two possible necessary parts

$$\{\alpha\beta^2, \alpha^2\gamma\}, \quad \{\alpha\beta^2, \gamma^3\}.$$

It can be easily verified that neither allow optional vertices.

Finally we assume that there are no $\alpha\beta\gamma$-type and $\alpha\beta^2$-type vertices. In other words, only α^3, β^3 and γ^3 can appear. There is no way for all three angles to appear in this way without forcing them to become equal. So we get no AVC.

Case (4)**.** There are four distinct angles α, β, γ and δ at degree 3 vertices.

If $\alpha\beta\gamma$ is a vertex, then up to symmetry, the angle δ must appear as $\alpha\delta^2$, $\alpha^2\delta$ or δ^3. This gives three possible necessary parts

$$\{\alpha\beta\gamma, \alpha\delta^2\}, \quad \{\alpha\beta\gamma, \alpha^2\delta\}, \quad \{\alpha\beta\gamma, \delta^3\}.$$

The appearance of $\alpha\beta\gamma$ excludes any other optional vertices of $\alpha\beta\gamma$-type. The necessary part $\{\alpha\beta\gamma, \alpha\delta^2\}$ only allows $\beta^2\delta$, $\gamma^2\delta$, β^3, γ^3 to be optional vertices, and the simultaneous appearance of any two from the four forces some angles to become equal. The necessary part $\{\alpha\beta\gamma, \alpha^2\delta\}$ only allows $\beta\delta^2$, $\gamma\delta^2$, β^3, γ^3 to be optional vertices, and the simultaneous appearance of any two forces some angles to become equal. The necessary part $\{\alpha\beta\gamma, \delta^3\}$ does not allow optional vertices. Therefore up to symmetry, we get five possible AVCs

$$\{\alpha\beta\gamma, \alpha\delta^2 \mid \beta^2\delta\}, \quad \{\alpha\beta\gamma, \alpha\delta^2 \mid \beta^3\},$$
$$\{\alpha\beta\gamma, \alpha^2\delta \mid \beta\delta^2\}, \quad \{\alpha\beta\gamma, \alpha^2\delta \mid \beta^3\}, \quad \{\alpha\beta\gamma, \delta^3\}.$$

Now we assume that there are no $\alpha\beta\gamma$-type vertices, and $\alpha\beta^2$ is a vertex. Then γ must appear as $\alpha^2\gamma$, $\beta\gamma^2$, $\gamma\delta^2$, $\gamma^2\delta$ or γ^3. Up to symmetry, we may drop $\beta\gamma^2$ and $\gamma^2\delta$. Moreover, for the combinations $\{\alpha\beta^2, \alpha^2\gamma\}$ and $\{\alpha\beta^2, \gamma^3\}$, we need to further consider the way δ appears. Up to symmetry, this leads to six possible necessary parts

$$\{\alpha\beta^2, \alpha^2\gamma, \beta\delta^2\}, \quad \{\alpha\beta^2, \alpha^2\gamma, \gamma^2\delta\}, \quad \{\alpha\beta^2, \alpha^2\gamma, \delta^3\},$$

$$\{\alpha\beta^2, \gamma\delta^2\}, \quad \{\alpha\beta^2, \gamma^3, \alpha^2\delta\}, \quad \{\alpha\beta^2, \gamma^3, \beta\delta^2\}.$$

The first is contained in the fourth via the transformation $\alpha \to \beta \to \gamma \to \alpha$. The second is contained in the fourth, and the fifth becomes the third via $\gamma \leftrightarrow \delta$. The sixth becomes the third via $\alpha \to \gamma \to \delta \to \beta \to \alpha$. So we only need to continue working with the third and the fourth necessary parts.

The third does not allow optional vertices. Under the assumption of no $\alpha\beta\gamma$-type vertices, the fourth only allows $\alpha^2\delta$ and $\beta\gamma^2$ to be optional vertices, and the two optional vertices cannot appear simultaneously. Up to symmetry, we get two possible AVCs

$$\{\alpha\beta^2, \alpha^2\gamma, \delta^3\}, \quad \{\alpha\beta^2, \gamma\delta^2 \mid \alpha^2\delta\}.$$

Finally, it is easy to see that we cannot have all vertices to be of α^3-type.

Case (5)**.** There are five distinct angles α, β, γ, δ, ϵ at degree 3 vertices.

If all vertices are of α^3-type, then all the angles must be equal. Therefore either there are $\alpha\beta\gamma$-type vertices, or there are $\alpha\beta^2$-type vertices. Moreover, given five distinct angles, there can be at most two $\alpha\beta\gamma$-type vertices. This leads to three subcases.

Case (5.1)**.** There are two $\alpha\beta\gamma$-type vertices.

Up to symmetry, we may assume that $\alpha\beta\gamma$ and $\alpha\delta\epsilon$ are all the $\alpha\beta\gamma$-type vertices. This gives one possible necessary part $\{\alpha\beta\gamma, \alpha\delta\epsilon\}$. It remains to find optional vertices, which are of either $\alpha\beta^2$-type or α^3-type.

It is easy to see that the only possible optional vertex involving α is α^3.

We look for optional vertices of $\alpha\beta^2$-type. Since such a vertex cannot involve α, up to symmetry (preserving the collection $\{\alpha\beta\gamma, \alpha\delta\epsilon\}$), such an optional vertex is $\beta\delta^2$. Next we ask whether $\{\alpha\beta\gamma, \alpha\delta\epsilon, \beta\delta^2\}$ (i.e., assuming all three vertices appear) allows any further optional vertices of $\alpha\beta^2$-type. It is easy to see that the only possibilities are $\beta^2\epsilon$, $\gamma\epsilon^2$, $\gamma^2\delta$, $\gamma^2\epsilon$. Up to symmetry, we only need to consider the combinations $\{\alpha\beta\gamma, \alpha\delta\epsilon, \beta\delta^2, \beta^2\epsilon\}$, $\{\alpha\beta\gamma, \alpha\delta\epsilon, \beta\delta^2, \gamma\epsilon^2\}$ or $\{\alpha\beta\gamma, \alpha\delta\epsilon, \beta\delta^2, \gamma^2\epsilon\}$, and the question becomes whether any of the three allows further optional vertices. It turns out that all three do not allow any further optional vertices of $\alpha\beta^2$-type. As for further optional vertices of α^3-type, only the second one allows α^3, and $\{\alpha\beta\gamma, \alpha\delta\epsilon, \beta\delta^2, \gamma\epsilon^2, \alpha^3\}$ does not allow any more optional vertices. So altogether we get three possible AVCs

$$\{\alpha\beta\gamma, \alpha\delta\epsilon \mid \beta\delta^2, \beta^2\epsilon\}, \quad \{\alpha\beta\gamma, \alpha\delta\epsilon \mid \beta\delta^2, \gamma\epsilon^2, \alpha^3\}, \quad \{\alpha\beta\gamma, \alpha\delta\epsilon \mid \beta\delta^2, \gamma^2\epsilon\}.$$

Having exhausted optional vertices of $\alpha\beta^2$-type for $\{\alpha\beta\gamma, \alpha\delta\epsilon, \beta\delta^2\}$, we still need to consider optional α^3-type vertices, which can only be α^3, γ^3 or ϵ^3. Since α^3 is already included in the second AVC above, we get two more possible AVCs

$$\{\alpha\beta\gamma, \alpha\delta\epsilon \mid \beta\delta^2, \gamma^3\}, \quad \{\alpha\beta\gamma, \alpha\delta\epsilon \mid \beta\delta^2, \epsilon^3\}.$$

Finally, we need to consider the case there are no $\alpha\beta^2$-type vertices, so that the only optional vertices are of α^3-type. Up to symmetry, we get the AVCs $\{\alpha\beta\gamma, \alpha\delta\epsilon \mid \alpha^3\}$ and $\{\alpha\beta\gamma, \alpha\delta\epsilon \mid \beta^3\}$. The first is included in one of the five AVCs above, and the second is also included via $\beta \leftrightarrow \gamma$.

Case (5.2)**.** There is only one $\alpha\beta\gamma$-type vertex.

Up to symmetry, we may assume that $\alpha\beta\gamma$ is the only $\alpha\beta\gamma$-type vertex.

Since δ^3 and ϵ^3 cannot appear simultaneously, one of δ and ϵ must appear in an $\alpha\beta^2$-type vertex. Up to symmetry, we may assume that one of $\alpha\delta^2$, $\alpha^2\delta$, $\delta\epsilon^2$ is a vertex. The first two cases really mean that either δ or ϵ is combined with α, β or γ to form a vertex. So in the third case, we may additionally assume that δ and ϵ are never combined with α, β or γ to form a vertex, which means that $\delta\epsilon^2$ is the only degree 3 vertex involving δ and ϵ.

If $\alpha\beta\gamma$ and $\alpha\delta^2$ are vertices, then under the assumption of no more $\alpha\beta\gamma$-type vertices and up to the symmetry of exchanging β and γ, the angle ϵ must appear as $\alpha^2\epsilon$, $\beta\epsilon^2$, $\beta^2\epsilon$, $\delta\epsilon^2$ or ϵ^3. Similarly, if $\alpha\beta\gamma$ and $\alpha^2\delta$ are vertices, then ϵ must appear as $\alpha\epsilon^2$, $\beta\epsilon^2$, $\beta^2\epsilon$, $\delta^2\epsilon$ or ϵ^3. So we get total of ten combinations in which all angles appear. Up to symmetry, the ten combinations are reduced to eight possible necessary parts, which we divide into four groups

$$\begin{aligned}
&\{\alpha\beta\gamma, \alpha\delta^2, \alpha^2\epsilon\},\\
&\{\alpha\beta\gamma, \alpha\delta^2, \delta\epsilon^2\}, \quad \{\alpha\beta\gamma, \alpha^2\delta, \delta^2\epsilon\},\\
&\{\alpha\beta\gamma, \alpha\delta^2, \beta\epsilon^2\}, \quad \{\alpha\beta\gamma, \alpha\delta^2, \beta^2\epsilon\}, \quad \{\alpha\beta\gamma, \alpha^2\delta, \beta^2\epsilon\},\\
&\{\alpha\beta\gamma, \alpha\delta^2, \epsilon^3\}, \quad \{\alpha\beta\gamma, \alpha^2\delta, \epsilon^3\}.
\end{aligned}$$

Under the assumption of no more $\alpha\beta\gamma$-type vertices, the necessary part $\{\alpha\beta\gamma, \alpha\delta^2, \alpha^2\epsilon\}$ only allows $\beta\epsilon^2$, $\beta^2\delta$, $\gamma\epsilon^2$, $\gamma^2\delta$, $\delta\epsilon^2$, β^3, γ^3 to be optional vertices. Up to the symmetry $\beta \leftrightarrow \gamma$ of the necessary part, the list may be reduced to $\beta\epsilon^2$, $\beta^2\delta$, $\delta\epsilon^2$, β^3. However, the appearance of $\alpha\delta^2, \alpha^2\epsilon, \delta\epsilon^2$ implies that $\alpha = \delta = \epsilon = \frac{2\pi}{3}$, so that the combination should be dismissed. Then we ask whether any of the remaining three combinations allows any one of the original seven to be further optional vertices. Since the answer is always negative, we get three possible AVCs

$$\{\alpha\beta\gamma, \alpha\delta^2, \alpha^2\epsilon \mid \beta\epsilon^2\}, \quad \{\alpha\beta\gamma, \alpha\delta^2, \alpha^2\epsilon \mid \beta^2\delta\}, \quad \{\alpha\beta\gamma, \alpha\delta^2, \alpha^2\epsilon \mid \beta^3\}.$$

The necessary part $\{\alpha\beta\gamma, \alpha\delta^2, \delta\epsilon^2\}$ only allows $\beta^2\epsilon$, $\gamma^2\epsilon$, β^3, γ^3 to be optional vertices. Up to symmetry, the list may be reduced to $\beta^2\epsilon$, β^3. The two combinations do not allow any further optional vertices, and we get two possible AVCs

$$\{\alpha\beta\gamma, \alpha\delta^2, \delta\epsilon^2 \mid \beta^2\epsilon\}, \quad \{\alpha\beta\gamma, \alpha\delta^2, \delta\epsilon^2 \mid \beta^3\}.$$

Applying similar argument to $\{\alpha\beta\gamma, \alpha^2\delta, \delta^2\epsilon\}$ gives two more possible AVCs

$$\{\alpha\beta\gamma, \alpha^2\delta, \delta^2\epsilon \mid \beta^2\epsilon\}, \quad \{\alpha\beta\gamma, \alpha^2\delta, \delta^2\epsilon \mid \beta^3\}.$$

The necessary part $\{\alpha\beta\gamma, \alpha\delta^2, \beta\epsilon^2\}$ only allows $\alpha^2\epsilon$, $\beta^2\delta$, $\gamma^2\delta$, $\gamma^2\epsilon$, γ^3 to be optional vertices. Up to symmetry, the list may be reduced to $\alpha^2\epsilon$, $\gamma^2\delta$, γ^3. The three combinations do not allow any further optional vertices, and we get three possible AVCs

$$\{\alpha\beta\gamma, \alpha\delta^2, \beta\epsilon^2 \mid \alpha^2\epsilon\}, \quad \{\alpha\beta\gamma, \alpha\delta^2, \beta\epsilon^2 \mid \gamma^2\delta\}, \quad \{\alpha\beta\gamma, \alpha\delta^2, \beta\epsilon^2 \mid \gamma^3\}.$$

The necessary part $\{\alpha\beta\gamma, \alpha\delta^2, \beta^2\epsilon\}$ only allows $\gamma\epsilon^2$, $\gamma^2\delta$, γ^3 to be optional vertices. The three combinations do not allow any further optional vertices, and we get three possible AVCs

$$\{\alpha\beta\gamma, \alpha\delta^2, \beta^2\epsilon \mid \gamma\epsilon^2\}, \quad \{\alpha\beta\gamma, \alpha\delta^2, \beta^2\epsilon \mid \gamma^2\delta\}, \quad \{\alpha\beta\gamma, \alpha\delta^2, \beta^2\epsilon \mid \gamma^3\}.$$

The necessary part $\{\alpha\beta\gamma, \alpha^2\delta, \beta^2\epsilon\}$ only allows $\alpha\epsilon^2$, $\beta\delta^2$, $\gamma\delta^2$, $\gamma\epsilon^2$, γ^3 to be optional vertices. Up to symmetry, the list may be reduced to $\alpha\epsilon^2$, $\gamma\delta^2$, γ^3. The three combinations do not allow any further optional vertices, and we get three possible AVCs

$$\{\alpha\beta\gamma, \alpha^2\delta, \beta^2\epsilon \mid \alpha\epsilon^2\}, \quad \{\alpha\beta\gamma, \alpha^2\delta, \beta^2\epsilon \mid \gamma\delta^2\}, \quad \{\alpha\beta\gamma, \alpha^2\delta, \beta^2\epsilon \mid \gamma^3\}.$$

The necessary part $\{\alpha\beta\gamma, \alpha\delta^2, \epsilon^3\}$ only allows $\beta^2\delta$, $\gamma^2\delta$ to be optional vertices. Up to symmetry, we only need to consider $\beta^2\delta$. The combination does not allow any further optional vertices, and we get one possible AVC

$$\{\alpha\beta\gamma, \alpha\delta^2, \epsilon^3 \mid \beta^2\delta\}.$$

Applying similar argument to $\{\alpha\beta\gamma, \alpha^2\delta, \epsilon^3\}$ gives another possible AVC

$$\{\alpha\beta\gamma, \alpha^2\delta, \epsilon^3 \mid \beta\delta^2\}.$$

It remains to consider the case $\delta\epsilon^2$ is the only $\alpha\beta^2$-type vertex involving δ and ϵ. The optional vertices allowed by the necessary part $\{\alpha\beta\gamma, \delta\epsilon^2\}$ can only involve α, β, γ. By the discussion about the AVC of three distinct angles with the necessary part $\{\alpha\beta\gamma\}$ in **Case** (3), we get one possible AVC

$$\{\alpha\beta\gamma, \delta\epsilon^2 \mid \alpha^3\}.$$

Case (5.3)**.** There are no $\alpha\beta\gamma$-type vertices.

Since there can be at most one α^3-type vertex, we may assume that $\alpha^3, \beta^3, \gamma^3, \delta^3$ are not vertices. Up to the symmetry of exchanging $\alpha, \beta, \gamma, \delta$ (and under the assumption of no $\alpha\beta\gamma$-type vertices), we may assume that $\alpha\beta^2$ and $\gamma\delta^2$ are vertices. Moreover, up to the symmetry of $\alpha \leftrightarrow \gamma$ and $\beta \leftrightarrow \delta$, we may further assume that ϵ appears as $\alpha^2\epsilon$, $\beta\epsilon^2$ or ϵ^3. This gives three possible necessary parts

$$\{\alpha\beta^2, \gamma\delta^2, \alpha^2\epsilon\}, \quad \{\alpha\beta^2, \gamma\delta^2, \beta\epsilon^2\}, \quad \{\alpha\beta^2, \gamma\delta^2, \epsilon^3\}.$$

Since the second becomes the first via $\alpha \to \epsilon \to \beta \to \alpha$, we only need to consider the first and the third.

Under the assumption of no $\alpha\beta\gamma$-type vertices, the necessary part $\{\alpha\beta^2, \gamma\delta^2, \alpha^2\epsilon\}$ only allows $\beta\gamma^2$, $\delta\epsilon^2$ to be optional vertices, and the two cannot appear simultaneously. Therefore we get two possible AVCs

$$\{\alpha\beta^2, \gamma\delta^2, \alpha^2\epsilon \mid \beta\gamma^2\}, \quad \{\alpha\beta^2, \gamma\delta^2, \alpha^2\epsilon \mid \delta\epsilon^2\},$$

The necessary part $\{\alpha\beta^2, \gamma\delta^2, \epsilon^3\}$ only allows $\alpha^2\delta$, $\beta\gamma^2$ to be optional vertices, and the two cannot appear simultaneously. Up to symmetry, we get one possible AVC

$$\{\alpha\beta^2, \gamma\delta^2, \epsilon^3 \mid \alpha^2\delta\}.$$

The following is the summary of our discussion.

Theorem 1. *For tilings of any surface with at most five distinct angles at degree* 3 *vertices, the anglewise vertex combinations at degree* 3 *vertices are classified by Table 1. Specifically, for any one such tiling, after suitable relabeling of the distinct angles, there is an AVC from the table, such that the collection of angle combinations at degree* 3 *vertices of the tiling contains the necessary part of the AVC and is contained in the whole (necessary plus optional parts) AVC.*

<table>
<tr><th colspan="3">Necessary</th><th>Optional</th></tr>
<tr><td>α^3</td><td colspan="2"></td><td></td></tr>
<tr><td>$\alpha\beta^2$</td><td colspan="2"></td><td></td></tr>
<tr><td colspan="2">$\alpha\beta\gamma$</td><td rowspan="3"></td><td>α^3</td></tr>
<tr><td rowspan="2">$\alpha\beta^2$</td><td>$\alpha^2\gamma$</td><td rowspan="2"></td></tr>
<tr><td>γ^3</td></tr>
<tr><td colspan="2" rowspan="5">$\alpha\beta\gamma$</td><td rowspan="2">$\alpha\delta^2$</td><td>$\beta^2\delta$</td></tr>
<tr><td>β^3</td></tr>
<tr><td rowspan="2">$\alpha^2\delta$</td><td>$\beta\delta^2$</td></tr>
<tr><td>β^3</td></tr>
<tr><td>δ^3</td><td></td></tr>
<tr><td rowspan="2">$\alpha\beta^2$</td><td colspan="2">$\gamma\delta^2$</td><td>$\alpha^2\delta$</td></tr>
<tr><td>$\alpha^2\gamma$</td><td>δ^3</td><td></td></tr>
<tr><td rowspan="8">$\alpha\beta\gamma$</td><td colspan="2" rowspan="5">$\alpha\delta\epsilon$</td><td>$\beta\delta^2, \beta^2\epsilon$</td></tr>
<tr><td>$\beta\delta^2, \gamma\epsilon^2, \alpha^3$</td></tr>
<tr><td>$\beta\delta^2, \gamma^2\epsilon$</td></tr>
<tr><td>$\beta\delta^2, \gamma^3$</td></tr>
<tr><td>$\beta\delta^2, \epsilon^3$</td></tr>
<tr><td rowspan="3">$\alpha\delta^2$</td><td rowspan="3">$\alpha^2\epsilon$</td><td>$\beta\epsilon^2$</td></tr>
<tr><td>$\beta^2\delta$</td></tr>
<tr><td>β^3</td></tr>
</table>

<table>
<tr><th colspan="3">Necessary</th><th>Optional</th></tr>
<tr><td rowspan="16">$\alpha\beta\gamma$</td><td rowspan="9">$\alpha\delta^2$</td><td rowspan="3">$\beta\epsilon^2$</td><td>$\alpha^2\epsilon$</td></tr>
<tr><td>$\gamma^2\delta$</td></tr>
<tr><td>γ^3</td></tr>
<tr><td rowspan="3">$\beta^2\epsilon$</td><td>$\gamma\epsilon^2$</td></tr>
<tr><td>$\gamma^2\delta$</td></tr>
<tr><td>γ^3</td></tr>
<tr><td rowspan="2">$\delta\epsilon^2$</td><td>$\beta^2\epsilon$</td></tr>
<tr><td>β^3</td></tr>
<tr><td>ϵ^3</td><td>$\beta^2\delta$</td></tr>
<tr><td rowspan="6">$\alpha^2\delta$</td><td rowspan="3">$\beta^2\epsilon$</td><td>$\alpha\epsilon^2$</td></tr>
<tr><td>$\gamma\delta^2$</td></tr>
<tr><td>γ^3</td></tr>
<tr><td rowspan="2">$\delta^2\epsilon$</td><td>$\beta^2\epsilon$</td></tr>
<tr><td>β^3</td></tr>
<tr><td>ϵ^3</td><td>$\beta\delta^2$</td></tr>
<tr><td colspan="2">$\delta\epsilon^2$</td><td>α^3</td></tr>
<tr><td colspan="2" rowspan="3">$\alpha\beta^2, \gamma\delta^2$</td><td rowspan="2">$\alpha^2\epsilon$</td><td>$\beta\gamma^2$</td></tr>
<tr><td>$\delta\epsilon^2$</td></tr>
<tr><td>ϵ^3</td><td>$\alpha^2\delta$</td></tr>
</table>

Table 1: Anglewise vertex combinations at degree 3 vertices.

Note that the argument leading to the table follows a specific sequence of cases, and the discussion of a case assumes the exclusion of the earlier cases. Following a different sequence of cases may lead to a different table, and not excluding earlier cases may introduce many overlappings between various cases.

3 Combinatorics of Spherical Pentagonal Tiling

Theorem 1 describes all the possible angle combinations at degree 3 vertices. For each combination in Theorem 1, we then try to further find angle combinations at vertices of degree ≥ 4, which we call *high degree vertices*. Unlike Theorem 1, we will only concentrate on tilings of the sphere by angle congruent pentagons. Moreover, we only consider tilings given naturally by embedded graphs, so that the tilings are edge-to-edge and all vertices have degree ≥ 3.

Let v, e, f be the numbers of vertices, edges and tiles in a spherical pentagonal tiling. Let v_k be the number of vertices of degree k. Then we have

$$v - e + f = 2, \quad v = v_3 + v_4 + \cdots, \quad 5f = 2e = 3v_3 + 4v_4 + \cdots. \tag{3.1}$$

It is easy to deduce (see [4, page 750], for example) that

$$v_3 = 20 + \sum_{k\geq 4}(3k-10)v_k, \quad \frac{f}{2} - 6 = \sum_{k\geq 4}(k-3)v_k. \tag{3.2}$$

The two equalities are easily equivalent by the third equality in (3.1). We call either equality the *vertex counting equation*.

The vertex counting equation implies that f is even and $f \geq 12$. Since spherical pentagonal tilings are completely understood for $f = 12$ by [1, 4], and we cannot have $f = 14$ by [8], we will assume that f is even and $f \geq 16$ throughout this paper.

For angle congruent tilings, the vertex counting equation also implies the following *angle sum equation for the pentagon*.

Lemma 2. *In a spherical tiling by f angle congruent pentagons, the sum of five angles in the pentagon is $3\pi + \frac{4\pi}{f}$.*

If the tiles are geometrically congruent, then the tiles have equal area $\frac{4\pi}{f}$. Since the area of a spherical pentagon (with great arc edges) is given by the sum of five angles subtracting 3π, we have the angle sum equation for the pentagon as stated in the lemma. What the lemma says is that the angle congruence (which is weaker than the geometrical congruence) is sufficient for the angle sum equation to hold.

Proof. Since the sum of angles at each vertex is 2π, the total sum of all angles is $2\pi v$. Since the angle congruence implies that the sum Σ of five angles is the same for all the tiles, the total sum of all angles is also $f\Sigma$. Therefore we have $2\pi v = f\Sigma$. It is also easy to derive $3f = 2v - 4$ from (3.1). Then we get

$$\Sigma = 2\pi\frac{v}{f} = 3\pi + \frac{4\pi}{f}. \qquad \square$$

By the vertex counting equation, we find that the way an angle appears (or not appears) at degree 3 vertices imposes constraints on the way the angle appears in the pentagon.

Lemma 3. *Suppose in a spherical tiling by angle congruent pentagons, an angle appears at every degree* 3 *vertex. Then the angle must appear at least twice in the pentagon.*

Proof. If an angle θ appears only once in the pentagon, then the total number of times θ appears in the whole tiling is f, and the total number of non-θ vertices is $4f$. If we also know that θ appears at every degree 3 vertex, then $f \geq v_3$ and the non-θ vertices appear $\leq 2v_3$ times at degree 3 vertices. Moreover, the non-θ vertices appear $\leq \sum_{k\geq 4} kv_k$ times at high degree vertices. Therefore we get

$$4v_3 \leq 4f \leq 2v_3 + \sum_{k\geq 4} kv_k.$$

This implies

$$v_3 \leq \sum_{k\geq 4} \frac{1}{2} kv_k.$$

Since $k \geq 4$ implies $\frac{1}{2}k \leq 3k - 10$, we get a contradiction to the first equality in (3.2). □

Lemma 4. *Suppose in a spherical tiling by angle congruent pentagons, an angle θ does not appear at degree* 3 *vertices.*

1. *There can be at most one such angle θ.*
2. *The angle θ appears only once in the pentagon.*
3. $2v_4 + v_5 \geq 12$.
4. *One of $\alpha\theta^3$, θ^4, θ^5 is a vertex, where $\alpha \neq \theta$.*

The first statement implies that the angle α in the fourth statement must appear at a degree 3 vertex.

Proof. Suppose two angles θ_1 and θ_2 do not appear at degree 3 vertices. Then the total number of times these two angles appear is at least $2f$, and is at most the total number $\sum_{k\geq 4} kv_k$ of angles at high degree vertices. Therefore we have $2f \le \sum_{k\ge 4} kv_k$. Since this contradicts with (the first equality uses the vertex counting equation)

$$\begin{aligned} 2f - \sum_{k\ge 4} kv_k &= 2\left(12 + 2\sum_{k\ge 4}(k-3)v_k\right) - \sum_{k\ge 4} kv_k \\ &= 24 + \sum_{k\ge 4} 3(k-4)v_k, \end{aligned}$$

the first statement is proved.

The argument above also applies to the case $\theta_1 = \theta_2$, which means the same angle appearing at least twice in the pentagon. Therefore the second statement is also proved.

The first two statements imply that the angle θ appears exactly f times. Since this should be no more than the total number $\sum_{k\ge 4} kv_k$ of angles at high degree vertices, we get

$$\begin{aligned} f - \sum_{k\ge 4} kv_k &= \left(12 + 2\sum_{k\ge 4}(k-3)v_k\right) - \sum_{k\ge 4} kv_k \\ &= 12 - 2v_4 - v_5 + \sum_{k\ge 6}(k-6)v_k \le 0. \end{aligned}$$

By $\sum_{k\ge 6}(k-6)v_k \ge 0$, we get the third statement.

For the last statement, we assume that $\alpha\theta^3, \theta^4, \theta^5$ are not vertices. The assumption means that θ appears at most twice at any degree 4 vertex, and at most four times at any degree 5 vertex. Since θ also does not appear at degree 3 vertices, the total number of times θ appears is $\le 2v_4 + 4v_5 + \sum_{k\ge 6} kv_k$. However, the number of times θ appears should also be f. Then we get the following contradiction

$$f = 12 + 2\sum_{k\ge 4}(k-3)v_k \le 2v_4 + 4v_5 + \sum_{k\ge 6} kv_k. \qquad \square$$

Lemmas 3 and 4 can be used to eliminate many combinations of angles in the pentagon and at degree 3 vertices. The following gives the complete answer for up to three distinct angles at degree 3 vertices.

Proposition 5. *For tilings of the sphere by more than* 12 *angle congruent pentagons with up to three distinct angles at degree* 3 *vertices, the combinations of angles in the pentagon and angles at all degree* 3 *vertices are, up to relabeling of angles, given by Table 2.*

all deg 3 vertices	possible pentagons
α^3	$\alpha^4\beta$
$\alpha\beta^2$	$\alpha^2\beta^3, \alpha^3\beta^2$
	$\alpha^2\beta^2\gamma$
$\alpha\beta\gamma, \alpha^3$	$\alpha^3\beta\gamma, \alpha^2\beta^2\gamma$
	$\alpha^2\beta\gamma\delta$
$\alpha\beta^2, \alpha^2\gamma$	$\alpha^3\beta\gamma, \alpha^2\beta^2\gamma, \alpha^2\beta\gamma^2$
	$\alpha^2\beta\gamma\delta$
$\alpha\beta^2, \gamma^3$	$\alpha^2\beta^2\gamma, \alpha^2\beta\gamma^2, \alpha\beta^3\gamma, \alpha\beta\gamma^3$
	$\alpha\beta^2\gamma\delta, \alpha\beta\gamma^2\delta$

Table 2: AVCs with angles in the pentagon and at all degree 3 vertices.

The left column of the table is the complete collections of angle combinations at degree 3 vertices. The right column is split into the case of all angles appearing at degree 3 vertices and the case one angle not appearing at degree 3 vertices. For example, if there are two distinct angles at degree 3 vertices, then the table tells us that, up to relabeling, the AVC is one of the following six

$$\{\alpha^2\beta^3\colon \alpha\beta^2\},\quad \{\alpha^3\beta^2\colon \alpha\beta^2\},\quad \{\alpha^2\beta^2\gamma\colon \alpha\beta^2, \alpha\delta^3\},$$
$$\{\alpha^2\beta^2\gamma\colon \alpha\beta^2, \beta\delta^3\},\quad \{\alpha^2\beta^2\gamma\colon \alpha\beta^2, \delta^4\},\quad \{\alpha^2\beta^2\gamma\colon \alpha\beta^2, \delta^5\}.$$

We get the necessary vertices $\alpha\delta^3$, $\beta\delta^3$, δ^4, δ^5 in the last four AVCs with the help of Lemma 4.

Proof. The proof follows the five AVCs in Table 1 for up to three distinct angles. We need to consider two cases

1. All angles appear at degree 3 vertices.

2. One angle does not appear at degree 3 vertices.

Consider $\{\alpha^3\}$ from the first row of Table 2. In the first case, the pentagon must be α^5, and the angle sum equations

$$5\alpha = 3\pi + \frac{4}{f}\pi, \quad 3\alpha = 2\pi,$$

imply $f = 12$. This is dismissed because we only consider $f \geq 16$. In the second case, let β be the extra angle appearing only at high degree vertices.

By Lemma 4, we have $\{\alpha^3, \alpha\beta^3\}$, $\{\alpha^3, \beta^4\}$ or $\{\alpha^3, \beta^5\}$, with the corresponding $\alpha = \frac{2}{3}\pi$ and $\beta = \frac{4}{9}\pi, \frac{1}{2}\pi, \frac{2}{5}\pi$. Then we consider the combinations $\alpha^n\beta^{5-n}$ for the pentagon such that the angle sum equation $n\alpha+(5-n)\beta = 3\pi+\frac{4}{f}\pi$ yields even $f \geq 16$. The only possibility is $\alpha^4\beta$, which corresponds to $f = 36, 24, 60$ for the three choices of β.

Consider $\{\alpha\beta^2\}$ from the second row of Table 1. Since both α and β appear at every degree 3 vertex, Lemma 3 implies that both α and β appear at least twice in the pentagon. In the first case, this means that the pentagon is either $\alpha^2\beta^3$ or $\alpha^3\beta^2$. In the second case, if γ is the extra angle at high degree vertices, then this means that the pentagon is $\alpha^2\beta^2\gamma$.

Next we assume three angles α, β, γ appearing at degree 3 vertices. In the second case, we denote by δ the extra angle at high degree vertices.

Consider $\{\alpha\beta\gamma \mid \alpha^3\}$ from Table 1. We note that $\alpha\beta\gamma$ cannot be the only degree 3 vertex because Lemma 3 would then imply that there are at least six angles (two each of α, β, γ) in the pentagon. Therefore the AVC becomes $\{\alpha\beta\gamma, \alpha^3\}$ (the devider $\mid$ is dropped to indicate the necessary appearance of α^3). Since the AVC contains all the degree 3 vertices, we see that α appears at every degree 3 vertex. By Lemma 3, therefore, α appears at least twice in the pentagon. In the first case, up to the symmetry of exchanging β and γ, this means that the pentagon is $\alpha^3\beta\gamma$ or $\alpha^2\beta^2\gamma$. In the second case, the pentagon must be $\alpha^2\beta\gamma\delta$.

Consider $\{\alpha\beta^2, \alpha^2\gamma\}$ from Table 1. Since α appears at every degree 3 vertex, Lemma 3 implies that α appears at least twice in the pentagon. In the first case, this means that the pentagon is $\alpha^3\beta\gamma, \alpha^2\beta^2\gamma$ or $\alpha^2\beta\gamma^2$. In the second case, the pentagon must be $\alpha^2\beta\gamma\delta$.

Consider $\{\alpha\beta^2, \gamma^3\}$ from Table 1. We claim that β and γ together must appear at least three times in the pentagon. If not, then they appear once each in the pentagon, so that the total number of β and γ is $2f$. From the AVC $\{\alpha\beta^2, \gamma^3\}$ that contains all the degree 3 vertices, we get $2f \geq 2v_3$. On the other hand, the non-(β, γ) angles appear three times in the pentagon, so that the total number of non-(β, γ) angles is $3f$. But we also know that the non-(β, γ) angles appear $\leq v_3$ times at degree 3 vertices, and appear $\leq \sum_{k\geq 4} kv_k$ times at high degree vertices. Therefore we get

$$3v_3 \leq 3f \leq v_3 + \sum_{k\geq 4} kv_k.$$

contradicting to the first equality in (3.2).

Once the claim is established, then it is easy to see that in the first case, the pentagon is $\alpha^2\beta^2\gamma$, $\alpha^2\beta\gamma^2$, $\alpha\beta^3\gamma$, $\alpha\beta^2\gamma^2$ or $\alpha\beta\gamma^3$, and in the second case, the pentagon is $\alpha\beta^2\gamma\delta$ or $\alpha\beta\gamma^2\delta$. However, the angle sum formula for the pentagon $\alpha\beta^2\gamma^2$ gives $3\pi + \frac{4\pi}{f} = \alpha + 2\beta + 2\gamma = 2\pi + 2\frac{2\pi}{3}$, so that $f = 12$, and the case is dismissed. □

4 AVCs with Finitely Many f

Using Proposition 5 as the starting point, we try to find all the angle combinations at the other (necessarily high degree) vertices. We may also calculate the numbers of tiles and vertices.

Let us consider the AVC $\{\alpha^2\beta\gamma\delta\colon \alpha\beta\gamma, \alpha^3, \beta\delta^3\}$, which means that α, α, β, γ, δ are the angles of the pentagon, $\alpha\beta\gamma$ and α^3 are all the degree 3 vertices, and $\beta\delta^3$ is also a vertex.

The angle sum equations for the pentagon $\alpha^2\beta\gamma\delta$ (Lemma 2) and at the vertices $\alpha\beta\gamma, \alpha^3, \beta\delta^3$ are

$$2\alpha + \beta + \gamma + \delta = 3\pi + \frac{4\pi}{f}, \quad \alpha + \beta + \gamma = 3\alpha = \beta + 3\delta = 2\pi.$$

Solving the equations, we get

$$\alpha = \frac{2}{3}\pi,\ \beta = \left(1 - \frac{12}{f}\right)\pi,\ \gamma = \left(\frac{1}{3} + \frac{12}{f}\right)\pi,\ \delta = \left(\frac{1}{3} + \frac{4}{f}\right)\pi.$$

By $f \geq 16$, all angles are positive. For the angles to be distinct, we also require $f \neq 24, 36$.

Besides the existing three, any other vertex $\alpha^a\beta^b\gamma^c\delta^d$ is given by a quadruple (a, b, c, d) of non-negative integers satisfying the angle sum equation

$$2 = \frac{2}{3}a + \left(1 - \frac{12}{f}\right)b + \left(\frac{1}{3} + \frac{12}{f}\right)c + \left(\frac{1}{3} + \frac{4}{f}\right)d,$$

and the high degree requirement

$$a + b + c + d \geq 4.$$

The angle sum equation can be rephrased as an expression for f

$$f = \frac{12(-3b + 3c + d)}{2a + 3b + c + d - 6}.$$

By $f \geq 16$, the angle sum equation also implies

$$2 \geq \frac{2}{3}a + \frac{1}{4}b + \frac{1}{3}c + \frac{1}{3}d.$$

We substitute the finitely many quadruples of non-negative integers satisfying the inequality above and the high degree requirement into the formula for f. We keep only those quadruples yielding even integers $f \geq 16$ that are not 24 and 36. They are listed in Table 3.

Note that we should also consider the possibility that the substitution yields $f = \frac{0}{0}$, which means $3b = 3c + d$ and $2a + 3b + c + d = 6$. Since this violates the high degree requirement, we are not concerned.

number	vertex $\alpha^a\beta^b\gamma^c\delta^d$				f
	a	b	c	d	
x_1	1	1	1	0	
x_2	3	0	0	0	
x_3	0	1	0	3	
y_1	0	0	2	2	48
y_1	1	0	1	2	60
y_2	0	0	3	1	
y_3	0	0	0	5	
y_1	0	0	4	0	72
y_1	1	0	3	0	84
y_2	0	0	2	3	
y_1	1	0	2	1	108
y_2	0	0	1	4	
y_1	0	0	3	2	132
y_1	0	0	4	1	156
y_1	0	0	0	5	180

Table 3: Vertices for $\{\alpha^2\beta\gamma\delta\colon \alpha\beta\gamma, \alpha^3, \beta\delta^3\}$.

Next we find the number of each vertex. In the table, x_1, x_2, x_3 are the numbers of existing $\alpha\beta\gamma, \alpha^3, \beta\delta^3$. For $f = 48$, y_1 is the number of $\gamma^2\delta^2$, and $\alpha\beta\gamma, \alpha^3, \beta\delta^3, \gamma^2\delta^2$ are all the vertices. Since the pentagon is $\alpha^2\beta\gamma\delta$, the total numbers of $\alpha, \beta, \gamma, \delta$ in the tiling are respectively $2 \times 48, 48, 48, 48$. On the other hand, we may count the total numbers from the viewpoint of the vertices. This gives the *angle counting equations*

$$\begin{aligned} 2\times 48 &= x_1+3x_2, \\ 48 &= x_1+x_3, \\ 48 &= x_1+2y_1, \\ 48 &= 3x_3+2y_1. \end{aligned}$$

The unique solution $x_1=36, x_2=20, x_3=12, y_1=6$ gives the full AVC (the specific values of the angles are obtained by substituting $f=48$)

$$\begin{aligned} &\{48\alpha^2\beta\gamma\delta\colon 36\alpha\beta\gamma,\ 20\alpha^3,\ 12\beta\delta^3 \mid 6\gamma^2\delta^2\}, \\ &\qquad \alpha=\tfrac{2}{3}\pi,\ \beta=\tfrac{3}{4}\pi,\ \gamma=\tfrac{7}{12}\pi,\ \delta=\tfrac{5}{12}\pi. \end{aligned}$$

For $f=60$, the angle counting equations are

$$\begin{aligned} 2\times 60 &= x_1+3x_2+y_1, \\ 60 &= x_1+x_3, \\ 60 &= x_1+y_1+3y_2, \\ 60 &= 3x_3+2y_1+y_2+5y_3. \end{aligned}$$

The solution is

$$x_1=60-y_1-3y_2,\ x_2=20+y_2,\ x_3=y_1+3y_2,\ y_1+2y_2+y_3=12.$$

The last equation is obtained by eliminating x_1, x_2, x_3 from the four equations, and is satisfied by finitely many (total number $=14^2=196$) triples (y_1, y_2, y_3) of non-negative integers. Among these triples, we only choose those yielding positive integers x_1, x_2, x_3 (total number $=14^2-18=178$). We denote the full AVC as (with the implicit understanding on the choices of (y_1, y_2, y_3))

$$\begin{aligned} &\{60\alpha^2\beta\gamma\delta\colon (48-y_2+3y_3)\alpha\beta\gamma,\ (20+y_2)\alpha^3,\ (12+y_2-3y_3)\beta\delta^3 \\ &\qquad \mid (12-2y_2-y_3)\alpha\gamma\delta^2,\ y_2\gamma^3\delta,\ y_3\delta^5\}, \\ &\qquad \alpha=\tfrac{2}{3}\pi,\ \beta=\tfrac{4}{5}\pi,\ \gamma=\tfrac{18}{15}\pi,\ \delta=\tfrac{2}{5}\pi. \end{aligned}$$

The full AVCs for all the f in Table 3 are given on page 43.

Strictly speaking, we still need to consider the possibility that, for f not listed in the table, the existing vertices may already form a viable full AVC.

This means solving the angle counting equations

$$\begin{aligned}2f &= x_1 + 3x_2,\\ f &= x_1 + x_3,\\ f &= x_1,\\ f &= 3x_3.\end{aligned}$$

Of course, the earlier solution for $f = 48$ implies that the system has no solution (because $y_1 \neq 0$ for $f = 48$).

Now we describe the general process. For an angle combination $\alpha^a\beta^b\gamma^c\cdots$, we call the collection of orders $(a, b, c, \dots)^T$ its *order vector* (we always write order vectors vertically). In the example above, the pentagon $\alpha^2\beta\gamma\delta$ has the order vector $P = (2, 1, 1, 1)^T$, and the existing vertices $\alpha\beta\gamma, \alpha^3, \beta\delta^3$ have the order vectors $X_1 = (1, 1, 1, 0)^T, X_2 = (3, 0, 0, 0)^T, X_3 = (0, 1, 0, 3)^T$. The given data form an *order matrix*

$$(P\ X) = \begin{pmatrix} 2 & 1 & 3 & 0\\ 1 & 1 & 0 & 1\\ 1 & 1 & 0 & 0\\ 1 & 0 & 0 & 3 \end{pmatrix}.$$

What is important for the example are the following two facts:

1. The order matrix is invertible.

2. The constant parts of all angles are positive.

Let (the horizontal vector) $\vec{\alpha} = (\alpha, \beta, \gamma, \dots)$ be the distinct angles in the pentagonal tiling. The angle sum equations are

$$\vec{\alpha}P = \left(3 + \frac{4}{f}\right)\pi, \quad \vec{\alpha}X_i = 2\pi.$$

The invertibility of the order matrix implies the unique solution

$$\vec{\alpha} = \vec{\alpha}_0 + \frac{1}{f}\vec{\alpha}_1, \quad \vec{\alpha}_0 = (\alpha_0, \beta_0, \gamma_0, \dots), \quad \vec{\alpha}_1 = (\alpha_1, \beta_1, \gamma_1, \dots),$$

where

$$\vec{\alpha}_0(P\ X) = \pi(3, 2, 2, 2, \dots), \quad \vec{\alpha}_1(P\ X) = \pi(4, 0, 0, 0, \dots).$$

The second fact above means that all the coordinates of $\vec{\alpha}_0$ are positive.

Next we verify the requirement that $\alpha, \beta, \gamma, \dots$ are distinct positive angles, which often means that f cannot take certain finitely many values. Moreover, we may estimate $\alpha = \alpha_0 + \frac{\alpha_1}{f}$ as follows.

1. If $\alpha_0 > 0$, then we have $\alpha_0 + \frac{\alpha_1}{f_0} > 0$ for all but finitely many f.
2. If $\alpha_0 < 0$ and $\alpha_1 > 0$, then $\alpha > 0$ for only finitely many f.
3. If $\alpha_0 = 0$, then $\alpha = \frac{\alpha_1}{f}$ can become arbitrarily small.

Up to the exception of finitely many f, therefore, the last is the only case α has no positive lower bound. The same estimation can be carried out for the other angles.

Next we try to find all the other vertices $\alpha^a\beta^b\gamma^c\cdots$. The order vectors $\vec{v} = (a, b, c, \dots)^T$ satisfy the angle sum equation

$$2\pi = a\alpha + b\beta + c\gamma + \cdots = \vec{\alpha}\vec{v} = \vec{\alpha}_0\vec{v} + \frac{1}{f}\vec{\alpha}_1\vec{v},$$

and the high degree requirement

$$\deg \vec{v} = a + b + c + \cdots \geq 4.$$

Given the positivity of all the coordinates of $\vec{\alpha}_0$, the lower bounds for all the angles translate into the upper bounds for the coordinates of $\vec{v}$. Therefore there are only finitely many choices for $\vec{v}$. We substitute those choices satisfying $\deg \vec{v} \geq 4$ into

$$f = \frac{\vec{\alpha}_1\vec{v}}{2\pi - \vec{\alpha}_0\vec{v}},$$

and keep only those yielding even integers $f \geq 16$ such that all angles are positive and distinct. Then we get finitely many possible f and for each f, finitely many optional vertices.

We need to consider the possibility that $\vec{\alpha}_0\vec{v} = 2\pi$ and $\vec{\alpha}_1\vec{v} = 0$. Since the order matrix is invertible, we may write $\vec{v} = (P\ X)\vec{u}$. Then we get

$$\begin{aligned} 2\pi &= \vec{\alpha}_0\vec{v} = \vec{\alpha}_0(P\ X)\vec{u} = \pi(3, 2, 2, 2, \dots)\vec{u}, \\ 0 &= \vec{\alpha}_1\vec{v} = \vec{\alpha}_1(P\ X)\vec{u} = \pi(4, 0, 0, 0, \dots)\vec{u}. \end{aligned}$$

Therefore $\vec{u} = (0, \lambda_1, \lambda_2, \dots)$, with $\lambda_1 + \lambda_2 + \cdots = 1$. If all the existing vertices (i.e., columns of X) have degree 3, then

$$\vec{v} = (P\ X)\vec{u} = \lambda_1 X_1 + \lambda_2 X_2 + \cdots ,$$

also has degree 3. Due to the high degree requirement, we are not concerned. Our experience also suggests that we should not be concerned when the existing vertices include a high degree vertex given in part 4 of Lemma 4.

For each possible f, the order vectors of the corresponding optional vertices form a matrix Y. The numbers x_i and y_j of all vertices satisfy the angle counting equations

$$fP = X\vec{x} + Y\vec{y}, \quad \vec{x} = (x_1, x_2, \dots)^T, \quad \vec{y} = (y_1, y_2, \dots)^T.$$

Since the order matrix $(P\ X)$ is invertible and f already has a specific value, the solution is an expression of $\vec{x}$ in terms of $\vec{y}$, and one equation for $\vec{y}$ that does not involve $\vec{x}$.

If all the existing vertices have degree 3, then we claim that the equation for $\vec{y}$ is exactly the vertex counting equation. Indeed, multiplying $\vec{\alpha}$ and $(1, 1, \dots)$ respectively to the angle counting equations, we get

$$\begin{aligned}
\left(3 + \frac{4}{f}\right) f\pi &= f\vec{\alpha}P = \vec{\alpha}X\vec{x} + \vec{\alpha}Y\vec{y} \\
&= 2\pi(1, 1, \dots)\vec{x} + 2\pi(1, 1, \dots)\vec{y}, \\
5f &= f(1, 1, \dots)P = (1, 1, \dots)X\vec{x} + (1, 1, \dots)Y\vec{y} \\
&= 3(1, 1, \dots)\vec{x} + (1, 1, \dots)Y\vec{y}.
\end{aligned}$$

Eliminating $\vec{x}$ from the two equalities gives exactly the second equation in (3.2)

$$\frac{f}{2} - 6 = [(1, 1, \dots)Y - 3(1, 1, \dots)]\vec{y}.$$

Therefore the vertex counting equation can be derived from the angle counting equations.

If there is an angle not appearing at degree 3 vertices, then the last column of X is given by part 4 of Lemma 4, and the vertex counting equation also includes the number of this vertex. For example, in the earlier example, for $f = 60$, the vertex counting equation is

$$x_3 + y_1 + y_2 + 3y_3 = \frac{60}{2} - 6 = 24.$$

On the other hand, the last angle counting equation gives us

$$x_3 = 20 - \frac{1}{3}(2y_1 + y_2 + 5y_3).$$

Substituting this into the vertex counting equation above, we get

$$y_1 + 2y_2 + y_3 = 12.$$

This is a general way of getting the relation among y_i.

5 AVCs with Variable f

The routine in Section 4 assumes that the order matrix $(P\ X)$ is invertible, and the constant part of every angle is positive. In this section, we still assume that the order matrix is invertible, but allow one angle to have non-positive constant part. As pointed out in Section 4, if the constant is negative, then there are only finitely many f making the angle positive, and the problem of deriving the full AVC becomes a finite one. So we will only consider the case that the constant is zero for one angle.

Let us consider the AVC $\{\alpha\beta\gamma^2\delta\colon \alpha\beta^2, \gamma^3, \alpha\delta^3\}$, which means that α, β, γ, γ, δ are the angles of the pentagon, $\alpha\beta^2$ and γ^3 are all the degree 3 vertices, and $\alpha\delta^3$ is also a vertex.

The order matrix

$$(P\ X) = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 2 & 2 & 0 & 0 \\ 1 & 0 & 3 & 0 \\ 1 & 0 & 0 & 3 \end{pmatrix}$$

is invertible. The solution of the angle sum equations is

$$\alpha = \frac{24}{f}\pi,\ \beta = \left(1 - \frac{12}{f}\right)\pi,\ \gamma = \frac{2}{3}\pi,\ \delta = \left(\frac{2}{3} - \frac{8}{f}\right)\pi.$$

We note that α has zero constant part. By $f \ge 16$, all angles are positive. For the angles to be distinct, we require $f \ne 36, 48$.

The angle sum equation at a vertex $\alpha^a\beta^b\gamma^c\delta^d$ implies

$$a = \frac{f}{72}(6 - 3b - 2c - 2d) + \frac{1}{6}(3b + 2d), \quad 2 \ge \frac{1}{4}b + \frac{2}{3}c + \frac{1}{6}d.$$

Substituting the finitely many triples (b, c, d) of non-negative integers satisfying the inequality into the expression for a, we get all the possible optional vertices in Tables 4 and 5.

Specifically, if $6 - 3b - 2c - 2d = 0$, then we need to consider the cases that $a = \frac{1}{6}(3b + 2d)$ is a non-negative integer, which would give vertices valid for all f. It turns out that we only get all the existing vertices.

Table 4 consists of those (f, b, c, d) satisfying $6 - 3b - 2c - 2d < 0$, f is an even integer ≥ 16 and $\neq 36, 48$, a is a non-negative integer, and the degree of the vertex is ≥ 4. It turns out we have $a = 0$ in all cases.

number	a	b	c	d	f
y_1	0	2	2	1	$f = 16$
y_2	0	0	2	4	
y_3	0	4	1	2	
y_4	0	8	0	0	
y_5	0	2	1	5	
y_6	0	6	0	3	
y_7	0	0	1	8	
y_8	0	4	0	6	
y_9	0	2	0	9	
y_{10}	0	0	0	12	
y_1	0	2	2	0	$f = 18$
y_2	0	4	1	0	
y_3	0	0	2	3	
y_4	0	6	0	0	
y_5	0	2	1	3	
y_6	0	4	0	3	
y_7	0	0	1	6	
y_8	0	2	0	6	
y_9	0	0	0	9	

number	a	b	c	d	f
y_1	0	1	2	1	$f = 20$
y_2	0	5	0	0	
y_3	0	2	1	2	
y_4	0	3	0	3	
y_5	0	0	1	5	
y_6	0	1	0	6	
y_1	0	4	0	0	$f = 24$
y_2	0	2	1	1	
y_3	0	0	2	2	
y_4	0	2	0	3	
y_5	0	0	1	4	
y_6	0	0	0	6	
y_1	0	1	1	2	$f = 28$
y_1	0	2	0	2	$f = 30$
y_2	0	0	0	5	

Table 4: Vertices with fixed f for $\{\alpha\beta\gamma^2\delta\colon \alpha\beta^2, \gamma^3, \alpha\delta^3\}$.

Table 5 consists of those (b, c, d) satisfying $6 - 3b - 2c - 2d > 0$ (and the three existing vertices $\alpha\beta^2, \gamma^3, \alpha\delta^3$). Such vertices allow f to be arbitrarily large, but some modulus conditions must be satisfied in order for a to be non-negative integers. Moreover, the high degree requirement becomes the lower bounds for f.

number	vertex $\alpha^a\beta^b\gamma^c\delta^d$				condition	f mod 72
	a	b	c	d		
x_1	1	2	0	0		
x_2	0	0	3	0		
x_3	1	0	0	3		
z_1	$\frac{f+60}{72}$	1	0	1	$f = 12(72)$ $f \geq 84$	12
z_2	$\frac{f+36}{72}$	1	1	0	$f = 36(72)$ $f \geq 108$	36
z_3	$\frac{f}{36}$	0	2	0	$f = 0(36)$ $f \geq 72$	$0, 36$
z_4	$\frac{f+24}{36}$	0	0	2	$f = 12(36)$ $f \geq 84$	$12, 48$
z_5	$\frac{f+12}{36}$	0	1	1	$f = 24(36)$ $f \geq 60$	$24, 60$
z_6	$\frac{f+12}{24}$	1	0	0	$f = 12(24)$ $f \geq 60$	$12, 36, 60$
z_7	$\frac{f}{18}$	0	1	0	$f = 0(18)$ $f \geq 54$	$0, 18, 36, 54$
z_8	$\frac{f+6}{18}$	0	0	1	$f = 12(18)$ $f \geq 66$	$12, 30, 48, 66$
z_9	$\frac{f}{12}$	0	0	0	$f = 0(12)$ $f \geq 60$	$0, 12, 24, 36, 48, 60$

Table 5: Vertices with variable f for $\{\alpha\beta\gamma^2\delta\colon \alpha\beta^2, \gamma^3, \alpha\delta^3\}$.

There is no overlapping between the two tables. For example, although $f = 18$ appears in Table 4, we require $f \geq 54$ for $f = 0(18)$ in Table 5. For each f in Table 4, we may try to solve the angle counting equations as in Section 4. It turns out that we always get a contradiction.

The angle counting equations for all the vertices in Table 5 are

$$\begin{aligned} f &= x_1 + x_3 + \tfrac{f+60}{72} z_1 + \tfrac{f+36}{72} z_2 + \tfrac{f}{36} z_3 + \tfrac{f+24}{36} z_4 \\ &\quad + \tfrac{f+12}{36} z_5 + \tfrac{f+12}{24} z_6 + \tfrac{f}{18} z_7 + \tfrac{f+6}{18} z_8 + \tfrac{f}{12} z_9, \\ f &= 2x_1 + z_1 + z_2 + z_6, \\ 2f &= 3x_2 + z_2 + 2z_3 + z_5 + z_7, \\ f &= 3x_3 + z_1 + 2z_4 + z_5 + z_8. \end{aligned}$$

It is then easy to get x_i in terms of f and z_j

$$\begin{aligned} x_1 &= \tfrac{1}{2} f - \tfrac{1}{2} z_1 - \tfrac{1}{2} z_2 - \tfrac{1}{2} z_6, \\ x_2 &= \tfrac{1}{3} f - \tfrac{1}{3} z_2 - \tfrac{2}{3} z_3 - \tfrac{1}{3} z_5 - \tfrac{1}{3} z_7, \\ x_3 &= \tfrac{1}{3} f - \tfrac{1}{3} z_1 - \tfrac{2}{3} z_4 - \tfrac{1}{3} z_5 - \tfrac{1}{3} z_8. \end{aligned}$$

Like the routine in Section 4, we also expect an equality relating f and z_j (and not involving x_i). We can get this by noting that the angle counting equations imply the vertex counting equation

$$\begin{aligned} \tfrac{f}{2} - 6 = x_3 &+ \left(\tfrac{f+60}{72} - 1\right) z_1 + \left(\tfrac{f+36}{72} - 1\right) z_2 + \left(\tfrac{f}{36} - 1\right) z_3 \\ &+ \left(\tfrac{f+24}{36} - 1\right) z_4 + \left(\tfrac{f+12}{36} - 1\right) z_5 + \left(\tfrac{f+12}{24} - 2\right) z_6 \\ &+ \left(\tfrac{f}{18} - 2\right) z_7 + \left(\tfrac{f+6}{18} - 2\right) z_8 + \left(\tfrac{f}{12} - 3\right) z_9. \end{aligned}$$

Substituting the formula for x_3, we get "modified vertex counting equation"

$$\begin{aligned} \tfrac{f}{2} - \tfrac{f}{3} - 6 = &\left(\tfrac{f+60}{72} - 1 - \tfrac{1}{3}\right) z_1 + \left(\tfrac{f+36}{72} - 1\right) z_2 + \left(\tfrac{f}{36} - 1\right) z_3 \\ &+ \left(\tfrac{f+24}{36} - 1 - \tfrac{2}{3}\right) z_4 + \left(\tfrac{f+12}{36} - 1 - \tfrac{1}{3}\right) z_5 + \left(\tfrac{f+12}{24} - 2\right) z_6 \\ &+ \left(\tfrac{f}{18} - 2\right) z_7 + \left(\tfrac{f+6}{18} - 2 - \tfrac{1}{3}\right) z_8 + \left(\tfrac{f}{12} - 3\right) z_9. \end{aligned}$$

It turns out that $f - 36$ is a common factor on both sides. Since $f \neq 36, 48$, we may divide the factor and get the relation for z_j

$$z_1 + z_2 + 2z_3 + 2z_4 + 2z_5 + 3z_6 + 4z_7 + 4z_8 + 6z_9 = 12.$$

The relation does not involve f! We will explain that this is a general phenomenon at the end of the section.

We summarize the full AVC as (see page 47)

$$
\begin{aligned}
\{f\alpha\beta\gamma^2\delta\colon & x_1\alpha\beta^2,\ x_2\gamma^3,\ x_3\alpha\delta^3 \\
& \mid z_1\alpha^{\frac{f+60}{72}}\beta\delta,\ z_2\alpha^{\frac{f+36}{72}}\beta\gamma,\ z_3\alpha^{\frac{f}{36}}\gamma^2,\ z_4\alpha^{\frac{f+24}{36}}\delta^2, \\
& z_5\alpha^{\frac{f+12}{36}}\gamma\delta,\ z_6\alpha^{\frac{f+12}{24}}\beta,\ z_7\alpha^{\frac{f}{18}}\gamma,\ z_8\alpha^{\frac{f+6}{18}}\delta,\ z_9\alpha^{\frac{f}{12}}\}, \\
& x_1 = \tfrac{1}{2}f - \tfrac{1}{2}z_1 - \tfrac{1}{2}z_2 - \tfrac{1}{2}z_6, \\
& x_2 = \tfrac{1}{3}f - \tfrac{1}{3}z_2 - \tfrac{2}{3}z_3 - \tfrac{1}{3}z_5 - \tfrac{1}{3}z_7, \\
& x_3 = \tfrac{1}{3}f - \tfrac{1}{3}z_1 - \tfrac{2}{3}z_4 - \tfrac{1}{3}z_5 - \tfrac{1}{3}z_8, \\
& z_1 + z_2 + 2z_3 + 2z_4 + 2z_5 + 3z_6 + 4z_7 + 4z_8 + 6z_9 = 12, \\
& \alpha = \tfrac{24}{f}\pi,\ \beta = \left(1 - \tfrac{12}{f}\right)\pi,\ \gamma = \tfrac{2}{3}\pi,\ \delta = \left(\tfrac{2}{3} - \tfrac{8}{f}\right)\pi.
\end{aligned}
$$

We note that the relation for z_j implies that we cannot have all $z_j = 0$. This means that the existing vertices $\alpha\beta^2, \gamma^3, \alpha\delta^3$ only cannot form a full AVC.

To read the full AVC, we need to consider the modulus condition because failing the condition means that the corresponding vertex cannot appear (i.e., the corresponding $z_j = 0$). For example, if $f = 0 \bmod 72$, then only z_3, z_7, z_9 can be non-trivial, so that the AVC becomes ($f = 72k$ for positive integer k)

$$
\begin{aligned}
\{72k\alpha\beta\gamma^2\delta\colon & x_1\alpha\beta^2,\ x_2\gamma^3,\ x_3\alpha\delta^3 \mid z_3\alpha^{2k}\gamma^2,\ z_7\alpha^{4k}\gamma,\ z_9\alpha^{6k}\}, \\
& x_1 = 36k,\ x_2 = 24k - \tfrac{2}{3}z_3 - \tfrac{1}{3}z_7,\ x_3 = 24k,\ 2z_3 + 4z_7 + 6z_9 = 12, \\
& \alpha = \tfrac{1}{3k}\pi,\ \beta = \left(1 - \tfrac{1}{6k}\right)\pi,\ \gamma = \tfrac{2}{3}\pi,\ \delta = \left(\tfrac{2}{3} - \tfrac{1}{9k}\right)\pi.
\end{aligned}
$$

The possible non-trivial z_j for the other modulus classes of f are listed in Table 6.

For each modulus class of f, there are finitely many combinations of the corresponding non-trivial z_j satisfying the relation derived from the vertex counting equation. Among these, we choose only those combinations such that x_1, x_2, x_3 are positive integers. Moreover, all vertices with nonzero z_j should have degree ≥ 4.

Now we describe the general process, which assumes two basic facts:

1. The order matrix is invertible.

2. The constant part of one angle is zero. The constant parts of all other angles are positive.

f mod 72	allowable variables
0	z_3, z_7, z_9
12	z_1, z_4, z_6, z_8, z_9
18,54	z_7
24	z_5, z_9
30,66	z_8
36	z_2, z_3, z_6, z_7, z_9
48	z_4, z_8, z_9
60	z_5, z_6, z_9

Table 6: Non-trivial z_j for various modulus classes of f.

Let α be the special angle in the second fact. Then

$$\alpha = \frac{\alpha_1}{f}, \quad \vec{\alpha}_0 = (0, \beta_0, \gamma_0, \dots) = (0, \vec{\beta}_0), \quad \vec{\alpha}_1 = (\alpha_1, \beta_1, \gamma_1, \dots) = (\alpha_1, \vec{\beta}_1).$$

The order vectors $\vec{v} = (a, b, c, \dots)^T = (a, \vec{u})^T$ of the other vertices $\alpha^a\beta^b\gamma^c\cdots$ satisfy the angle sum equation

$$2\pi = \vec{\alpha}_0\vec{v} + \frac{1}{f}\vec{\alpha}_1\vec{v} = \vec{\beta}_0\vec{u} + \frac{1}{f}\alpha_1 a + \frac{1}{f}\vec{\beta}_1\vec{u}.$$

We solve the equation for a

$$a = \frac{f}{\alpha_1}(2\pi - \vec{\beta}_0\vec{u}) - \frac{1}{\alpha_1}\vec{\beta}_1\vec{u}.$$

The lower bounds for the angles $\beta, \gamma, \dots$ translate into the upper bounds for the coordinates of $\vec{u}$ and therefore finitely many choices for $\vec{u}$. Substituting these choices into the formula for a, we get three possibilities.

1. If $2\pi - \vec{\beta}_0\vec{u} < 0$, then only finitely many f yields non-negative integer a.
2. If $2\pi - \vec{\beta}_0\vec{u} = 0$, then in case $a = -\frac{1}{\alpha_1}\vec{\beta}_1\vec{u}$ is a non-negative integer, we get a vertex valid for all f.
3. If $2\pi - \vec{\beta}_0\vec{u} > 0$, then we get non-negative integers a for f beyond a lower bound and satisfying a modulus condition.

The case $2\pi - \vec{\beta}_0\vec{u} = 0$ includes all the existing vertices. If there is any besides the existing ones, then we should include the new vertex in all cases. As shown in Section 4, if all the existing vertices have degree 3, then the case does not yield any new vertex. Our experience suggests that we do not get new vertices even if the existing vertices include a special one from Lemma 4.

From the case $2\pi - \vec{\beta}_0\vec{u} < 0$, we get a finite collection of f, and for each f, a finite collection of all optional vertices. Our experience suggests that there is always no overlapping between this case and the case $2\pi - \vec{\beta}_0\vec{u} > 0$, and the corresponding angle counting equations always lead to contradiction.

So our experience suggests that only the case $2\pi - \vec{\beta}_0\vec{u} > 0$ leads to full AVC. The order vectors of the vertices in this case form the columns of a matrix

$$Z = Z_0 + fZ_1,$$

where Z_0 and Z_1 are matrices with constant entries. The angle sum equation for Z becomes

$$2\pi(1, 1, \dots) = \vec{\alpha}Z = \vec{\alpha}_0Z_0 + \vec{\alpha}_1Z_1 + f\vec{\alpha}_0Z_1 + \frac{1}{f}\vec{\alpha}_1Z_0.$$

Since this is satisfied by infinitely many f, we get

$$\vec{\alpha}_0Z_0 + \vec{\alpha}_1Z_1 = 2\pi(1, 1, \dots), \quad \vec{\alpha}_0Z_1 = \vec{\alpha}_1Z_0 = (0, 0, \dots).$$

Like Section 4, we solve the angle counting equations

$$fP = X\vec{x} + Z\vec{z}.$$

Since $(P\ X)$ is invertible, the solution is an expression of $\vec{x}$ in terms of f, $\vec{z}$, and one equation involving $f, \vec{z}$ but not $\vec{x}$.

Our example suggests that f can be canceled to get one equation involving $\vec{z}$ only. This is no accident, because multiplying $\vec{\alpha}_1$ to the angle counting equations gives

$$4\pi f = (\vec{\alpha}_1P)f = (\vec{\alpha}_1X)\vec{x} + (\vec{\alpha}_1Z_0 + \vec{\alpha}_1Z_1f)\vec{z} = (\vec{\alpha}_1Z_1\vec{z})f.$$

Therefore $\vec{z}$ does satisfy an equation $4\pi = (\vec{\alpha}_1Z_1)\vec{z}$ with constant coefficients. This has to be equivalent to the equation involving $f, \vec{z}$ but not $\vec{x}$, because both are derived from the angle counting equations.

6 Some Complicated Examples

The routine processes in Sections 4 and 5 cover all except two of the AVCs in Proposition 5. In this section, we first compute these two AVCs. Then we compute two more examples not covered by Proposition 5.

Example (1). $\{\alpha\beta^2\gamma\delta\colon \alpha\beta^2, \gamma^3\}$, no other degree 3 vertices.

The AVC is from the last row of Table 2. By Lemma 4, the angle δ appears as $\alpha\delta^3, \beta\delta^3, \gamma\delta^3, \delta^4$ or δ^5. The AVC $\{\alpha\beta^2\gamma\delta\colon \alpha\beta^2, \gamma^3, \alpha\delta^3\}$ can be handled by Section 4, and the result is given on page 47. The AVC $\{\alpha\beta^2\gamma\delta\colon \alpha\beta^2, \gamma^3, \beta\delta^3\}$ can be handled by Section 5, and the result is given on page 48. It remains to consider the AVCs $\{\alpha\beta^2\gamma\delta\colon \alpha\beta^2, \gamma^3, \gamma\delta^3\}$, $\{\alpha\beta^2\gamma\delta\colon \alpha\beta^2, \gamma^3, \delta^4\}$ and $\{\alpha\beta^2\gamma\delta\colon \alpha\beta^2, \gamma^3, \delta^5\}$. What is special about the three cases is that the order matrices are singular.

We deal with the last AVC $\{\alpha\beta^2\gamma\delta\colon \alpha\beta^2, \gamma^3, \delta^5\}$ first. The AVC means that $\alpha, \beta, \beta, \gamma, \delta$ are the angles of the pentagon, $\alpha\beta^2$ and γ^3 are all the degree 3 vertices, and δ^5 is also a vertex. The angle sum equations are

$$\alpha + 2\beta + \gamma + \delta = 3\pi + \frac{4}{f}\pi, \quad \alpha + 2\beta = 3\gamma = 5\delta = 2\pi.$$

The solution is

$$\alpha + 2\beta = 2\pi,\ \gamma = \frac{2}{3}\pi,\ \delta = \frac{2}{5}\pi, \quad f = 60.$$

Since the pentagon is $\alpha\beta^2\gamma\delta$, the total number of β should be twice of the total number of α. Among the existing $\alpha\beta^2, \gamma^3, \delta^5$, the equality $2a = b$ always holds. Therefore to maintain this doubling relation between the total numbers of α and β, we either have $2a = b$ for all the vertices, or have vertices with $2a > b$ as well as vertices with $2a < b$.

If $\alpha < \beta$, then $\alpha < \frac{2}{3}\pi$ and $\beta > \frac{2}{3}\pi$, and the angle sum equation for a vertex $\alpha^a\beta^b\gamma^c\delta^d$ implies

$$2 > \frac{2}{3}b + \frac{2}{3}c + \frac{2}{5}d.$$

If there is a vertex with $2a < b$, then we have $b \geq 1$ for this vertex. The following are all the triples (b, c, d) satisfying $b \geq 1$ and the inequality above

$$(2,0,1), (2,0,0), (1,1,1), (1,1,0), (1,0,3), (1,0,2), (1,0,1), (1,0,0).$$

Since optional vertices have degree ≥ 4, the only ways we can add $a < \frac{b}{2}$ to (b, c, d) to get a vertex of degree ≥ 4 are $(1, 2, 0, 1)$ and $(0, 1, 0, 3)$. We note that $(1, 2, 0, 1)$ cannot be a vertex because

$$\alpha + 2\beta + \delta > \alpha + 2\beta = 2\pi.$$

On the other hand, if $(0, 1, 0, 3)$ is a vertex, then

$$\beta = 2\pi - 3\delta = \frac{4}{5}\pi, \quad \alpha = 2\pi - 2\beta = \frac{2}{5}\pi = \delta,$$

contradicting to the distinct angle assumption. We conclude that, if $\alpha < \beta$, then the existing $\alpha\beta^2, \gamma^3, \delta^5$ are all the vertices.

If $\alpha > \beta$, then $\alpha > \frac{2}{3}\pi$ and $\beta < \frac{2}{3}\pi$, and the angle sum equation for a vertex $\alpha^a\beta^b\gamma^c\delta^d$ implies

$$2 > \frac{2}{3}a + \frac{2}{3}c + \frac{2}{5}d.$$

Like the case $\alpha < \beta$, we try to look for vertices of degree ≥ 4 satisfying $2a > b$. Such a vertex must have $a \geq 1$, and the possible list of (a, c, d) is the same as the case $\alpha < \beta$ (except a and b are switched)

$$(2, 0, 1), (2, 0, 0), (1, 1, 1), (1, 1, 0), (1, 0, 3), (1, 0, 2), (1, 0, 1), (1, 0, 0).$$

Now we try to add $b < 2a$ to the list to get vertices of degree ≥ 4. The only such possible quadruples (a, b, c, d) are

$$(2, 3, 0, 1), (2, 2, 0, 1), (2, 1, 0, 1), (2, 3, 0, 0), (2, 2, 0, 0),$$

$$(1, 1, 1, 1), (1, 1, 0, 3), (1, 0, 0, 3), (1, 1, 0, 2).$$

By $\alpha + 2\beta = 2\pi$ and $\alpha > \frac{2}{3}\pi$, we have

$$\alpha + \beta + 2\delta = \frac{1}{2}\alpha + \frac{1}{2}(\alpha + 2\beta) + 2\delta > \frac{1}{2} \cdot \frac{2}{3}\pi + \frac{1}{2} \cdot 2\pi + 2 \cdot \frac{2}{5}\pi > 2\pi.$$

This implies that $(1, 1, 0, 2)$ is not a vertex. The similar reason excludes all except $(1, 0, 0, 3)$. If $(1, 0, 0, 3)$ is indeed a vertex, then we have the AVC $\{\alpha\beta^2\gamma\delta\colon \alpha\beta^2, \gamma^3, \alpha\delta^3, \delta^5\}$. However, the AVC contains $\{\alpha\beta^2\gamma\delta\colon \alpha\beta^2, \gamma^3, \alpha\delta^3\}$, which can be handled by Section 4. The resulting full AVC is given on page 47. The current case is exactly the case $f = 60$ in that full AVC.

If $(1,0,0,3)$ is not a vertex, then the argument so far implies that we must have $2a=b$ in all the optional vertices. Using $\alpha+2\beta=2\pi$, the angle sum equation then becomes

$$2=2a+\frac{2}{3}c+\frac{2}{5}d,\quad a=2b.$$

The solutions give exactly the existing vertices $\alpha\beta^2,\gamma^3,\delta^5$, and there are no optional vertices. The solution of the angle counting equations is

$$x_1=f=60,\quad x_2=\frac{1}{3}f=20,\quad x_3=\frac{1}{5}f=12,$$

and we get the full AVC on page 49

$$\{60\alpha\beta^2\gamma\delta\colon 60\alpha\beta^2,20\gamma^3,12\delta^5\}.$$

Now we turn to the AVC $\{\alpha\beta^2\gamma\delta\colon \alpha\beta^2,\gamma^3,\gamma\delta^3\}$. It is easy to show that

$$\alpha+2\beta=2\pi,\ \gamma=\frac{2}{3}\pi,\ \delta=\frac{4}{9}\pi,\quad f=36.$$

If $\alpha<\beta$, then $\alpha<\frac{2}{3}\pi,\beta>\frac{2}{3}\pi$, and the angle sum equation for a vertex $\alpha^a\beta^b\gamma^c\delta^d$ implies

$$2>\frac{2}{3}b+\frac{2}{3}c+\frac{4}{9}d.$$

Since

$$\frac{2}{3}b+\frac{2}{3}c+\frac{4}{9}d\geq\frac{2}{3}b+\frac{2}{3}c+\frac{2}{5}d,$$

we have tighter constraint than the AVC $\{\alpha\beta^2\gamma\delta\colon \alpha\beta^2,\gamma^3,\delta^5\}$. Then by the same argument, we find no vertices satisfying $2a<b$. The case of $\alpha>\beta$ can be handled similarly, and we find no vertices satisfying $2a>b$. Therefore all optional vertices must satisfy $2a=b$, and the angle sum equations show that the existing vertices are the only ones. Then from the angle counting equations, we get the full AVC on page 49

$$\{36\alpha\beta^2\gamma\delta\colon 36\alpha\beta^2,8\gamma^3,12\gamma\delta^3\}.$$

Finally, for the AVC $\{\alpha\beta^2\gamma\delta\colon \alpha\beta^2,\gamma^3,\delta^4\}$, we have

$$\alpha+2\beta=2\pi,\ \gamma=\frac{2}{3}\pi,\ \delta=\frac{1}{2}\pi,\quad f=24.$$

We get even tighter inequality

$$2 > \frac{2}{3}b + \frac{2}{3}c + \frac{1}{2}d \geq \frac{2}{3}b + \frac{2}{3}c + \frac{4}{9}d.$$

The same argument leads to the full AVC on page 48

$$\{24\alpha\beta^2\gamma\delta\colon 24\alpha\beta^2, 8\gamma^3, 6\delta^4\}.$$

Example (2)**.** $\{\alpha^2\beta\gamma\delta\colon \alpha\beta^2, \alpha^2\gamma\}$, no other degree 3 vertices.

By Lemma 4, the angle δ appears as $\alpha\delta^3, \beta\delta^3, \gamma\delta^3, \delta^4$ or δ^5. The cases of $\alpha\delta^3$ and δ^5 can be handled by Section 4, and the results are given on pages 44 and 46. The cases of $\beta\delta^3$ and δ^4 can be handled by Section 5, and the results are given on pages 45 and 46.

It remains to study $\{\alpha^2\beta\gamma\delta\colon \alpha\beta^2, \alpha^2\gamma, \gamma\delta^3\}$. The order matrix is still invertible, and we can solve the angle sum equations to get

$$\alpha = \frac{24}{f}\pi,\ \beta = \left(1 - \frac{12}{f}\right)\pi,\ \gamma = \left(2 - \frac{48}{f}\right)\pi,\ \delta = \frac{16}{f}\pi.$$

The problem is that two angles have zero constant part.

For the angles to be positive, we require $f > 24$. For the angles to be distinct, we require $f \neq 28, 32, 36$. By $f \geq 26$, the angle sum equation for a vertex $\alpha^a\beta^b\gamma^c\delta^d$ implies

$$2 \geq b\frac{\beta}{\pi} + c\frac{\gamma}{\pi} \geq \left(1 - \frac{12}{26}\right)b + \left(2 - \frac{48}{26}\right)c = \frac{7}{13}b + \frac{1}{13}c.$$

This allows too many possible choices of (b, c). So we treat the first couple of f separately, and then consider those f with bigger lower bound. It turns out that $f > 32$ is convenient enough.

For $f = 26, 30$, the angles are

$$f = 26\colon \alpha = \frac{12}{13}\pi,\ \beta = \frac{7}{13}\pi,\ \gamma = \frac{2}{13}\pi,\ \delta = \frac{8}{13}\pi;$$
$$f = 30\colon \alpha = \frac{4}{5}\pi,\ \beta = \frac{3}{5}\pi,\ \gamma = \frac{2}{5}\pi,\ \delta = \frac{8}{15}\pi.$$

Then it is easy to find the corresponding optional vertices in Table 7.

Since $f \neq 28, 32, 36$, next we should consider $f > 32$. Then the angle sum equation for a vertex $\alpha^a\beta^b\gamma^c\delta^d$ implies

$$3a + 2d = \frac{f}{8}(2 - b - 2c) + \frac{3}{2}(b + 4c) = \lambda f + \mu, \quad 2 > \frac{5}{8}b + \frac{1}{2}c.$$

Substituting couples (b, c) of non-negative integers satisfying the inequality into the formula for $3a + 2d$, we get the following:

1. For $(b, c) = (2, 0)$ or $(0, 1)$, we get $\lambda = 0$ and the three existing vertices $\alpha\beta^2, \alpha^2\gamma, \gamma\delta^3$.

2. For $b + c = 2$ or 3 and $(b, c) \neq (2, 0)$, we have $\lambda < 0$, so that the number of possible (f, b, c) is finite. Actually we can only have $(f, b, c) = (40, 0, 2)$ or $(44, 1, 1)$. The full details are

$$\begin{aligned} f = 40,\ \gamma^2\delta\colon \alpha = \frac{3}{5}\pi,\ \beta = \frac{7}{10}\pi,\ \gamma = \frac{4}{5}\pi,\ \delta = \frac{2}{5}\pi; \\ f = 44,\ \beta\gamma\delta\colon \alpha = \frac{6}{11}\pi,\ \beta = \frac{8}{11}\pi,\ \gamma = \frac{10}{11}\pi,\ \delta = \frac{4}{11}\pi. \end{aligned}$$

   Then it is easy to find the corresponding optional vertices in Table 7.

3. For $b + c = 0$ or 1 and $(b, c) \neq (0, 1)$, we get two families of optional vertices

$$\begin{aligned} \alpha^a\beta\delta^d\colon 3a + 2d &= \frac{f + 12}{8},\ f = 4(8); \\ \alpha^{a'}\delta^{d'}\colon 3a' + 2d' &= \frac{f}{4},\ f = 0(4). \end{aligned}$$

   Considering $f > 24$, $f \neq 28, 32, 36$, that the cases $f = 26, 30, 40, 44$ are to be separately treated, and that $f = 0(4)$ for the two vertex families above, this remaining case is only for $f \geq 48$. The vertices for this remaining case are the three existing vertices $\alpha\beta^2, \alpha^2\gamma, \gamma\delta^3$ and the two families above.

Solve the angle counting equations, we get contradictions for $f = 26, 30$ and full AVCs for $f = 40, 44$ on page 7. For $f \geq 48$, the angle counting

number	vertex $\alpha^a\beta^b\gamma^c\delta^d$				condition
	a	b	c	d	
x_1	1	2	0	0	
x_2	2	0	1	0	
x_3	0	0	1	3	
y_1	1	0	3	1	$f=26$
y_2	0	2	2	1	
y_3	0	0	5	2	
y_4	1	0	7	0	
y_5	0	2	6	0	
y_6	0	0	9	1	
y_7	0	0	13	0	
y_1	1	0	3	0	$f=30$
y_2	0	2	2	0	
y_3	0	0	5	0	
y_1	0	0	2	1	$f=40$
y_2	2	0	0	2	
y_3	0	0	0	5	
y_1	0	1	1	1	$f=44$
y_2	3	0	0	1	
y_3	1	0	0	4	
z_i	a_i	1	0	d_i	$3a_i+2d_i=\frac{f+12}{8}$ $f=4(8), f\geq 48$
z_i'	a_i'	0	0	d_i'	$3a_i'+2d_i'=\frac{f}{4}$ $f=0(4), f\geq 48$

Table 7: Vertices for $\{\alpha^2\beta\gamma\delta\colon \alpha\beta^2, \alpha^2\gamma, \gamma\delta^3\}$.

equations are

$$
\begin{aligned}
2f &= x_1 + 2x_2 + \sum a_i z_i + \sum a_i' z_i', \\
f &= 2x_1 + \sum z_i, \\
f &= x_2 + x_3, \\
f &= 3x_3 + \sum d_i z_i + \sum d_i' z_i'.
\end{aligned}
$$

The solution is expressions of x_i in terms of f, z_i, z_i'

$$
\begin{aligned}
x_1 &= \tfrac{1}{2}f - \tfrac{1}{2}\sum z_i, \\
x_2 &= \tfrac{2}{3}f + \tfrac{1}{3}\sum d_i z_i + \tfrac{1}{3}\sum d_i' z_i', \\
x_3 &= \tfrac{1}{3}f - \tfrac{1}{3}\sum d_i z_i - \tfrac{1}{3}\sum d_i' z_i',
\end{aligned}
$$

together with an equality involving only f, z_i, z_i'. To get this equality, we may start with the vertex counting equation

$$
\frac{f}{2} - 6 = x_3 + \sum (a_i + d_i - 2) z_i + \sum (a_i' + d_i' - 3) z_i.
$$

Substituting the formula for x_3, we get

$$
\begin{aligned}
\tfrac{f}{2} - \tfrac{f}{3} - 6 &= \sum \left(a_i + d_i - \tfrac{1}{3}d_i - 2\right) z_i + \sum \left(a_i' + d_i' - \tfrac{1}{3}d_i' - 3\right) z_i \\
&= \sum \left(\tfrac{1}{3}(3a_i + 2d_i) - 2\right) z_i + \sum \left(\tfrac{1}{3}(3a_i' + 2d_i') - 3\right) z_i \\
&= \sum \left(\tfrac{f+12}{3\cdot 8} - 2\right) z_i + \sum \left(\tfrac{f}{3\cdot 4} - 3\right) z_i \\
&= \tfrac{1}{24}\sum (f - 36) z_i + \tfrac{1}{12}\sum (f - 36) z_i.
\end{aligned}
$$

Canceling $f - 36$ on both sides, we get

$$
\sum z_i + 2\sum z_i', = 4.
$$

Example (3)**.** $\{\alpha^2\beta\gamma\delta \colon \alpha\beta\gamma, \alpha\delta^2\}$.

According to Table 1, for $\{\alpha^2\beta\gamma\delta \colon \alpha\beta\gamma, \alpha\delta^2\}$, we need to consider the possibility that $\beta^2\delta$ or β^3 appears as a vertex. The order matrix is invertible in these two cases, so that the routines in Sections 4 and 5 can be applied. We omit the details.

So we assume that there are no other degree 3 vertices. The order matrix

$$(P\ X) = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 2 \end{pmatrix}$$

is singular due to lack of enough existing vertices. The angle sum equations give

$$\alpha = \frac{8}{f}\pi,\ \beta + \gamma = \left(2 - \frac{8}{f}\right)\pi,\ \delta = \left(1 - \frac{4}{f}\right)\pi.$$

Example 1 of this section shows that singular order matrix itself is not a big obstacle for computing the AVC. The difference here is that the constant part of α is 0, so that the angle has no positive lower bound. From Sections 4 and 5, we expect the solution to involve one more discrete variable.

By $f \geq 16$, we get

$$\beta + \gamma \geq \frac{3}{2}\pi, \quad \delta \geq \frac{3}{4}\pi.$$

Due to the symmetry of exchanging β and γ, we assume $\beta < \gamma$ without loss of generality. Then

$$\gamma > \frac{\beta + \gamma}{2} = \delta \geq \frac{3}{4}\pi > \beta, \quad \beta + \gamma + \delta = 3\delta > 2\pi.$$

The angle sum equation at a vertex $\alpha^a\beta^b\gamma^c\delta^d$ implies

$$2 \geq \frac{3}{4}c + \frac{3}{4}d.$$

Then we have $c + d \leq 2$ and the following possible high degree vertices (a, b, c, d):

1. $(c, d) = (0, 2)$. By $\alpha + 2\delta = 2\pi$, we must have $a = 0$, so that the vertex is $(0, b, 0, 2)$ with $b \geq 2$.

2. $(c, d) = (1, 1)$. By $\alpha + \gamma + \delta > \alpha + 2\delta = 2\pi$ and $\beta + \gamma + \delta > 2\pi$, there is no such vertex.

3. $(c, d) = (2, 0)$. By $\alpha + 2\gamma > \alpha + \beta + \gamma = 2\pi$, we must have $a = 0$, so that the vertex is $(0, b, 2, 0)$ with $b \geq 2$. But this implies $b\beta + 2\gamma = 2(\beta + \gamma) > 2\pi$, a contradiction.

4. $(c,d)=(0,1)$. We get $(a,b,0,1)$ with $a+b\ge 3$.

5. $(c,d)=(1,0)$. By $\alpha+\beta+\gamma=2\pi$, either a or b is 0. So we get $(a,0,1,0)$ with $a\ge 3$, or $(0,b,1,0)$ with $b\ge 3$.

6. $(c,d)=(0,0)$. We get $(a,b,0,0)$ with $a+b\ge 4$.

In summary, we get the following list of possible high degree vertices

$$(0,b,0,2),\ (a,b,0,1),\ (a,0,1,0),\ (0,b,1,0),\ (a,b,0,0).$$

The pentagon $\alpha^2\beta\gamma\delta$ implies that β and γ appear the same number of times in the tiling. Since among the existing vertices $\alpha\beta\gamma$ and $\alpha\delta^2$, β and γ always appear the same number of times, they must also appear the same number of times at high degree vertices. This implies that among five families of high degree vertices above, either only those with $b=c$ can appear, or at least one with $b<c$ must appear.

First consider the case that only those with $b=c$ can appear. This means that the high degree vertices are either $(a,0,0,1)$ with $a\ge 3$ or $(a,0,0,0)$ with $a\ge 4$. Solving the angle sum equations at the vertices, we find that they are $\alpha^{\frac{f+4}{8}}\delta$ and $\alpha^{\frac{f}{4}}$. Then we solve the angle counting equations for $\{\alpha^2\beta\gamma\delta\colon \alpha\beta\gamma,\alpha\delta^2 \mid \alpha^{\frac{f+4}{8}}\delta,\alpha^{\frac{f}{4}}\}$ and get the full AVC

$$\{f\alpha^2\beta\gamma\delta\colon f\alpha\beta\gamma,\ (\tfrac{1}{2}f-2+y_2)\alpha\delta^2 \mid (4-2y_2)\alpha^{\frac{f+4}{8}}\delta,\ y_2\alpha^{\frac{f}{4}}\},$$
$$\alpha=\tfrac{8}{f}\pi,\ \beta+\gamma=\left(2-\tfrac{8}{f}\right)\pi,\ \delta=\left(1-\tfrac{4}{f}\right)\pi.$$

Next consider the case that a vertex with $b<c$ appears. From the list, we find the only possibility is $(a,0,1,0)$, $a\ge 3$. We use g (another variable in addition to f) in place of a and get an updated AVC $\{\alpha^2\beta\gamma\delta\colon \alpha\beta\gamma,\alpha\delta^2,\alpha^g\gamma\}$ in which all vertices are necessary (i.e., must appear). The updated order matrix

$$(P\ X')=\begin{pmatrix}2&1&1&g\\1&1&0&0\\1&1&0&1\\1&0&2&0\end{pmatrix}$$

is now invertible, so that we can calculate all the angles

$$\alpha=\frac{8}{f}\pi,\ \beta=\frac{8(g-1)}{f}\pi,\ \gamma=\left(2-\frac{8g}{f}\right)\pi,\ \delta=\left(1-\frac{4}{f}\right)\pi.$$

The remaining four families in the list above are the possible optional vertices. In fact, $(0, b, 0, 2)$ cannot be a vertex because $b \geq 2$ implies

$$b\beta + 2\delta > \alpha + 2\delta = 2\pi.$$

Moreover, $(0, b, 1, 0)$ is also not a vertex because $b \geq 3$ implies

$$b\beta + \gamma > 2\beta + \gamma > \alpha + \beta + \gamma = 2\pi.$$

On the other hand, $(a, b, 0, 1)$ and $(a', b', 0, 0)$ can be optional vertices, and solving the angle sum equations at the vertices gives

$$(a, b, 0, 1)\colon a + (g-1)b = \frac{f+4}{8},\ f = 4(8);$$
$$(a', b', 0, 0)\colon a' + (g-1)b' = \frac{f}{4},\ f = 0(4).$$

Then we solve the angle counting equations similar to Example 2 of this section and get the full AVC

$$\begin{aligned}
&\{f\alpha^2\beta\gamma\delta\colon x_1\alpha\beta\gamma,\ x_2\alpha\delta^2,\ x_3\alpha^g\gamma \mid y_i\alpha^{a_i}\beta^{b_i}\delta,\ y_i'\alpha^{a_i'}\beta^{b_i'}\},\\
&\quad a_i + (g-1)d_i = \tfrac{f+4}{8},\ a_i' + (g-1)d_i' = \tfrac{f}{4},\\
&\quad x_1 = f - \sum b_i y_i - \sum b_i' y_i',\\
&\quad x_2 = \tfrac{1}{2}f - \tfrac{1}{2}\sum y_i,\\
&\quad x_3 = \sum b_i y_i + \sum b_i' y_i',\\
&\quad \sum y_i + 2\sum y_i' = 4,\\
&\quad \alpha = \tfrac{8}{f}\pi,\ \beta = \tfrac{8(g-1)}{f}\pi,\ \gamma = \left(2 - \tfrac{8g}{f}\right)\pi,\ \delta = \left(1 - \tfrac{4}{f}\right)\pi.
\end{aligned} \tag{6.1}$$

Example (4)**.** $\{\alpha\beta\gamma\delta\epsilon\colon \alpha\beta\gamma, \delta^3\}$, no other degree 3 vertices.

The assumption of no other degree 3 vertices follows from Table 1. By Lemma 4 and up to symmetry, we may assume that one of $\alpha\epsilon^3, \delta\epsilon^3, \epsilon^4, \epsilon^5$ is also a vertex. If $\alpha\epsilon^3$ is a vertex, then we get

$$\alpha = \left(1 - \frac{12}{f}\right)\pi,\ \beta + \gamma = \left(1 + \frac{12}{f}\right)\pi,\ \delta = \frac{2}{3}\pi,\ \epsilon = \left(\frac{1}{3} + \frac{4}{f}\right)\pi.$$

Using to the positive lower bounds for α, δ, ϵ, the study of $\{\alpha\beta\gamma\delta\epsilon\colon \alpha\beta\gamma, \delta^3, \alpha\epsilon^3\}$ can proceed similar to Example 1 in this section. We omit the details.

For $\{\alpha\beta\gamma\delta\epsilon\colon \alpha\beta\gamma, \delta^3, \epsilon^5\}$, we have

$$\alpha+\beta+\gamma=2\pi,\ \delta=\frac{2}{3}\pi,\ \epsilon=\frac{2}{5}\pi,\quad f=60.$$

This is similar to $\{\alpha\beta^2\gamma\delta\colon \alpha\beta^2, \gamma^3, \delta^5\}$ in Example 1 of this section, except $\alpha+2\beta=2\pi$ is changed to $\alpha+\beta+\gamma=2\pi$. It is easy to see that the only vertices of the form $\delta^d\epsilon^e$ are δ^3 and ϵ^5. Therefore if $\alpha\beta\gamma$ is the only vertex involving the three angles, then $\alpha\beta\gamma, \delta^3, \epsilon^5$ are all the vertices, and we can easily get the full AVC

$$\{60\alpha\beta\gamma\delta\epsilon\colon 60\alpha\beta\gamma,\ 20\delta^3,\ 12\epsilon^5\}. \tag{6.2}$$

So we assume that α, β, γ appear at vertices other than $\alpha\beta\gamma$. Since the total number of times they appear are the same, and they appear the same number of times at degree 3 vertices, each of them appears at some high degree vertices. By symmetry, we may further assume $\alpha<\beta<\gamma$. Then $\gamma>\frac{2\pi}{3}$ and $\alpha\gamma^2\cdots$, $\beta\gamma^2\cdots$, $\beta^2\gamma\cdots$, $\beta\gamma\delta\cdots$ cannot be vertices because the angle sums are $>2\pi$. We will study how γ appears at a high degree vertex.

If α appears at all the vertices where γ appears, then by α and γ appearing the same total number of times, and $\alpha\gamma^2\cdots$ not being a vertex, such vertices must all be of the form $\alpha\gamma\cdots$, with the remainder $\cdots$ not containing α and γ. Since $\alpha\beta\gamma$ is already a vertex, the high degree vertices $\alpha\gamma\cdots$ must be $\alpha\gamma\delta^d\epsilon^e$, $d+e\ge 2$. With the exception of $d=0, e=2$, we always have

$$\beta=2\pi-\alpha-\gamma=d\delta+e\epsilon\ge\frac{16}{15}\pi,$$

so that

$$\gamma<\alpha+\gamma=2\pi-(d\delta+e\epsilon)\le\frac{14}{15}\pi<\beta,$$

contradiction to the assumption $\beta<\gamma$. We conclude that the only high degree vertex involving α or γ is $\alpha\gamma\epsilon^2$. Then we get

$$\alpha+\gamma=\frac{6}{5}\pi,\ \beta=\frac{4}{5}\pi,\ \delta=\frac{2}{3}\pi,\ \epsilon=\frac{2}{5}\pi,$$

and find that the only other possible high degree vertex is $\beta\epsilon^3$. A simple calculation then determines the full AVC

$$\{60\alpha\beta\gamma\delta\epsilon\colon (60-y_1)\alpha\beta\gamma,\ 20\delta^3,\ (12-y_1)\epsilon^5,\ y_1\alpha\gamma\epsilon^2 \mid y_1\beta\epsilon^3\}.$$

Note that $y_1 = 0$ reduces to (6.2). Moreover, $y_1 > 0$ belongs to the case $\{\alpha\beta\gamma\delta\epsilon\colon \alpha\beta\gamma, \delta^3, \alpha\epsilon^3\}$ after exchanging α and β.

It remains to consider the case that γ appears at a high degree vertex without α. By the similar angle sum consideration, this vertex must be $\gamma\epsilon^3$. So this belongs to the case $\{\alpha\beta\gamma\delta\epsilon\colon \alpha\beta\gamma, \delta^3, \alpha\epsilon^3\}$ after exchanging α and γ.

We conclude that the full AVC derived from $\{\alpha\beta\gamma\delta\epsilon\colon \alpha\beta\gamma, \delta^3, \epsilon^5\}$ is either (6.2), or belongs to the full AVC derived from $\{\alpha\beta\gamma\delta\epsilon\colon \alpha\beta\gamma, \delta^3, \alpha\epsilon^3\}$ after some exchange among α, β, γ.

Next we consider $\{\alpha\beta\gamma\delta\epsilon\colon \alpha\beta\gamma, \delta^3, \delta\epsilon^3\}$. We have

$$\alpha+\beta+\gamma = 2\pi,\ \delta = \frac{2}{3}\pi,\ \epsilon = \frac{4}{9}\pi, \quad f = 36.$$

Like Example 1 in this section, this is tighter than the case $f = 60$. In fact, the similar argument shows that, if $\alpha < \beta < \gamma$, then the only high degree vertex where γ may appear is $\alpha\gamma\epsilon^2$. If $\alpha\gamma\epsilon^2$ is a vertex, then

$$\alpha+\gamma = \frac{10}{9}\pi,\ \beta = \frac{8}{9}\pi,\ \delta = \frac{2}{3}\pi,\ \epsilon = \frac{4}{9}\pi,$$

and α and γ appear only in $\alpha\beta\gamma$ and $\alpha\gamma\epsilon^2$. However, this implies that the only possible vertices involving β are $\alpha\beta\gamma$ and $\beta\delta\epsilon$. Since $\beta\delta\epsilon$ has degree 3 and is therefore excluded, comparing the total numbers of α, β, γ shows that $\alpha\gamma\epsilon^2$ is not a vertex. So we conclude that $\alpha\gamma\epsilon^2$ cannot not appear, and $\alpha\beta\gamma, \delta^3, \delta\epsilon^3$ are all the vertices. A simple calculation determines the full AVC

$$\{36\alpha\beta\gamma\delta\epsilon\colon 36\alpha\beta\gamma,\ 8\delta^3,\ 12\delta\epsilon^3\}. \tag{6.3}$$

Finally, for $\{\alpha\beta\gamma\delta\epsilon\colon \alpha\beta\gamma, \delta^3, \epsilon^4\}$, we have

$$\alpha+\beta+\gamma = 2\pi,\ \delta = \frac{2}{3}\pi,\ \epsilon = \frac{1}{2}\pi, \quad f = 24.$$

The argument similar to the cases $f = 60, 36$ shows that $\alpha\beta\gamma, \delta^3, \epsilon^4$ are the only vertices, and the full AVC is

$$\{24\alpha\beta\gamma\delta\epsilon\colon 24\alpha\beta\gamma,\ 8\delta^3,\ 6\epsilon^4\}. \tag{6.4}$$

7 Classification of AVCs

Up to three distinct angles at degree 3 vertices

Starting from Table 2 and applying Sections 4, 5, 6, we get the complete list of all the AVCs with up to three distinct angles at degree 3 vertices. One should keep the following in mind while reading the list.

1. The list assumes $f \ge 16$ (equivalent to $f \ne 12$). Any anglewise vertex combination with up to three distinct angles or with four distinct angles but one not appearing at degree 3 vertices is contained in one of the AVCs in the list. There are some overlappings between the AVCs in the list.

2. All coefficients in the necessary part (i.e., before the divider $\mid$) are positive integers. All coefficients in the optional part (i.e., after the divider $\mid$) are non-negative integers.

3. All angles are distinct. This implies that f should not take certain specific values. Although allowing angles to be equal can still give an AVC, such an AVC is contained in another AVC in the list.

4. All optional vertices should have non-negative integer exponents and have degree ≥ 4. This implies that, for the corresponding coefficient to be nonzero, f should satisfy some modulus condition and has some lower bound.

The following are the full AVCs from $\{\alpha^3\}$, which must include an extra angle β not appearing at degree 3 vertices.

$$\begin{aligned}
&\{24\alpha^4\beta\colon 32\alpha^3,\ 6\beta^4\},\ \alpha = \tfrac{2}{3}\pi,\ \beta = \tfrac{1}{2}\pi;\\
&\{36\alpha^4\beta\colon 44\alpha^3,\ 12\alpha\beta^3\},\ \alpha = \tfrac{2}{3}\pi,\ \beta = \tfrac{4}{9}\pi;\\
&\{60\alpha^4\beta\colon 80\alpha^3,\ 12\beta^5\},\ \alpha = \tfrac{2}{3}\pi,\ \beta = \tfrac{2}{5}\pi.
\end{aligned}$$

The following is the full AVC from $\{\alpha\beta^2\}$.

$$\begin{aligned}
&\{f\alpha^2\beta^3\colon (\tfrac{3}{2}f - 2 + y_2)\alpha\beta^2 \mid (4 - 2y_2)\alpha^{\frac{f+4}{8}}\beta,\ y_2\alpha^{\frac{f}{4}}\},\\
&\qquad \alpha = \tfrac{8}{f}\pi,\ \beta = \left(1 - \tfrac{4}{f}\right)\pi.
\end{aligned}$$

For three angles α,β,γ at degree 3 vertices, we first list the full AVCs with α,β,γ as the only angles, then separately list the full AVCs with an angle δ appearing as $\alpha\delta^3,\beta\delta^3,\gamma\delta^3,\delta^4$ or δ^5.

The following are the full AVCs from $\{\alpha\beta\gamma,\alpha^3\}$.

$$\{24\alpha^2\beta^2\gamma\colon 24\alpha\beta\gamma,\ 8\alpha^3,\ 6\beta^4\},\ \alpha=\tfrac{2}{3}\pi,\ \beta=\tfrac{1}{2}\pi,\ \gamma=\tfrac{5}{6}\pi;$$
$$\{36\alpha^2\beta^2\gamma\colon 36\alpha\beta\gamma,\ 8\alpha^3,\ 12\alpha\beta^3\},\ \alpha=\tfrac{2}{3}\pi,\ \beta=\tfrac{4}{9}\pi,\ \gamma=\tfrac{8}{9}\pi;$$
$$\{60\alpha^2\beta^2\gamma\colon 60\alpha\beta\gamma,\ 20\alpha^3,\ 12\beta^5\},\ \alpha=\tfrac{2}{3}\pi,\ \beta=\tfrac{2}{5}\pi,\ \gamma=\tfrac{14}{15}\pi.$$

The following is the full AVC from $\{\alpha\beta\gamma,\alpha^3,\alpha\delta^3\}$.

$$\{36\alpha^2\beta\gamma\delta\colon 36\alpha\beta\gamma,\ 8\alpha^3,\ 12\alpha\delta^3\},\ \alpha=\tfrac{2}{3}\pi,\ \beta+\gamma=\tfrac{4}{3}\pi,\ \gamma=\tfrac{4}{9}\pi.$$

The following are the full AVCs from $\{\alpha\beta\gamma,\alpha^3,\beta\delta^3\}$. The transformation $\beta\leftrightarrow\gamma$ gives the full AVCs from $\{\alpha\beta\gamma,\alpha^3,\gamma\delta^3\}$.

$$\{48\alpha^2\beta\gamma\delta\colon 36\alpha\beta\gamma,\ 20\alpha^3,\ 12\beta\delta^3 \mid 6\gamma^2\delta^2\},$$
$$\alpha=\tfrac{2}{3}\pi,\ \beta=\tfrac{3}{4}\pi,\ \gamma=\tfrac{7}{12}\pi,\ \delta=\tfrac{5}{12}\pi;$$
$$\{60\alpha^2\beta\gamma\delta\colon (60-y_1-3y_2)\alpha\beta\gamma,\ (20+y_2)\alpha^3,\ (y_1+3y_2)\beta\delta^3$$
$$\mid y_1\alpha\gamma\delta^2,\ y_2\gamma^3\delta,\ (12-y_1-2y_2)\delta^5\},$$
$$\alpha=\tfrac{2}{3}\pi,\ \beta=\tfrac{4}{5}\pi,\ \gamma=\tfrac{18}{15}\pi,\ \delta=\tfrac{2}{5}\pi;$$
$$\{72\alpha^2\beta\gamma\delta\colon 48\alpha\beta\gamma,\ 32\alpha^3,\ 24\beta\delta^3 \mid 6\gamma^4\},$$
$$\alpha=\tfrac{2}{3}\pi,\ \beta=\tfrac{5}{6}\pi,\ \gamma=\tfrac{1}{2}\pi,\ \delta=\tfrac{7}{18}\pi;$$
$$\{84\alpha^2\beta\gamma\delta\colon (72-y_1)\alpha\beta\gamma,\ 32\alpha^3,\ (12+y_1)\beta\delta^3 \mid y_1\alpha\gamma^3,\ (12-y_1)\gamma^2\delta^3\},$$
$$\alpha=\tfrac{2}{3}\pi,\ \beta=\tfrac{6}{7}\pi,\ \gamma=\tfrac{10}{21}\pi,\ \delta=\tfrac{8}{21}\pi;$$
$$\{108\alpha^2\beta\gamma\delta\colon (84-y_1)\alpha\beta\gamma,\ 44\alpha^3,\ (24+y_1)\beta\delta^3 \mid y_1\alpha\gamma^2\delta,\ (12-y_1)\gamma\delta^4\},$$
$$\alpha=\tfrac{2}{3}\pi,\ \beta=\tfrac{8}{9}\pi,\ \gamma=\tfrac{4}{9}\pi,\ \delta=\tfrac{10}{27}\pi;$$
$$\{132\alpha^2\beta\gamma\delta\colon 96\alpha\beta\gamma,\ 56\alpha^3,\ 36\beta\delta^3 \mid 12\gamma^3\delta^2\},$$
$$\alpha=\tfrac{2}{3}\pi,\ \beta=\tfrac{10}{11}\pi,\ \gamma=\tfrac{14}{33}\pi,\ \delta=\tfrac{4}{11}\pi;$$
$$\{156\alpha^2\beta\gamma\delta\colon 108\alpha\beta\gamma,\ 68\alpha^3,\ 48\beta\delta^3 \mid 12\gamma^4\delta\},$$
$$\alpha=\tfrac{2}{3}\pi,\ \beta=\tfrac{12}{13}\pi,\ \gamma=\tfrac{16}{39}\pi,\ \delta=\tfrac{14}{39}\pi;$$
$$\{180\alpha^2\beta\gamma\delta\colon 120\alpha\beta\gamma,\ 80\alpha^3,\ 60\beta\delta^3 \mid 12\gamma^5\},$$
$$\alpha=\tfrac{2}{3}\pi,\ \beta=\tfrac{14}{15}\pi,\ \gamma=\tfrac{2}{5}\pi,\ \delta=\tfrac{16}{45}\pi.$$

The following is the full AVC from $\{\alpha\beta\gamma,\alpha^3,\delta^4\}$.

$$\{24\alpha^2\beta\gamma\delta\colon 24\alpha\beta\gamma,\ 8\alpha^3,\ 6\delta^4\},\ \alpha=\tfrac{2}{3}\pi,\ \beta+\gamma=\tfrac{4}{3}\pi,\ \delta=\tfrac{1}{2}\pi.$$

The following is the full AVC from $\{\alpha\beta\gamma, \alpha^3, \delta^5\}$.

$$\{60\alpha^2\beta\gamma\delta\colon 60\alpha\beta\gamma,\ 20\alpha^3, 12\delta^5\},\ \alpha=\tfrac{2}{3}\pi,\ \beta+\gamma=\tfrac{4}{3}\pi,\ \delta=\tfrac{2}{5}\pi.$$

The following are the full AVCs from $\{\alpha\beta^2, \alpha^2\gamma\}$.

$$\begin{aligned}
&\{32\alpha^2\beta\gamma^2\colon 16\alpha\beta^2,\ 24\alpha^2\gamma \mid 10\gamma^4\},\ \alpha=\tfrac{3}{4}\pi,\ \beta=\tfrac{5}{8}\pi,\ \gamma=\tfrac{1}{2}\pi;\\
&\{52\alpha^2\beta\gamma^2\colon 16\alpha\beta^2,\ 44\alpha^2\gamma \mid 20\beta\gamma^3\},\ \alpha=\tfrac{10}{13}\pi,\ \beta=\tfrac{8}{13}\pi,\ \gamma=\tfrac{6}{13}\pi;\\
&\{f\alpha^2\beta^2\gamma\colon x_1\alpha\beta^2,\ x_2\alpha^2\gamma\\
&\qquad \mid y_1\alpha\beta\gamma^{\frac{f+4}{16}},\ y_2\beta^3\gamma^{\frac{f-12}{16}},\ y_3\alpha\gamma^{\frac{f+4}{8}},\ y_4\beta^2\gamma^{\frac{f-4}{8}},\ y_5\beta\gamma^{\frac{3f-4}{16}},\ y_6\gamma^{\frac{f}{4}}\},\\
&\qquad x_1=f-\tfrac{1}{2}y_1-\tfrac{3}{2}y_2-y_4-\tfrac{1}{2}y_5,\\
&\qquad x_2=\tfrac{1}{2}f-\tfrac{1}{4}y_1+\tfrac{3}{4}y_2-\tfrac{1}{2}y_3+\tfrac{1}{2}y_4+\tfrac{1}{4}y_5,\\
&\qquad y_1+y_2+2y_3+2y_4+3y_5+4y_6=8,\\
&\qquad \alpha=\left(1-\tfrac{4}{f}\right)\pi,\ \beta=\left(\tfrac{1}{2}+\tfrac{2}{f}\right)\pi,\ \gamma=\tfrac{8}{f}\pi;\\
&\{f\alpha^3\beta\gamma\colon (\tfrac{1}{2}f-2+y_2)\alpha\beta^2,\ f\alpha^2\gamma \mid (4-2y_2)\alpha^{\frac{f+4}{8}}\beta,\ y_2\alpha^{\frac{f}{4}}\},\\
&\qquad \alpha=\tfrac{8}{f}\pi,\ \beta=\left(1-\tfrac{4}{f}\right)\pi,\ \gamma=\left(2-\tfrac{16}{f}\right)\pi.
\end{aligned}$$

The following are the full AVCs from $\{\alpha\beta^2, \alpha^2\gamma, \alpha\delta^3\}$.

$$\begin{aligned}
&\{56\alpha^2\beta\gamma\delta\colon 28\alpha\beta^2,\ 36\alpha^2\gamma,\ 12\alpha\delta^3 \mid 10\gamma^2\delta^2\},\\
&\qquad \alpha=\tfrac{22}{35}\pi,\ \beta=\tfrac{24}{35}\pi,\ \gamma=\tfrac{26}{35}\pi,\ \delta=\tfrac{16}{35}\pi;\\
&\{76\alpha^2\beta\gamma\delta\colon (28+y_2)\alpha\beta^2,\ (56-y_2)\alpha^2\gamma,\ (12+y_2)\alpha\delta^3\\
&\qquad \mid (20-2y_2)\beta\gamma\delta^2,\ y_2\gamma^3\delta\},\\
&\qquad \alpha=\tfrac{14}{19}\pi,\ \beta=\tfrac{12}{19}\pi,\ \gamma=\tfrac{10}{19}\pi,\ \delta=\tfrac{8}{19}\pi;\\
&\{96\alpha^2\beta\gamma\delta\colon 48\alpha\beta^2,\ 56\alpha^2\gamma,\ 32\alpha\delta^3 \mid 10\gamma^4\},\\
&\qquad \alpha=\tfrac{3}{4}\pi,\ \beta=\tfrac{5}{8}\pi,\ \gamma=\tfrac{1}{2}\pi,\ \delta=\tfrac{5}{12}\pi;\\
&\{116\alpha^2\beta\gamma\delta\colon 48\alpha\beta^2,\ 76\alpha^2\gamma,\ 32\alpha\delta^3 \mid 20\beta\gamma^2\delta\},\\
&\qquad \alpha=\tfrac{22}{29}\pi,\ \beta=\tfrac{18}{29}\pi,\ \gamma=\tfrac{14}{29}\pi,\ \delta=\tfrac{12}{29}\pi;\\
&\{156\alpha^2\beta\gamma\delta\colon 68\alpha\beta^2,\ 96\alpha^2\gamma,\ 52\alpha\delta^3 \mid 20\beta\gamma^3\},\\
&\qquad \alpha=\tfrac{10}{13}\pi,\ \beta=\tfrac{8}{13}\pi,\ \gamma=\tfrac{6}{13}\pi,\ \delta=\tfrac{16}{39}\pi.
\end{aligned}$$

The following are the full AVCs from $\{\alpha\beta^2, \alpha^2\gamma, \beta\delta^3\}$.

$$
\begin{aligned}
\{f\alpha^2\beta\gamma\delta\colon & x_1\alpha\beta^2,\ x_2\alpha^2\gamma,\ x_3\beta\delta^3 \\
& \mid y_1\beta\gamma^{\frac{f-4}{48}}\delta^2,\ y_2\alpha\gamma^{\frac{f+28}{48}}\delta,\ y_3\beta^2\gamma^{\frac{f-20}{48}}\delta,\ y_4\alpha\beta\gamma^{\frac{f+12}{48}}, \\
& y_5\gamma^{\frac{f+12}{48}}\delta^3,\ y_6\beta^3\gamma^{\frac{f-36}{48}},\ y_7\beta\gamma^{\frac{f-4}{24}}\delta,\ y_8\alpha\gamma^{\frac{f+12}{24}}, \\
& y_9\beta^2\gamma^{\frac{f-12}{24}},\ y_{10}\gamma^{\frac{f+4}{24}}\delta^2,\ y_{11}\beta\gamma^{\frac{f-4}{16}},\ y_{12}\gamma^{\frac{f}{12}}\}, \\
& x_1 = \tfrac{1}{3}f - \tfrac{1}{6}y_1 + \tfrac{1}{6}y_2 - \tfrac{5}{6}y_3 - \tfrac{1}{2}y_4 + \tfrac{1}{2}y_5 - \tfrac{3}{2}y_6 \\
& \quad - \tfrac{1}{3}y_7 - y_9 + \tfrac{1}{3}y_{10} - \tfrac{1}{2}y_{11}, \\
& x_2 = \tfrac{5}{6}f + \tfrac{1}{12}y_1 - \tfrac{7}{12}y_2 + \tfrac{5}{12}y_3 - \tfrac{1}{4}y_4 - \tfrac{1}{4}y_5 + \tfrac{3}{4}y_6 \\
& \quad + \tfrac{1}{6}y_7 - \tfrac{1}{2}y_8 + \tfrac{1}{12}y_9 - \tfrac{1}{6}y_{10} + \tfrac{1}{4}y_{11}, \\
& x_3 = \tfrac{1}{3}f - \tfrac{2}{3}y_1 - \tfrac{1}{3}y_2 - \tfrac{1}{3}y_3 - y_5 - \tfrac{1}{3}y_7 - \tfrac{2}{3}y_{10}, \\
& y_1 + \cdots + y_6 + 2(y_7 + \cdots + y_{10}) + 3y_{11} + 4y_{12} = 8, \\
& \alpha = \left(1 - \tfrac{12}{f}\right)\pi,\ \beta = \left(\tfrac{1}{2} + \tfrac{6}{f}\right)\pi,\ \gamma = \tfrac{24}{f}\pi,\ \delta = \left(\tfrac{1}{2} - \tfrac{2}{f}\right)\pi.
\end{aligned}
$$

The following are the full AVCs from $\{\alpha\beta^2, \alpha^2\gamma, \gamma\delta^3\}$.

$$
\begin{aligned}
\{40\alpha^2\beta\gamma\delta\colon & 20\alpha\beta^2,\ (30 - y_2)\alpha^2\gamma,\ (14 - y_2 - 2y_3)\gamma\delta^3 \\
& \mid (y_2 + y_3 - 2)\gamma^2\delta,\ y_2\alpha^2\delta^2,\ y_3\delta^5\}, \\
& \alpha = \tfrac{3}{5}\pi,\ \beta = \tfrac{7}{10}\pi,\ \gamma = \tfrac{4}{5}\pi,\ \delta = \tfrac{2}{5}\pi; \\
\{44\alpha^2\beta\gamma\delta\colon & (24 - y_2 - y_3)\alpha\beta^2,\ (32 - y_2)\alpha^2\gamma,\ (16 - y_2 - 2y_3)\gamma\delta^3 \\
& \mid (2y_2 + 2y_3 - 4)\beta\gamma\delta,\ y_2\alpha^3\delta,\ y_3\alpha\delta^4\}, \\
& \alpha = \tfrac{6}{11}\pi,\ \beta = \tfrac{8}{11}\pi,\ \gamma = \tfrac{10}{11}\pi,\ \delta = \tfrac{4}{11}\pi; \\
\{f\alpha^2\beta\gamma\delta\colon & x_1\alpha\beta^2,\ x_2\alpha^2\gamma,\ x_3\gamma\delta^3 \mid y_i\alpha^{a_i}\beta\delta^{d_i},\ y_i'\alpha^{a_i'}\delta^{d_i'}\}, \\
& 3a_i + 2d_i = \tfrac{f+12}{8},\ 3a_i' + 2d_i' = \tfrac{f}{4},\ f \ge 48, \\
& x_1 = \tfrac{1}{2}f - \tfrac{1}{2}\sum y_i, \\
& x_2 = \tfrac{2}{3}f + \tfrac{1}{3}\sum d_i y_i + \tfrac{1}{3}\sum d_i' y_i', \\
& x_3 = \tfrac{1}{3}f - \tfrac{1}{3}\sum d_i y_i - \tfrac{1}{3}\sum d_i' y_i', \\
& \sum y_i + 2\sum y_i' = 4, \\
& \alpha = \tfrac{24}{f}\pi,\ \beta = \left(1 - \tfrac{12}{f}\right)\pi,\ \gamma = \left(2 - \tfrac{48}{f}\right)\pi,\ \delta = \tfrac{16}{f}\pi.
\end{aligned}
$$

The following are the full AVCs from $\{\alpha\beta^2, \alpha^2\gamma, \delta^4\}$.

$$
\begin{aligned}
\{f\alpha^2\beta\gamma\delta\colon\ & x_1\alpha\beta^2,\ x_2\alpha^2\gamma,\ x_3\delta^4 \\
& \mid y_1\gamma^{\frac{f}{32}}\delta^3,\ y_2\beta\gamma^{\frac{f-8}{32}}\delta^2,\ y_3\alpha\gamma^{\frac{f+16}{32}}\delta,\ y_4\beta^2\gamma^{\frac{f-16}{32}}\delta, \\
& y_5\alpha\beta\gamma^{\frac{f+8}{32}},\ y_6\beta^3\gamma^{\frac{f-24}{32}},\ y_7\gamma^{\frac{f}{16}}\delta^2,\ y_8\beta\gamma^{\frac{f-4}{16}}\delta, \\
& y_9\alpha\gamma^{\frac{f+8}{16}},\ y_{10}\beta^2\gamma^{\frac{f-8}{16}},\ y_{11}\gamma^{\frac{3f}{32}}\delta,\ y_{12}\beta\gamma^{\frac{3f-8}{32}},\ y_{13}\gamma^{\frac{f}{8}}\}, \\
& x_1 = \tfrac{1}{2}f - \tfrac{1}{2}y_2 - y_4 - \tfrac{1}{2}y_5 - \tfrac{3}{2}y_6 - \tfrac{1}{2}y_8 - y_{10} - \tfrac{1}{2}y_{12}, \\
& x_2 = \tfrac{3}{4}f + \tfrac{1}{4}y_2 - \tfrac{1}{2}y_3 + \tfrac{1}{2}y_4 - \tfrac{1}{4}y_5 + \tfrac{3}{4}y_6 + \tfrac{1}{4}y_8 - \tfrac{1}{2}y_9 + \tfrac{1}{2}y_{10} + \tfrac{1}{4}y_{12}, \\
& x_3 = \tfrac{1}{4}f - \tfrac{3}{4}y_1 - \tfrac{1}{2}y_2 - \tfrac{1}{4}y_3 - \tfrac{1}{4}y_4 - \tfrac{1}{2}y_7 - \tfrac{1}{4}y_8 - \tfrac{3}{4}y_{11}, \\
& y_1 + \cdots + y_6 + 2(y_7 + \cdots + y_{10}) + 3(y_{11} + y_{12}) + 4y_{13} = 8, \\
& \alpha = \left(1 - \tfrac{8}{f}\right)\pi,\ \beta = \left(\tfrac{1}{2} + \tfrac{4}{f}\right)\pi,\ \gamma = \tfrac{16}{f}\pi,\ \delta = \tfrac{1}{2}\pi.
\end{aligned}
$$

The following are the full AVCs from $\{\alpha\beta^2, \alpha^2\gamma, \delta^5\}$.

$$
\begin{aligned}
& \{40\alpha^2\beta\gamma\delta\colon 20\alpha\beta^2,\ (30-y_1)\alpha^2\gamma,\ (2-y_1)\delta^5 \mid y_1\alpha^2\delta^2,\ (10+y_1)\gamma\delta^3\}, \\
& \qquad \alpha = \tfrac{3}{5}\pi,\ \beta = \tfrac{7}{10}\pi,\ \gamma = \tfrac{4}{5}\pi,\ \delta = \tfrac{2}{5}\pi; \\
& \{80\alpha^2\beta\gamma\delta\colon 40\alpha\beta^2,\ 60\alpha^2\gamma,\ 12\delta^5 \mid 10\gamma^2\delta^2\}, \\
& \qquad \alpha = \tfrac{7}{10}\pi,\ \beta = \tfrac{13}{20}\pi,\ \gamma = \tfrac{3}{5}\pi,\ \delta = \tfrac{2}{5}\pi; \\
& \{100\alpha^2\beta\gamma\delta\colon 40\alpha\beta^2,\ 80\alpha^2\gamma,\ 12\delta^5 \mid 20\beta\gamma\delta^2\}, \\
& \qquad \alpha = \tfrac{18}{25}\pi,\ \beta = \tfrac{16}{25}\pi,\ \gamma = \tfrac{14}{25}\pi,\ \delta = \tfrac{2}{5}\pi; \\
& \{120\alpha^2\beta\gamma\delta\colon 60\alpha\beta^2,\ 90\alpha^2\gamma,\ 22\delta^5 \mid 10\gamma^3\delta\}, \\
& \qquad \alpha = \tfrac{11}{15}\pi,\ \beta = \tfrac{19}{30}\pi,\ \gamma = \tfrac{8}{15}\pi,\ \delta = \tfrac{2}{5}\pi; \\
& \{160\alpha^2\beta\gamma\delta\colon 80\alpha\beta^2,\ 120\alpha^2\gamma,\ 32\delta^5 \mid 10\gamma^4\}, \\
& \qquad \alpha = \tfrac{3}{4}\pi,\ \beta = \tfrac{5}{8}\pi,\ \gamma = \tfrac{1}{2}\pi,\ \delta = \tfrac{2}{5}\pi; \\
& \{180\alpha^2\beta\gamma\delta\colon 80\alpha\beta^2,\ 140\alpha^2\gamma,\ 32\delta^5 \mid 20\beta\gamma^2\delta\}, \\
& \qquad \alpha = \tfrac{34}{45}\pi,\ \beta = \tfrac{28}{45}\pi,\ \gamma = \tfrac{22}{45}\pi,\ \delta = \tfrac{2}{5}\pi; \\
& \{260\alpha^2\beta\gamma\delta\colon 120\alpha\beta^2,\ 200\alpha^2\gamma,\ 52\delta^5 \mid 20\beta\gamma^3\}, \\
& \qquad \alpha = \tfrac{10}{13}\pi,\ \beta = \tfrac{8}{13}\pi,\ \gamma = \tfrac{6}{13}\pi,\ \delta = \tfrac{2}{5}\pi.
\end{aligned}
$$

The following are the full AVCs from $\{\alpha\beta^2,\gamma^3\}$.

$$\{f\alpha\beta\gamma^3\colon x_1\alpha\beta^2,\ x_2\gamma^3 \mid y_1\alpha^{\frac{f+12}{24}}\beta\gamma,\ y_2\alpha^{\frac{f}{12}}\gamma^2,\ y_3\alpha^{\frac{f+4}{8}}\beta,\ y_4\alpha^{\frac{f}{6}}\gamma,\ y_5\alpha^{\frac{f}{4}}\},$$
$$x_1=\tfrac{1}{2}f-\tfrac{1}{2}y_1-\tfrac{1}{2}y_3,$$
$$x_2=f-\tfrac{1}{3}y_1-\tfrac{2}{3}y_2-\tfrac{1}{3}y_4,$$
$$y_1+2y_2+3y_3+4y_4+6y_5=12,$$
$$\alpha=\tfrac{8}{f}\pi,\ \beta=\left(1-\tfrac{4}{f}\right)\pi,\ \gamma=\tfrac{2}{3}\pi.$$
$$\{24\alpha\beta^3\gamma\colon 24\alpha\beta^2,\ 8\gamma^3 \mid 6\beta^4\},\ \alpha=\pi,\ \beta=\tfrac{1}{2}\pi,\ \gamma=\tfrac{2}{3}\pi;$$
$$\{36\alpha\beta^3\gamma\colon 36\alpha\beta^2,\ 8\gamma^3 \mid 12\beta^3\gamma\},\ \alpha=\tfrac{10}{9}\pi,\ \beta=\tfrac{4}{9}\pi,\ \gamma=\tfrac{2}{3}\pi;$$
$$\{60\alpha\beta^3\gamma\colon 60\alpha\beta^2,\ 20\gamma^3 \mid 12\beta^5\},\ \alpha=\tfrac{6}{5}\pi,\ \beta=\tfrac{2}{5}\pi,\ \gamma=\tfrac{2}{3}\pi;$$
$$\{48\alpha^2\beta\gamma^2\colon 24\alpha\beta^2,\ 32\gamma^3 \mid 18\alpha^4\},\ \alpha=\tfrac{1}{2}\pi,\ \beta=\tfrac{3}{4}\pi,\ \gamma=\tfrac{2}{3}\pi;$$
$$\{24\alpha^2\beta^2\gamma\colon 24\alpha\beta^2,\ 8\gamma^3 \mid 6\alpha^4\},\ \alpha=\tfrac{1}{2}\pi,\ \beta=\tfrac{3}{4}\pi,\ \gamma=\tfrac{2}{3}\pi;$$
$$\{36\alpha^2\beta^2\gamma\colon 36\alpha\beta^2,\ 8\gamma^3 \mid 12\alpha^3\gamma\},\ \alpha=\tfrac{4}{9}\pi,\ \beta=\tfrac{7}{9}\pi,\ \gamma=\tfrac{2}{3}\pi;$$
$$\{60\alpha^2\beta^2\gamma\colon (48+y_2)\alpha\beta^2,\ 20\gamma^3 \mid (24-2y_2)\alpha^3\beta,\ y_2\alpha^5\},$$
$$\alpha=\tfrac{2}{5}\pi,\ \beta=\tfrac{4}{5}\pi,\ \gamma=\tfrac{2}{3}\pi.$$

The following are the full AVCs from $\{\alpha\beta^2,\gamma^3,\alpha\delta^3\}$.

$$\{60\alpha\beta^2\gamma\delta\colon (60-y_1)\alpha\beta^2,\ 20\gamma^3,\ y_1\alpha\delta^3 \mid y_1\beta^2\delta^2,\ (12-y_1)\delta^5\},$$
$$\alpha=\tfrac{4}{5}\pi,\ \beta=\tfrac{3}{5}\pi,\ \gamma=\tfrac{2}{3}\pi,\ \delta=\tfrac{2}{5}\pi;$$
$$\{84\alpha\beta^2\gamma\delta\colon 72\alpha\beta^2,\ 20\gamma^3,\ 12\alpha\delta^3 \mid 24\beta\gamma\delta^2\},$$
$$\alpha=\tfrac{6}{7}\pi,\ \beta=\tfrac{4}{7}\pi,\ \gamma=\tfrac{2}{3}\pi,\ \delta=\tfrac{8}{21}\pi;$$
$$\{132\alpha\beta^2\gamma\delta\colon (120-y_1)\alpha\beta^2,\ 20\gamma^3,\ (12+y_1)\alpha\delta^3 \mid y_1\beta^3\delta,\ (24-y_1)\beta\delta^4\},$$
$$\alpha=\tfrac{10}{11}\pi,\ \beta=\tfrac{6}{11}\pi,\ \gamma=\tfrac{2}{3}\pi,\ \delta=\tfrac{4}{11}\pi;$$
$$\{f\alpha\beta\gamma^2\delta\colon x_1\alpha\beta^2,\ x_2\gamma^3,\ x_3\alpha\delta^3$$
$$\mid y_1\alpha^{\frac{f+60}{72}}\beta\delta,\ y_2\alpha^{\frac{f+36}{72}}\beta\gamma,\ y_3\alpha^{\frac{f}{36}}\gamma^2,\ y_4\alpha^{\frac{f+24}{36}}\delta^2,\ y_5\alpha^{\frac{f+12}{36}}\gamma\delta,$$
$$y_6\alpha^{\frac{f+12}{24}}\beta,\ y_7\alpha^{\frac{f}{18}}\gamma,\ y_8\alpha^{\frac{f+6}{18}}\delta,\ y_9\alpha^{\frac{f}{12}}\},$$
$$x_1=\tfrac{1}{2}f-\tfrac{1}{2}y_1-\tfrac{1}{2}y_2-\tfrac{1}{2}y_6,$$
$$x_2=\tfrac{1}{3}f-\tfrac{1}{3}y_2-\tfrac{2}{3}y_3-\tfrac{1}{3}y_5-\tfrac{1}{3}y_7,$$
$$x_3=\tfrac{1}{3}f-\tfrac{1}{3}y_1-\tfrac{2}{3}y_4-\tfrac{1}{3}y_5-\tfrac{1}{3}y_8,$$
$$y_1+y_2+2y_3+2y_4+2y_5+3y_6+4y_7+4y_8+6y_9=12,$$
$$\alpha=\tfrac{24}{f}\pi,\ \beta=\left(1-\tfrac{12}{f}\right)\pi,\ \gamma=\tfrac{2}{3}\pi,\ \delta=\left(\tfrac{2}{3}-\tfrac{8}{f}\right)\pi.$$

The following are the full AVCs from $\{\alpha\beta^2, \gamma^3, \beta\delta^3\}$.

$$
\begin{aligned}
&\{f\alpha\beta^2\gamma\delta\colon x_1\alpha\beta^2,\ x_2\gamma^3,\ x_3\beta\delta^3 \\
&\qquad \mid y_1\alpha^{\frac{f-12}{72}}\gamma^2\delta,\ y_2\alpha^{\frac{f+36}{72}}\beta\gamma,\ y_3\alpha^{\frac{f-36}{72}}\gamma\delta^3,\ y_4\alpha^{\frac{f+12}{72}}\beta\delta^2,\ y_5\alpha^{\frac{f-60}{72}}\delta^5, \\
&\qquad y_6\alpha^{\frac{f}{36}}\beta^2,\ y_7\alpha^{\frac{f-12}{36}}\gamma\delta^2,\ y_8\alpha^{\frac{f+12}{36}}\beta\delta,\ y_9\alpha^{\frac{f-24}{36}}\delta^4, \\
&\qquad y_{10}\alpha^{\frac{f-4}{24}}\gamma\delta,\ y_{11}\alpha^{\frac{f+12}{24}}\beta,\ y_{12}\alpha^{\frac{f-12}{24}}\delta^3, \\
&\qquad y_{13}\alpha^{\frac{f}{18}}\gamma,\ y_{14}\alpha^{\frac{f-6}{18}}\delta^2,\ y_{15}\alpha^{\frac{5f-12}{72}}\delta,\ y_{16}\alpha^{\frac{f}{12}}\}, \\
&\qquad x_1 = \tfrac{5}{6}f + \tfrac{1}{6}y_1 - \tfrac{1}{2}y_2 + \tfrac{1}{6}y_3 - \tfrac{1}{6}y_4 + \tfrac{5}{6}y_5 + \tfrac{1}{3}y_7 - \tfrac{1}{3}y_8 + \tfrac{2}{3}y_9 \\
&\qquad\quad + \tfrac{1}{6}y_{10} - \tfrac{1}{2}y_{11} + \tfrac{1}{2}y_{12} + \tfrac{1}{3}y_{14} + \tfrac{1}{6}y_{15}, \\
&\qquad x_2 = \tfrac{1}{3}f - \tfrac{2}{3}y_1 - \tfrac{1}{3}y_2 - \tfrac{1}{3}y_3 - \tfrac{2}{3}y_6 - \tfrac{1}{3}y_7 - \tfrac{1}{3}y_{10} - \tfrac{1}{3}y_{13}, \\
&\qquad x_3 = \tfrac{1}{3}f - \tfrac{1}{3}y_1 - y_3 - \tfrac{2}{3}y_4 - \tfrac{5}{3}y_5 - \tfrac{2}{3}y_7 - \tfrac{1}{3}y_8 - \tfrac{4}{3}y_9 \\
&\qquad\quad - \tfrac{1}{3}y_{10} - y_{12} - \tfrac{2}{3}y_{14} - \tfrac{1}{3}y_{15}, \\
&\qquad y_1 + \cdots + y_5 + 2(y_6 + \cdots + y_9) + 3(y_{10} + y_{11} + y_{12}) \\
&\qquad\quad + 4(y_{13} + y_{14}) + 5y_{15} + 6y_{16} = 12, \\
&\qquad \alpha = \tfrac{24}{f}\pi,\ \beta = \left(1 - \tfrac{12}{f}\right)\pi,\ \gamma = \tfrac{2}{3}\pi,\ \delta = \left(\tfrac{1}{3} + \tfrac{4}{f}\right)\pi; \\
&\{84\alpha\beta\gamma^2\delta\colon 36\alpha\beta^2,\ 56\gamma^3,\ 12\beta\delta^3 \mid 24\alpha^2\delta^2\}, \\
&\qquad \alpha = \tfrac{4}{7}\pi,\ \beta = \tfrac{9}{14}\pi,\ \gamma = \tfrac{2}{3}\pi,\ \delta = \tfrac{1}{7}\pi; \\
&\{228\alpha\beta\gamma^2\delta\colon 84\alpha\beta^2,\ 152\gamma^3,\ 60\beta\delta^3 \mid 48\alpha^3\delta\}, \\
&\qquad \alpha = \tfrac{10}{19}\pi,\ \beta = \tfrac{14}{19}\pi,\ \gamma = \tfrac{2}{3}\pi,\ \delta = \tfrac{8}{19}\pi.
\end{aligned}
$$

The following are the full AVCs from $\{\alpha\beta^2, \gamma^3, \gamma\delta^3\}$.

$$
\begin{aligned}
&\{36\alpha\beta^2\gamma\delta\colon 36\alpha\beta^2,\ 8\gamma^3,\ 12\gamma\delta^3\},\ \alpha + 2\beta = 2\pi,\ \gamma = \tfrac{2}{3}\pi,\ \delta = \tfrac{4}{9}\pi; \\
&\{72\alpha\beta\gamma^2\delta\colon 36\alpha\beta^2,\ 44\gamma^3,\ 12\gamma\delta^3 \mid 18\alpha^2\delta^2\}, \\
&\qquad \alpha = \tfrac{5}{9}\pi,\ \beta = \tfrac{13}{18}\pi,\ \gamma = \tfrac{2}{3}\pi,\ \delta = \tfrac{4}{9}\pi; \\
&\{108\alpha\beta\gamma^2\delta\colon 54\alpha\beta^2,\ 62\gamma^3,\ 30\gamma\delta^3 \mid 18\alpha^3\delta\}, \\
&\qquad \alpha = \tfrac{14}{27}\pi,\ \beta = \tfrac{20}{27}\pi,\ \gamma = \tfrac{2}{3}\pi,\ \delta = \tfrac{4}{9}\pi; \\
&\{144\alpha\beta\gamma^2\delta\colon 72\alpha\beta^2,\ 80\gamma^3,\ 48\gamma\delta^3 \mid 18\alpha^4\}, \\
&\qquad \alpha = \tfrac{1}{2}\pi,\ \beta = \tfrac{3}{4}\pi,\ \gamma = \tfrac{2}{3}\pi,\ \delta = \tfrac{4}{9}\pi.
\end{aligned}
$$

The following are the full AVCs from $\{\alpha\beta^2,\gamma^3,\delta^4\}$.

$$\{24\alpha\beta^2\gamma\delta\colon 24\alpha\beta^2,\ 8\gamma^3,\ 6\delta^4\},\ \alpha+2\beta=2\pi,\ \gamma=\tfrac{2}{3}\pi,\ \delta=\tfrac{1}{2}\pi;$$
$$\{72\alpha\beta\gamma^2\delta\colon 36\alpha\beta^2,\ 44\gamma^3,\ 18\delta^4\ \mid\ 12\alpha^3\gamma\},$$
$$\alpha=\tfrac{4}{9}\pi,\ \beta=\tfrac{7}{9}\pi,\ \gamma=\tfrac{2}{3}\pi,\ \delta=\tfrac{1}{2}\pi;$$
$$\{96\alpha\beta\gamma^2\delta\colon 48\alpha\beta^2,\ 56\gamma^3,\ 18\delta^4\ \mid\ 24\alpha^2\gamma\delta\},$$
$$\alpha=\tfrac{5}{12}\pi,\ \beta=\tfrac{19}{24}\pi,\ \gamma=\tfrac{2}{3}\pi,\ \delta=\tfrac{1}{2}\pi;$$
$$\{120\alpha\beta\gamma^2\delta\colon (48+y_2)\alpha\beta^2,\ 80\gamma^3,\ 30\delta^4\ \mid (24-2y_2)\alpha^3\beta,\ y_2\alpha^5\},$$
$$\alpha=\tfrac{2}{5}\pi,\ \beta=\tfrac{4}{5}\pi,\ \gamma=\tfrac{2}{3}\pi,\ \delta=\tfrac{1}{2}\pi;$$
$$\{192\alpha\beta\gamma^2\delta\colon 96\alpha\beta^2,\ 128\gamma^3,\ 42\delta^4\ \mid\ 24\alpha^4\delta\},$$
$$\alpha=\tfrac{3}{8}\pi,\ \beta=\tfrac{13}{16}\pi,\ \gamma=\tfrac{2}{3}\pi,\ \delta=\tfrac{1}{2}\pi.$$

The following and the case of $f=60$ for the AVC on page 47 are the full AVCs from $\{\alpha\beta^2,\gamma^3,\delta^5\}$.

$$\{60\alpha\beta^2\gamma\delta\colon 60\alpha\beta^2,\ 20\gamma^3,\ 12\delta^5\},\ \alpha+2\beta=2\pi,\ \gamma=\tfrac{2}{3}\pi,\ \delta=\tfrac{2}{5}\pi;$$
$$\{120\alpha\beta\gamma^2\delta\colon 60\alpha\beta^2,\ 80\gamma^3,\ 12\delta^5\ \mid\ 30\alpha^2\delta^2\},$$
$$\alpha=\tfrac{3}{5}\pi,\ \beta=\tfrac{7}{10}\pi,\ \gamma=\tfrac{2}{3}\pi,\ \delta=\tfrac{2}{5}\pi.$$

Four distinct angles at degree 3 vertices

If there are four distinct angles at degree 3 vertices, then the pentagon is $\alpha^2\beta\gamma\delta$, $\alpha\beta^2\gamma\delta$, $\alpha\beta\gamma^2\delta$, $\alpha\beta\gamma\delta^2$ or $\alpha\beta\gamma\delta\epsilon$. The possible degree 3 vertices are listed in Table 1.

For $\{\alpha\beta\gamma,\alpha\delta^2\}$, by Lemma 3, we get the following possible combinations of the pentagon and degree 3 vertices:

1. The pentagon is $\alpha^2\beta\gamma\delta$, and degree 3 vertices are $\{\alpha\beta\gamma,\alpha\delta^2\}$.
2. The pentagon is $\alpha^2\beta\gamma\delta$, $\alpha\beta^2\gamma\delta$, $\alpha\beta\gamma^2\delta$, or $\alpha\beta\gamma\delta^2$, and degree 3 vertices are $\{\alpha\beta\gamma,\alpha\delta^2,\beta^2\delta\}$ or $\{\alpha\beta\gamma,\alpha\delta^2,\beta^3\}$.
3. The pentagon is $\alpha\beta\gamma\delta\epsilon$, and degree 3 vertices are $\{\alpha\beta\gamma,\alpha\delta^2,\beta^2\delta\}$ or $\{\alpha\beta\gamma,\alpha\delta^2,\beta^3\}$, and one of $\alpha\epsilon^3,\beta\epsilon^3,\gamma\epsilon^3,\delta\epsilon^3,\epsilon^4,\epsilon^5$ is also a vertex.

The first combination is Example 3 in Section 6. The order matrix is invertible in the later two combinations, and the routines in Sections 4 and 5 can be applied.

The AVC $\{\alpha\beta\gamma, \alpha\delta^2\}$ is completely similar to $\{\alpha\beta\gamma, \alpha\delta^2\}$. One combination is similar to Example 3 in Section 6, and two combinations can be treated by Sections 4 and 5.

For $\{\alpha\beta\gamma, \delta^3\}$, we note that $\{\alpha\beta\gamma\delta^2 \colon \alpha\beta\gamma, \delta^3\}$ is impossible because the combination implies $f = 12$. Then up to symmetry, we get the following possible combinations of the pentagon and degree 3 vertices:

1. The pentagon is $\alpha^2\beta\gamma\delta$, and degree 3 vertices are $\{\alpha\beta\gamma, \delta^3\}$.
2. The pentagon is $\alpha\beta\gamma\delta\epsilon$, and degree 3 vertices are $\{\alpha\beta\gamma, \delta^3\}$, and $\alpha\epsilon^3$ is also a vertex.
3. The pentagon is $\alpha\beta\gamma\delta\epsilon$, and degree 3 vertices are $\{\alpha\beta\gamma, \delta^3\}$, and one of $\delta\epsilon^3, \epsilon^4, \epsilon^5$ is also a vertex.

The first combination has positive lower bounds for α, δ, and the second combination has positive lower bounds for α, δ, ϵ. Both can be treated similar to Example 1 in Section 6. The third combination is Example 4 in Section 6.

For $\{\alpha\beta^2, \gamma\delta^2\}$, if the optional vertex $\alpha^2\delta$ also appears, then Sections 4 and 5 can be applied. If $\alpha\beta^2$ and $\gamma\delta^2$ are the only degree 3 vertices, then by an argument similar to the case $\{\alpha\beta^2, \gamma^3\}$ in Proposition 5, we find that β and δ together must appear at least three times in the pentagon. Therefore, up to the symmetry of exchanging β and γ, we may assume the pentagon is $\alpha\beta^2\gamma\delta$. The combination $\{\alpha\beta^2\gamma\delta \colon \alpha\beta^2, \gamma\delta^2\}$ has

$$\alpha + 2\beta = 2\pi,\ \gamma = \frac{8\pi}{f},\ \delta = \pi - \frac{4\pi}{f},$$

and can be treated similar to Example 3 in Section 6.

For $\{\alpha\beta^2, \alpha^2\gamma, \delta^3\}$, Sections 4 and 5 can be applied to all the cases.

So we know how to find all the full AVCs when there are four distinct angles at degree 3 vertices.

Five distinct angles at degree 3 vertices

If there are five distinct angles at degree 3 vertices, then the pentagon is $\alpha\beta\gamma\delta\epsilon$. The possible degree 3 vertices are listed in Table 1.

The AVC $\{\alpha\beta\gamma, \alpha\delta^2, \alpha^2\epsilon\}$ is the simplest. By Lemma 3, one of the optional vertices $\beta\epsilon^2, \beta^2\delta, \beta^3$ must appear. Then Sections 4 and 5 can be applied.

For $\{\alpha\beta\gamma, \alpha\delta\epsilon\}$, by Lemma 3, at least one of the optional vertices must appear. If $\beta\delta^2$ is a vertex, then

$$\alpha = \left(1 - \frac{4}{f}\right)\pi,\ \gamma = \left(1 + \frac{4}{f}\right)\pi - \beta,\ \delta = \pi - \frac{1}{2}\beta,\ \epsilon = \frac{4}{f}\pi + \frac{1}{2}\beta.$$

The situation does not happen for up to four distinct angles at degree 3 vertices. The difficulty also appears for other AVCs. For example, for $\{\alpha\beta\gamma, \alpha\delta^2, \beta\epsilon^2\}$, we have

$$\beta = \left(2 - \frac{8}{f}\right)\pi - \alpha,\ \gamma = \frac{8}{f}\pi,\ \delta = \pi - \frac{1}{2}\alpha,\ \epsilon = \frac{4}{f}\pi + \frac{1}{2}\alpha.$$

For $\{\alpha\beta\gamma, \alpha\delta^2, \delta\epsilon^2\}$, we have

$$\alpha = \left(2 - \frac{16}{f}\right)\pi,\ \beta + \gamma = \frac{16}{f}\pi,\ \delta = \frac{8}{f}\pi,\ \epsilon = \left(1 - \frac{4}{f}\right)\pi.$$

Although some of the techniques we developed so far may still be used to compute the full AVC, we choose do not pursue further. We expect the other considerations (so far we only used the numerical constraints), such as the topological configurations of the tiling, should be used to provide more restrictions on the possible AVCs.

We remark that the examples above allow free continuous choice of at most one angle. Among the AVCs with five distinct angles at degree 3 vertices, the only possible one allowing free choices of two angles is $\{\alpha\beta\gamma, \delta\epsilon^2\}$. However, we already have

$$\alpha + \beta + \gamma = 2\pi,\ \delta = \frac{8}{f}\pi,\ \epsilon = \left(1 - \frac{4}{f}\right)\pi,$$

and the appearance of one more vertex will cut the number of free choices. For example, according to Table 1, we should consider the case α^3 is also a vertex. In this case, we get

$$\alpha = \frac{2}{3}\pi,\ \beta + \gamma = \frac{4}{3}\pi,\ \delta = \frac{8}{f}\pi,\ \epsilon = \left(1 - \frac{4}{f}\right)\pi.$$

Now assume α^3 is not a vertex, so that $\alpha\beta\gamma$ and $\delta\epsilon^2$ are the only degree 3 vertices. Since $f > 12$, there must be a high degree vertex. Up to symmetry, we may also assume $\alpha < \beta < \gamma$. If $\alpha\gamma\cdots$ is a high degree vertex, then the vertex must be $\alpha\gamma\delta^g$, and we get

$$\alpha + \gamma = \left(2 - \frac{8g}{f}\right)\pi,\ \beta = \frac{8g}{f}\pi,\ \delta = \frac{8}{f}\pi,\ \epsilon = \left(1 - \frac{4}{f}\right)\pi.$$

If γ appears at a high degree vertex without α, then by an argument similar to Example 4 in Section 6, we get all such possible vertices and the corresponding angles

$$\begin{aligned}
\beta\gamma\delta^g &: \alpha = \tfrac{8g}{f}\pi,\ \beta + \gamma = \left(2 - \tfrac{8g}{f}\right)\pi; \\
\gamma\delta^g &: \alpha + \beta = \tfrac{8g}{f}\pi,\ \gamma = \left(2 - \tfrac{8g}{f}\right)\pi; \\
\gamma^2\delta^g &: \alpha + \beta = \left(1 + \tfrac{4g}{f}\right)\pi,\ \gamma = \left(1 - \tfrac{4g}{f}\right)\pi; \\
\gamma\delta^g\epsilon &: \alpha + \beta = \left(1 + \tfrac{8g-4}{f}\right)\pi,\ \gamma = \left(1 - \tfrac{8g-4}{f}\right)\pi.
\end{aligned}$$

We conclude that, if there are five distinct angles at degree 3 vertices, then the AVC does not allow free continuous choice of two angles.

8 Tilings Allowing Free Continuous Choice of Two Angles

Since only numerical constraints are used to obtain our AVCs, we expect that many cannot be realized by angle congruent tilings. In [4], we saw that only some AVCs from the complete list in [4, Proposition 17] can be realized as angle congruent tilings in [4, Section 6], and there is only one family of geometrically congruent tilings [1].

The family of geometrically congruent tilings in [4] actually allows free choice of two angles (continuously and within some range). It can be constructed by the *pentagonal subdivision* illustrated in Figure 1. Start with any tiling of the sphere (or any compact oriented surface without boundary). We add one vertex at the center of each tile and add two vertices on each edge. For each tile, the orientation of the surface determines the direction of the boundary edges, and can be used to label the two new vertices on each boundary edge as the first and second. Then we connect the center vertex to the second new vertex of each boundary edge. The subdivision divides a tile with n edges into n pentagons.

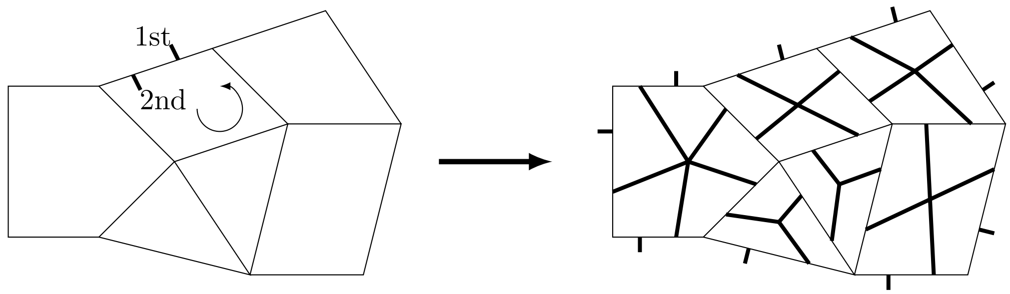

Figure 1: Pentagonal subdivision.

If we start with a regular tiling, then we may triple divide each edge in such a way that the first and third sub-edges have equal length a (see the left of Figure 2), to get a tiling by geometrically congruent pentagons (see the middle of Figure 2). Moreover, by only requiring $\alpha+\beta+\gamma=2\pi$ instead of $\alpha+\gamma=\beta=2\pi$ (i.e., we no longer keep the original edges straight), we get a tiling by geometrically congruent pentagons given on the right of Figure 2 (the original regular tiling is made up of n-gons and each vertex has degree m). By the same argument as in [4], the pentagonal tile allows free choice of two variables, which can be two angles from α, β, γ (a, b, c are determined by these two angles).

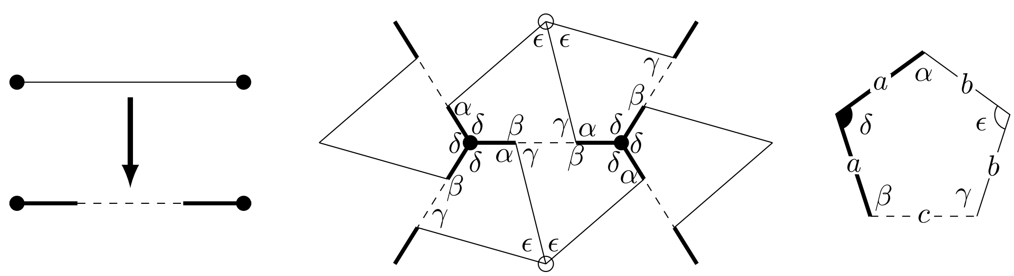

Figure 2: Geometrically congruent tiling: $\alpha+\beta+\gamma=2\pi$, $\delta=\frac{2}{m}\pi$, $\epsilon=\frac{2}{n}\pi$.

The pentagonal subdivision of the regular tetrahedron is the geometrically congruent tiling in [1, 4]. The pentagonal subdivision of the regular cube on the left of Figure 3 is a tiling of the sphere by 24 geometrically congruent pentagons. The pentagonal subdivision of the regular octahedron on the right of Figure 3 is the same tiling, with the duality between the cube and the octahedron exchanging a and b and the corresponding angles. Similarly, the pentagonal subdivisions of the regular dodecahedron and the regular icosahedron are the same tiling of the sphere by 60 geometrically congruent pentagons.

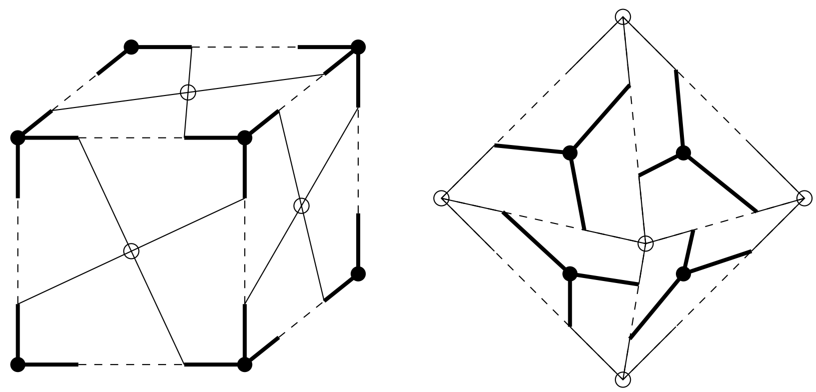

Figure 3: Pentagonal subdivisions of the cube and the octahedron.

The pentagonal subdivisions of the platonic solids give three families of the tilings of the sphere by $f = 12, 24, 60$ geometrically congruent pentagons. Each family allows free choice of two parameters that can be chosen to be two angles among α, β, γ. The natural problem is whether these are the only families of the tilings of the sphere by geometrically congruent pentagons that allow free choice of two angles.

Since this paper is mainly concerned with the angle congruent tilings, we will investigate the problem by first ignoring the edge lengths a, b, c in the pentagonal subdivisions. The pentagonal subdivision then gives us three families of angle congruent tilings that allow free choices of two angles. Now the question is whether the three families are the only angle congruent tilings that allow free choice of two angles.

For $f = 12$, by [4, Proposition 17], the only AVC allowing free choice of two angles is $\{12\alpha^2\beta\gamma\delta\colon 8\alpha^3, 12\beta\gamma\delta\}$ (the exchange $\alpha, \alpha, \beta, \gamma, \delta \to \delta, \epsilon, \alpha, \beta, \gamma$ translates into the notations used in the pentagonal subdivision). By [4, Proposition 19], the AVC can be realized (up to the symmetry of exchanging β, γ, δ) by the pentagonal subdivision of the tetrahedron, and by the further modification of six independent exchanges of β, γ.

For $f > 12$, by the discussion in Section 7, the only AVCs allowing free choice of two angles are (6.2), (6.3), (6.4) from Example 4 of Section 6, with respective $f = 60, 36, 24$. We already know that the AVCs (6.2) and (6.4) can be realized by pentagonal subdivisions. It turns out that no other realizations are possible (so there is no further modification similar to the case $f = 12$). Moreover, the AVC (6.3) cannot be realized.

Theorem 6. *The following are all the angle congruent tilings of the sphere that allow free continuous choice of two angles.*

1. *Pentagonal subdivision of tetrahedron, and the further modification of six independent exchanges of two angles. See [4, Figure 22].*
2. *Pentagonal subdivision of (cube, dodecahedron).*
3. *Pentagonal subdivision of (dodecahedron, icosahedron).*

We note that the proof does not use the free choice of two angles from α, β, γ, and only uses the consequence that the five angles are distinct.

Proof. We first consider the AVC $\{24\alpha\beta\gamma\delta\epsilon\colon 24\alpha\beta\gamma, 8\delta^3, 6\epsilon^4\}$ in (6.4). To simplify the presentation, we introduce some notations. Denote by P_i the i-th tile, and illustrate the tile in the picture as circled i. Denote by θ_i the angle θ in P_i. Denote by $V_{\theta,i}$ the vertex of P_i where the angle θ is located. The notations are unambiguous because the pentagon has only one θ.

We start with a vertex ϵ^4. In Figure 4, two of the four tiles around the vertex are denoted P_1, P_2. If ϵ and δ are adjacent in the pentagon, then up to symmetry, we may assume that δ_1 is located as on the left of Figure 4. By the AVC, we get $V_{\delta,1} = \delta^3$. This gives a tile P_3 out of P_1, P_2, and we know the locations of δ_2, δ_3. Up to symmetry, we may assume that the angle ϵ_3 adjacent to δ_3 in P_3 is located as indicated. Then by the AVC (we will not repeat saying "by the AVC" again), we get $V_{\epsilon,3} = \epsilon^4$. This implies that there are two ϵ in P_1, a contradiction.

So we conclude that δ and ϵ cannot be adjacent in the pentagon. Up to the symmetry of exchanging α, β, γ, we may assume that all the angles in the pentagon are located as on the right of Figure 2 (edge lengths are ignored). We may also assume that all the angles of P_1 are located as indicated on the right of Figure 4.

By $V_{\alpha,1} = \alpha\beta\gamma$ and the non-adjacency of β_2, ϵ_2, we get the locations of γ_2, β_3. Then ϵ_2, γ_2 determine all the angles of P_2. By $V_{\delta,1} = \delta^3$, we get (the location of) δ_3. Then β_3, δ_3 determine all the angles of P_3. By $V_{\gamma,3} = \alpha\beta\gamma$ and $V_{\epsilon,3} = \epsilon^4$, we get a tile P_4 outside P_2, P_3. We also get α_4, ϵ_4, which determine all the angles of P_4.

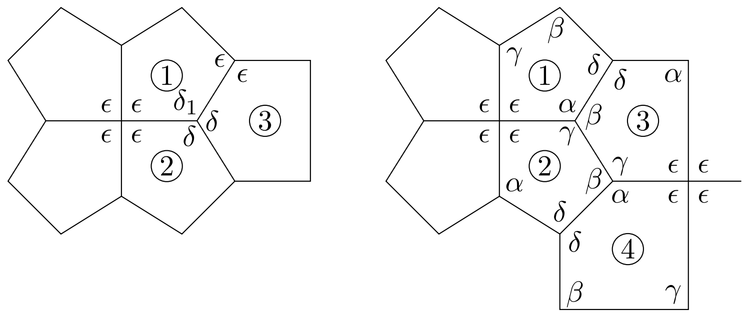

Figure 4: Derive tiling from $\{\alpha\beta\gamma\delta\epsilon\colon \alpha\beta\gamma, \delta^3, \epsilon^4\}$.

Note that the starting point for deriving the right of Figure 4 is the full angle information about P_1 at a vertex ϵ^4. Now we know the same full angle information about P_2, P_4 at their respective ϵ^4 vertices. So we can repeat the same argument by starting with P_2, P_4 and get all the angles of more tiles. More repetition of the same argument gives the pentagonal subdivision of (cube, dodecahedron).

The proof for $f = 60$ is completely similar to $f = 24$. The key point is that our argument for $f = 24$ only concerns two adjacent tiles around a degree 4 vertex ϵ^4. Such argument is clearly still valid for two adjacent tiles around a degree 5 vertex ϵ^4 in the AVC $\{60\alpha\beta\gamma\delta\epsilon\colon 60\alpha\beta\gamma, 20\delta^3, 12\epsilon^5\}$ in (6.2). The only difference is more complicated combinatorial structure of the tiling we get at the end, which is the pentagonal subdivision of (dodecahedron, icosahedron).

Finally, we prove that the AVC $\{36\alpha\beta\gamma\delta\epsilon\colon 36\alpha\beta\gamma, 8\delta^3, 12\delta\epsilon^3\}$ in (6.3) cannot be realized.

We start with four tiles P_1, P_2, P_3, P_4 around a vertex $\delta\epsilon^3$. If ϵ and δ are adjacent in the pentagon, then up to symmetry, we may assume that δ_1 is located as on the left of Figure 5. By $V_{\delta,1} = \delta^3$ or $\delta\epsilon^3$ and only one ϵ in P_2, we get $V_{\delta,1} = \delta^3$, a tile P_3 outside P_1, P_2, and the location of δ_3. Up to symmetry, we may assume that the angle ϵ_3 adjacent to δ_3 is located as indicated. Then $V_{\epsilon,3} = \delta\epsilon^3$, so that P_1 has either two ϵ or two δ, a contradiction.

Since δ and ϵ cannot be adjacent, up to the symmetry of exchanging α, β, γ, we may assume that the angles in the pentagon are arranged as on the right of Figure 2, and all the angles of P_2 are located as indicated on the right of Figure 5.

By $V_{\alpha,2} = \alpha\beta\gamma$ and the non-adjacency of γ_1, ϵ_1, we get a tile P_5 and

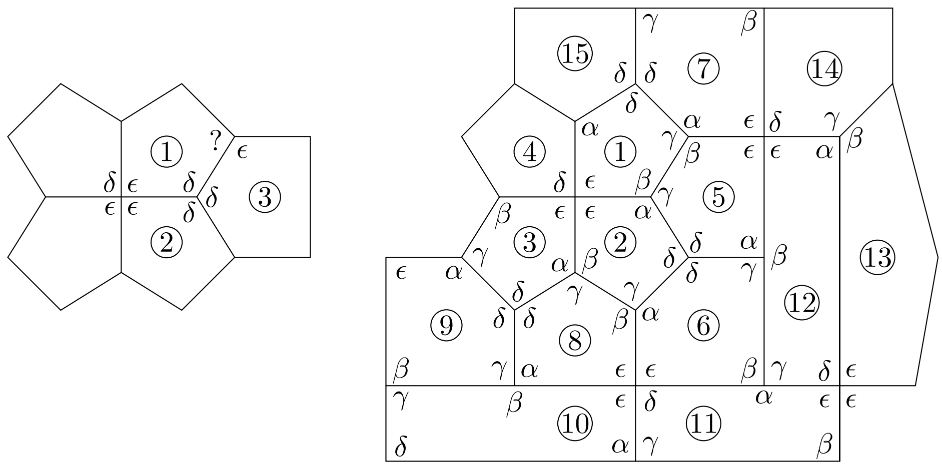

Figure 5: Derive tiling from $\{\alpha\beta\gamma\delta\epsilon\colon \alpha\beta\gamma, \delta^3, \delta\epsilon^3\}$.

(the locations of) β_1, γ_5. Then β_1, ϵ_1 determine (all the angles of) P_1. By $V_{\delta,2} = \delta^3$ or $\delta\epsilon^3$ and the non-adjacency of γ_5, ϵ_5, we get $V_{\delta,2} = \delta^3$, a tile P_6 and δ_5, δ_6. Then γ_5, δ_5 determine P_5. By $V_{\gamma,1} = V_{\beta,5} = \alpha\beta\gamma$, we get P_7 and α_7. By $V_{\gamma,2} = \alpha\beta\gamma$ and the fact that δ_6 is adjacent only to α_6, γ_6, we get P_8 and $\alpha_6, \gamma_6, \beta_8$. The angles $\alpha_6, \delta_6, \gamma_6$ determine P_6. By $V_{\beta,2} = \alpha\beta\gamma$ and the non-adjacency of α_8, β_8, we get α_3, γ_8. Then α_3, ϵ_3 determine P_3 and β_8, γ_8 determine P_8. From δ_3, δ_8, we get $V_{\delta,3} = V_{\delta,8} = \delta^3$, a tile P_9 and δ_9. Since α_9 is adjacent to δ_9 and cannot appear at the same vertex as α_8, we get α_9. Then α_9, δ_9 determine P_9. By $V_{\alpha,8} = V_{\gamma,9} = \alpha\beta\gamma$, we get P_{10} and β_{10}. By $V_{\epsilon,6} = V_{\epsilon,8} = \delta\epsilon^3$ and the non-adjacency of β_{10}, δ_{10}, we get ϵ_{10}. From $\epsilon_6, \epsilon_8, \epsilon_{10}$ at the vertex $\delta\epsilon^3$, we get P_{12} and δ_{12}.

By $V_{\alpha,5} = V_{\gamma,6} = \alpha\beta\gamma$, we get P_{12} and β_{12}. By $V_{\beta,6} = \alpha\beta\gamma$ and the non-adjacency of α_{12}, β_{12}, we get α_{11}, γ_{12}. Then α_{11}, δ_{11} determine P_{11} and β_{12}, γ_{12} determine P_{12}. By $V_{\epsilon,11} = \delta_{12} = \delta\epsilon^3$, we get P_{13}, ϵ_{13}. By $V_{\alpha,12} = \alpha\beta\gamma$ and the non-adjacency of $\gamma_{13}, \epsilon_{13}$, we get P_{14} and β_{13}, γ_{14}. By $V_{\epsilon,5} = V_{\epsilon,12} = \delta\epsilon^3$ and the non-adjacency of $\gamma_{14}, \epsilon_{14}$, we get ϵ_7, δ_{14}. Then α_7, ϵ_7 determine P_7. From δ_1, δ_7, we get $V_{\delta,1} = V_{\delta,7} = \delta^3$, a tile P_{15} and δ_{15}. Since $V_{\alpha,1} = \alpha\beta\gamma$ has degree 3, P_{15} is also glued to P_1 as indicated. Now the angle β at $V_{\alpha,1} = \alpha\beta\gamma$ must be either β_4 or β_{15}, so that β and δ must be adjacent in either P_4 or P_{15}, a contradiction. □

Now we add the edge length consideration to the conclusion of Theorem 6 and get geometrically congruent tilings allowing free choice of two angles.

The five distinct angles enable us to label edges in a tile by the two angles at the ends of the edges. For example, the pentagon on the right of Figure 2 has $\alpha\delta$-edge, $\beta\delta$-edge, $\beta\gamma$-edge, $\gamma\epsilon$-edge, and $\alpha\epsilon$-edge. Then in the pentagonal subdivision in the middle of Figure 2, the $\alpha\delta$-edge of one tile is identified with the $\beta\delta$-edge of an adjacent tile. This implies that the $\alpha\delta$-edge and the $\beta\delta$-edge have equal length. By the same reason, the $\alpha\epsilon$-edge and the $\gamma\epsilon$-edge have equal length. Applying such argument to the conclusion of Theorem 6, we get the following result.

Theorem 7. *The geometrically congruent tilings of the sphere that allow free continuous choice of two angles are the pentagonal subdivisions of platonic solids.*

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