Extension of Convex Functions

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Abstract

We study the extension of a locally convex function or a function with positive definite second order derivative on a not necessarily convex domain to a function of the same type on a bigger domain. The extension problem is separated in two steps: extension to the convex hull of the domain and the extension out of the convex hull. Useful criteria are given for various types of related problems.

Contents

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1 Introduction

Convex functions appear in many important problems in pure and applied mathematics. In many applications of convex analysis, convex (strictly convex) functions appear as functions with semipositive definite (positive definite) second order derivatives. Such functions may or may not be defined on a convex domain. However, classical literatures such as [2] [3] on convex analysis usually assume the domain of convex functions to be convex. Moreover, convex functions are often assumed to be defined on the whole linear space and have extended value (i.e., including ∞). Finite valued convex functions on convex domains are often extended to convex functions on the whole linear space by setting the value to be $+\infty$ out of the convex domain. While such treatment of convex functions is preferred in some applications (e.g., optimization, convex programming) and some theories (e.g., duality), it may not be the desirable thing to do if the problem is more analytic (e.g., variational problems). Therefore it is useful to know how finite valued convex functions on a not necessarily convex domain can be extended to a finite valued convex function on the whole space. We would also like to know the answer for strictly convex functions, and functions with semipositive definite or positive definite second order derivatives.

One immediately realizes that, for functions on a not necessarily convex domain Ω in $\mathbf{R}^n$, there are two notions of convexity (see Definition 1): *local* and *global* convexity. Functions with semipositive definite second order derivatives are (by the local nature of derivative) only locally convex, while global convexity is a necessary condition for the existence of convex extensions (see Theorem 1). Local and global convexity are equivalent if Ω is convex.

A closer analysis of the convex extension problem indicates that the extension may be separated into two steps. First there is the problem of the extension of a locally convex function on Ω to a (locally=globally) convex function on the convex hull Ω^{co}. Second there is the problem of the extension of a locally convex function on Ω to a locally convex function on $\Omega \cup (\mathbf{R}^k - \Omega^{co})$ (i.e., out of convex hull). If one can successfully solve the two problems consecutively, then one can obtain convex extension on the whole of $\mathbf{R}^k$.

In Section 2, we study the convex extension problem for general functions. We introduce the notion of global convexity in Definition 1 and prove in Theorem 1 that the extendability of a function on Ω to a convex function on the convex hull Ω^{co} is equivalent to global convexity. We also prove in Theorem 2 that a convex function on a bounded convex Ω can be extended to a convex function on $\mathbf{R}^k$ if and only if it is Lipschitz. In principle, the two theorems can be combined to yield conditions for convex extensions from Ω to $\mathbf{R}^k$. However, there is no good control on the Lipschitz condition in general. The following is one case when the Lipschitz condition can be controlled.

Theorem A *Suppose Ω is a bounded subset of $\mathbf{R}^k$, such that $\overline{\Omega^{co} - \Omega}$ is contained in the relative interior of Ω^{co}. Then a function on Ω can be extended to a convex function on $\mathbf{R}^k$ if and only if it is globally convex and is a Lipschitz function.*

The condition on Ω is equivalent to $\Omega = C - K$ for some convex C and compact K, such that K is contained in the relative interior of C. Intuitively, such Ω has a "thick convex frontier", as illustrated in Fig.1.

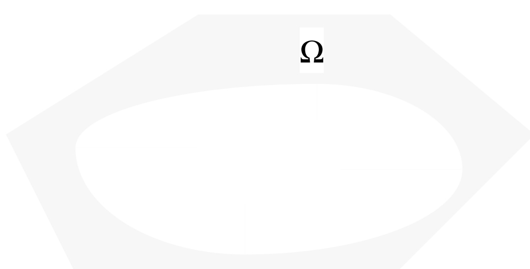

Fig. 1 $\overline{\Omega^{co} - \Omega}$ contained in relative interior of Ω^{co}

In Theorem 5, we reduce various types (local or global, strict or not strict) of convexity on an open domain into many one dimensional problems. Since convexity along the real line is well understood, the theorem provides practical criteria for convexity. Theorem 4 gives further relations between different convexities.

In Section 3, we study the problem of extending functions with positive definite second order derivatives to functions of the same type. We prove in Theorem 5 that in some sense, the extendability of such functions on a compact Ω to the convex hull Ω^{co} is equivalent to global convexity. We also prove in Theorem 6 that such functions can always be extended out of the convex hull. Combining the theorems together, we obtain

Theorem B *Suppose Ω is a compact subset of $\mathbf{R}^k$. Then for any globally convex C^r-function ($r \geq 2$) f with positive definite second order derivative on a neighborhood of Ω, $f|_\Omega$ can be extended to a C^r-function on $\mathbf{R}^k$ with positive definite second order derivative.*

On a convex domain, local and global convexity are equivalent. Theorems 7 and 8 provide some sort of converse to this statement. They indicate that in some sense, the equivalence of local and global convexity on an open domain Ω means that Ω is convex up to deleting a codimension ≥ 2 subset.

This paper came out of a variational problem from Professor V.A. Vladimirov [1] [4]. In fact, [1] was done only for the linear case, due to the lack of appropriate convex extension theorems. The results of this paper make it possible to extend [1] to the general nonlinear case.

For more practical purpose, such as certain specific estimations in fluid dynamics, it would be even more useful if we can control the second derivative of the extension. This leads to the following problem.

Conjecture: *Suppose f is a C^2-function on a neighborhood of a compact convex subset Ω of $\mathbf{R}^k$, such that $A\|v\|^2 < < H_f v, v > < B\|v\|^2$ for some positive constants A and B. Then $f|_\Omega$ can be extended to a C^2-function $\tilde{f}$ on $\mathbf{R}^k$, such that $A\|v\|^2 < < H_{\tilde{f}} v, v > < B\|v\|^2$.*

Even something less, such as $0.2A\|v\|^2 < < H_{\tilde{f}} v, v > < 5B\|v\|^2$, could be practically useful. Related to this weaker problem is the following.

Conjecture: *Suppose f is a C^2-function on a neighborhood of a compact convex subset Ω of $\mathbf{R}^k$, such that $| < H_f v, v > | < A\|v\|^2$ for some positive constant A. Then $f|_\Omega$ can be extended to a C^2-function $\tilde{f}$ on $\mathbf{R}^k$, such that $| < H_{\tilde{f}} v, v > | < A\|v\|^2$.*

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2 Convex Extension of General Functions

2.1 Extension to the Convex Hull

A convex combination of points $x_1, \cdots, x_n \in \mathbf{R}^k$ is a point $x = \lambda_1 x_1 + \cdots + \lambda_n x_n$, where $\lambda_i \geq 0$ and $\lambda_1 + \cdots + \lambda_n = 1$. A function f is usually called convex if

$$\lambda_1 f(x_1) + \cdots + \lambda_n f(x_n) \geq f(x) \tag{1}$$

for any convex combination. The function is *strictly* convex if strict inequality holds in case x is none of the x_i.

However, there is some ambiguity in the definition above if the domain Ω of f is not convex. The ambiguity is clarified by the following definition.

Definition 1 *A function f on $\Omega \subset \mathbf{R}^k$ is* globally convex *if for any convex combination $x \in \Omega$ of points $x_1, \cdots, x_n \in \Omega$,* (1) *is satisfied. It is* locally convex *if at any point of Ω there is a neighborhood such that the restriction of f to the neighborhood is globally convex.*

It is not difficult to see that the two concepts are the same if the domain Ω is convex. Therefore we will not use adjectives local and global when the domain is convex. Moreover, we also have the strict versions of the two concepts, and for functions over convex domains, local strict convexity is equivalent to global strict convexity.

Theorem 1 *A function on Ω may be extended to a convex function on the convex hull Ω^{co} if and only if f is globally convex.*

Proof: The necessity is obvious. To prove the sufficiency, we consider the epigraph of f:

$$\mathrm{epi}(f) = \{(x, \mu) : x \in \Omega, \mu \geq f(x)\} \subset \mathbf{R}^{k+1}.$$

It can be verified that the convex hull $\mathrm{epi}^{co}(f)$ of $\mathrm{epi}(f)$ consists of all points (x, μ), such that $x = \lambda_1 x_1 + \cdots + \lambda_n x_n$ for some $x_1, \cdots, x_n \in \Omega$, $\lambda_i \geq 0$, $\lambda_1 + \cdots + \lambda_n = 1$, and $\mu \geq \lambda_1 f(x_1) + \cdots + \lambda_n f(x_n)$. Thus

$$f^{co}(x) = \inf\{\mu : (x, \mu) \in \mathrm{epi}^{co}(f)\}$$

is a function on Ω^{co}. The convexity of $\mathrm{epi}^{co}(f)$ implies that f^{co} is a convex function on Ω^{co} [3]. The global convexity of f implies that f^{co} extends f.

$\square$

In contrast, global strict convexity does not necessarily imply the extendability of f to a strictly convex function on the convex hull of the domain. For example, the function $f(x, y) = e^x y^2$ is globally strictly convex on $\Omega = \mathbf{R}^2 - x$-axis. If g is a convex extension of f to $\Omega^{co} = \mathbf{R}^2$, then for any fixed x, $g_x(y) = g(x, y)$ is a convex extension of the quadratic function $f_x(y) = e^x y^2$ for $y \neq 0$. The continuity (which is a consequence of convexity [3]) of g_x implies that $g_x(0) = 0$. Therefore we must have $g(x, y) = e^x y^2$ throughout $\mathbf{R}^2$. However, g is not strictly convex because $g = 0$ along the x-axis.

I believe the following version of extension to the convex hull is true for globally strictly convex functions. Unfortunately I do not know how to prove it.

Conjecture: *Suppose f is a globally strictly convex function on a neighborhood of a compact subset Ω of $\mathbf{R}^k$. Then $f|_\Omega$ can be extended to a strictly convex function $\tilde{f}$ on a neighborhood of Ω^{co}.*

The smooth analogue of this conjecture will be proved in Theorem 5.

2.2 Extension out of the Convex Hull

Not every convex function on a convex set Ω can be extended to a finite convex function on $\mathbf{R}^k$. For example, it is not hard to see that a convex continuous function $f(x)$ on $[a,b]$ can be extended to a convex function on $\mathbf{R}$ if and only if the derivatives $f'(a+)$ and $f'(b-)$ are finite. Equivalently, this means that f is Lipschitz on the whole interval: There is a number R such that $|f(x_1) - f(x_2)| \leq R|x_1 - x_2|$.

Theorem 2 *A convex function f on a bounded convex set Ω can be extended to a convex function on the whole space if and only if it is a Lipschitz function.*

Proof: The necessity follows from Theorem 10.4 of [3]. For the sufficiency, we first consider the case that Ω affinely spans the whole space. Suppose f is Lipschitz. Define

$$F(x) = \sup\{(1-\lambda)f(y) + \lambda f(z) : x = (1-\lambda)y + \lambda z, \lambda \geq 1, y, z \in \Omega\}.$$

Then we have (with x, y, z as in the definition of F(x))

$$|(1-\lambda)f(y)+\lambda f(z)| \leq |f(y)|+\lambda|f(y)-f(z)| \leq |f(y)|+\lambda R|y-z| = |f(y)|+R|x-y|.$$

The boundedness of Ω and the Lipschitz property of f implies that, for fixed x, the right side is bounded. Therefore F is a finite valued function. By the convexity of f, $f(x) \geq (1-\lambda)f(y) + \lambda f(z)$ in case $x, y, z \in \Omega$ (and $x = (1-\lambda)y + \lambda z, \lambda \geq 1$). Thus F extends f.

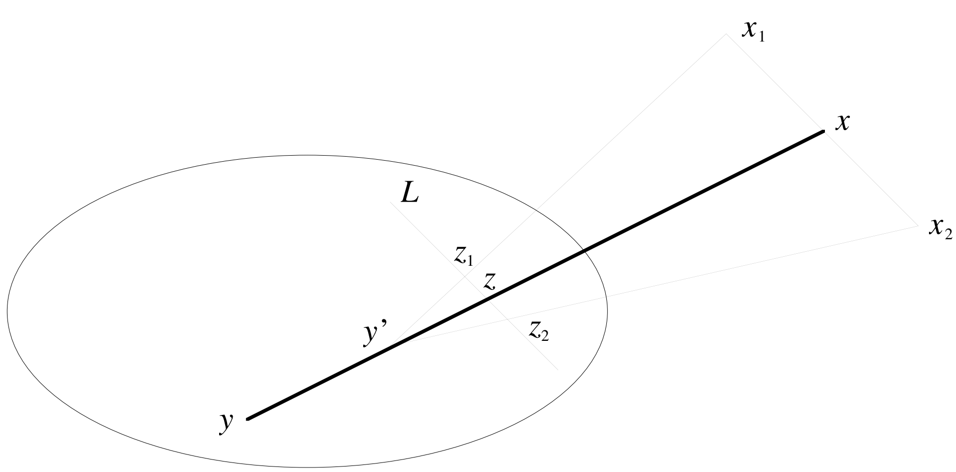

Fig. 2 proving convexity of the extension F(x)

Now assume $x = tx_1 + (1-t)x_2$ for some $0 < t < 1$. Suppose $x = (1-\lambda)y + \lambda z$ for some $\lambda \geq 1$ and y, z in the relative interior of Ω. Since z is in the relative interior of Ω and Ω spans the space, we have a small line segment L in Ω such that L is parallel to $[x_1, x_2]$ and z is in the interior of L (see Fig.2). Then we may choose a point y' on $[y, z)$ sufficiently close to z, such that the line segments $[y', x_1]$ and $[y', x_2]$ intersect L at z_1 and z_2. Write $x = (1-\lambda')y' + \lambda' z$. Then $\lambda' > 1$, and by applying the convexity of f to

$$y' = \frac{\lambda - 1}{\lambda' - 1} y + \frac{\lambda' - \lambda}{\lambda' - 1} z,$$

we have

$$(1-\lambda')f(y') + \lambda' f(z) \geq (1-\lambda)f(y) + \lambda f(z). \tag{2}$$

Since L is parallel to $[x_1, x_2]$, we have $z = tz_1 + (1-t)z_2$, $x_1 = (1-\lambda')y' + \lambda' z_1$, $x_2 = (1-\lambda')y' + \lambda' z_2$. Therefore by the definition of F we have

$$F(x_i) \geq (1-\lambda')f(y') + \lambda' f(z_i), \quad i = 1, 2. \tag{3}$$

Consequently,

$$\begin{aligned}
& tF(x_1) + (1-t)F(x_2) \\
\overset{(3)}{\geq}\ & t(1-\lambda')f(y') + t\lambda' f(z_1) + (1-t)(1-\lambda')f(y') + (1-t)\lambda' f(z_2) \\
=\ & (1-\lambda')f(y') + \lambda'(tf(z_1) + (1-t)f(z_2)) \\
\geq\ & (1-\lambda')f(y') + \lambda' f(z) \qquad\qquad (4) \\
\overset{(2)}{\geq}\ & (1-\lambda)f(y) + \lambda f(z).
\end{aligned}$$

By the continuity of f, the supremum of the right side of (4), for y, z in the relative interior of Ω, is the same as for y, z in Ω. Hence the supremum equals $F(x)$, and we conclude that $tF(x_1) + (1-t)F(x_2) \geq F(x)$.

Finally, if Ω does not affinely span the whole space, then the argument above produces an extension to a convex function on the affine span of Ω. Then it is very easy to extend the convex function on an affine subspace to a convex function on the whole space.

$\square$

Theorem A in the introduction is a consequence of Theorems 1 and 2. The only nontrivial part is to show that the extension f^{co} of f to the convex hull Ω^{co} is still Lipschitz.

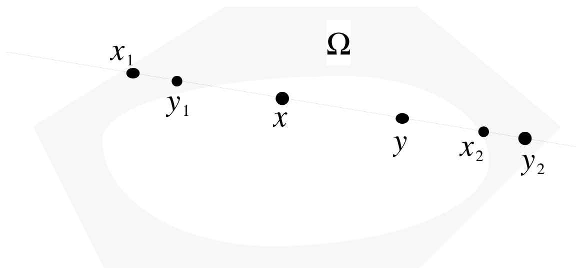

Fig. 3 slope on $[x,y]$ is controlled for convex functions

Suppose x, y are in Ω. Then the assumption on Ω implies that there are points x_1, y_1, x_2, y_2 on the straight line passing through x and y, such that (see Fig.3)

1. x_1, y_1, x_2, y_2 are in Ω;
2. x, y, y_1, x_2 are on the interval $[x_1, y_2]$;
3. y_1 is on the interval $[x_1, y]$;
4. x_2 is on the interval $[x, y_2]$.

Then by the convexity of f^{co} we have

$$\frac{|f^{co}(x)-f^{co}(y)|}{|x-y|} \leq \max\{\frac{|f^{co}(x_1)-f^{co}(y_1)|}{|x_1-y_1|}, \frac{|f^{co}(x_2)-f^{co}(y_2)|}{|x_2-y_2|}\} \tag{5}$$

Since x_1, y_1, x_2, y_2 are in Ω, and f^{co} extends f, we have

$$|f^{co}(x_1)-f^{co}(y_1)| \leq R|x_1-y_1|, \quad |f^{co}(x_2)-f^{co}(y_2)| \leq R|x_2-y_2|, \tag{6}$$

where R is the Lipschitz number for f (on Ω). Combining (5) and (6) together, we see that f^{co} is a Lipschitz function.

Finally, we note that by modifying the counterexample in section 2.1 a little bit, one can see that Theorem A is not true for strictly convex functions.

2.3 Relations Between Different Convexities

It is very easy to characterize convexity for functions defined on open subsets of $\mathbf{R}$. A function on an open subset of $\mathbf{R}$ is locally (strictly) convex if and only if it is continuous, has left and right derivatives, and the derivative is (strictly) increasing. If $\Omega = (a, b) \cup (c, d) \subset \mathbf{R}$ for $a < b < c < d$, then a locally convex function f on Ω is globally convex if and only if

$$f'(c+) \geq \frac{f(c+) - f(b-)}{b - c} \geq f'(b-). \tag{7}$$

If we require the inequalities to be strict, then (7) becomes a criterion for strict convexity. If f is defined on the closure $\bar{\Omega}$, then for global convexity we also require f to be continuous at b and c. The criterion can be easily generalized to locally finite disjoint union of intervals.

Given any (locally, globally, strictly) convex function f on Ω and an affine subspace H, the restriction of f to $\Omega \cap H$ is (locally, globally, strictly) convex. In particular, the restriction to any straight line is (locally, globally, strictly) convex. The converse is not true in general. For example, if no three points in Ω are colinear, then any function on Ω has the property that the restriction on any straight line is strictly convex. However, the converse is true in case Ω is an open subset. As a result, on an open domain the detection of convexity can be converted to many one dimensional convexity problems, for which we have already given the explicit answer above.

Theorem 3 *Suppose Ω is an open subset of $\mathbf{R}^k$. Then a function f on Ω is (locally, globally, strictly) convex if and only if for any straight line $L \subset \mathbf{R}^k$, the restriction of f to $L \cap \Omega$ is (locally, globally, strictly) convex.*

Proof: We only prove the nonstrict statements. If the inequalities in the proof are replaced by strict inequalities, then we obtain the proof of the strict case.

In a convex domain, a function is convex if and only if $f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$ for any x, y in the domain and $0 \leq t \leq 1$. Note that the condition is really about the restriction of f to the interval $[x, y]$. Therefore the theorem is true if Ω is convex. Since Ω is locally convex, we conclude that f is locally convex if and only if its restriction to any interval is locally convex.

The discussion also indicates that it remains to prove the following: Suppose f is locally convex and the restriction of f to any interval is convex. Suppose $x_i \in \Omega$ and $x = \lambda_1 x_1 + \cdots + \lambda_n x_n \in \Omega$ is a convex combination. Then $\lambda_1 f(x_1) + \cdots + \lambda_n f(x_n) \geq f(x)$.

Let $\epsilon > 0$ be very small. Denote $y_i = \epsilon x_i + (1-\epsilon)x$. Then by the convexity of f in the interval $[x_i, x]$, we have $\epsilon f(x_i) + (1-\epsilon) f(x) \geq f(y_i)$. The inequality may be rewritten as $f(y_i) - f(x) \leq \epsilon(f(x_i) - f(x))$. By the local convexity of f, for sufficiently small ϵ, the points y_i are in a neighborhood of x in which f is convex. Therefore, by $\lambda_1 y_1 + \cdots + \lambda_n y_n = \epsilon x + (1-\epsilon)x = x$ we have

$$\begin{aligned} 0 &\leq \lambda_1 f(y_1) + \cdots + \lambda_n f(y_n) - f(x) \\ &= \lambda_1 (f(y_1) - f(x)) + \cdots + \lambda_n (f(y_n) - f(x)) \\ &\leq \epsilon[\lambda_1 (f(x_1) - f(x)) + \cdots + \lambda_n (f(x_n) - f(x))] \qquad (8) \\ &= \epsilon[\lambda_1 f(x_1) + \cdots + \lambda_n f(x_n) - f(x)]. \end{aligned}$$

This completes the proof.

□

Theorem 4 *Suppose Ω is an open subset of $\mathbf{R}^k$. Then a function on Ω is globally strictly convex if and only if it is locally strictly convex and globally convex.*

Proof: The necessity is obvious. The proof of the last theorem also proves the sufficiency. In fact, local strict convexity allows us to change the first inequality in (8) into strict inequality.

□

3 Convex Extension of Differentiable Functions

3.1 Extension to the Convex Hull

A necessary condition for a function with positive definite second order derivative on Ω to be extendable to a function with positive definite second order derivative on Ω^{co} is global convexity. The next result shows that global convexity is also sufficient in some sense.

Theorem 5 *Suppose f is a globally convex C^r-function ($r \geq 2$) with positive definite second order derivative on a neighborhood of a compact subset Ω of $\mathbf{R}^k$. Then $f|_\Omega$ can be extended to a C^r-function $\tilde{f}$ with positive definite second order derivative on a neighborhood of Ω^{co}.*

Proof: Let $\epsilon > 0$ be very small, so that f is defined and has positive definite second order derivative on the open 3ϵ-neighborhood $\Omega^{3\epsilon}$, consisting of points within 3ϵ-distance from Ω. By global convexity of f, the function f^{co} constructed in the proof of Theorem 1 is an extension of f to the convex hull $\Omega^{3\epsilon,co}$ of $\Omega^{3\epsilon}$. Let $\delta \geq 0$ be a smooth function concentrated in an ϵ-neighborhood of the origin of $\mathbf{R}^k$, such that $\int \delta(x)dx = 1$. Then

$$g(x) = \int f^{co}(y)\delta(x-y)dy = \int f^{co}(x-y)\delta(y)dy$$

is a smooth function on $\Omega^{2\epsilon,co}$. The second expression for g and the convexity of f^{co} imply that g is convex (so that the second order derivative of g is semipositive definite) on $\Omega^{2\epsilon,co}$. By taking δ extremely concentrated around the origin and $\alpha > 0$ extremely small, we obtain a smooth function $h(x) = g(x) + \alpha\|x\|^2$ on $\Omega^{2\epsilon,co}$ such that the second order derivative of h is positive definite, and the restriction of h to Ω^ϵ can be as C^r-close to f as we wish. Let $0 \leq \theta \leq 1$ be a smooth function on $\Omega^{2\epsilon,co}$ such that $\theta = 0$ on Ω and $\theta = 1$ on $\Omega^{2\epsilon,co} - \Omega^\epsilon$. Then $\tilde{f} = (1-\theta)f + \theta h$ is a C^r-function on $\Omega^{2\epsilon,co}$ extending $f|_\Omega$. Moreover, $\tilde{f} - f = \theta(h - f)$ can have arbitrarily small second order derivative on Ω^ϵ. Since the second order derivative of f has a positive definite lower bound over Ω^ϵ, we can find $\tilde{f}$ such that the second order derivative of $\tilde{f}$ is still positive definite over Ω^ϵ. On the other hand, over $\Omega^{2\epsilon,co} - \Omega^\epsilon$ we have $\tilde{f} = h$, which has positive definite second order derivative. Thus $\tilde{f}$ is the extension of $f|_\Omega$ we are looking for.

□

3.2 Extension out of the Convex Hull

Our differentiable analogue of Theorem 2 is more precise. Here by extension out of the convex hull of Ω, we mean extension to a domain contained in the union of Ω and the complement of the convex hull of Ω.

Theorem 6 *Suppose $\Omega \subset \Omega'$ are subsets of $\mathbf{R}^k$. Suppose Ω is compact, Ω' is closed, and $\Omega' \subset \Omega \cup (\mathbf{R}^k - \Omega^{co})$. Then for any C^r-function ($r \geq 2$) f with positive definite second order derivative on a neighborhood of Ω, $f|_\Omega$ can be extended to a C^r-function $\tilde{f}$ with positive definite second order derivative on a neighborhood of Ω'.*

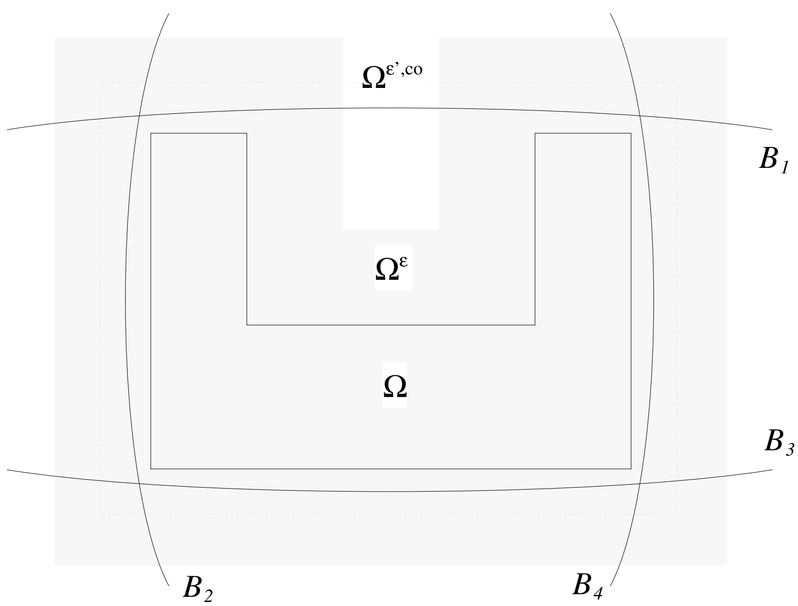

Fig. 4 neighborhoods and balls

Proof: Let f be defined on an open neighborhood U of Ω. Let $\epsilon > 0$ be sufficiently small, such that $\overline{\Omega^\epsilon} \subset U$. Then by the assumption, $\Omega' - \Omega^\epsilon$ is a closed subset of

$\mathbf{R}^k - \Omega^{co}$. Since Ω is assumed to be compact, there is another small $\epsilon' > 0$, such that $\epsilon > \epsilon' > 0$ and $\Omega' - \Omega^\epsilon \subset \mathbf{R}^k - \Omega^{\epsilon',co}$. In other words,

$$\Omega' \subset (\Omega^\epsilon \cap \Omega^{\epsilon',co}) \cup (\mathbf{R}^k - \Omega^{\epsilon',co}). \tag{9}$$

The compactness of Ω implies that there are finitely many open balls $B_i = B(x_i, R_i)$ such that

$$\Omega \subset \cap B_i, \quad \cap \bar{B}_i \subset \Omega^{\epsilon',co}. \tag{10}$$

The constructions are illustrated in Fig.4.

Let $\gamma : [0, \infty) \to \mathbf{R}$ be a smooth function such that $\gamma = 0$ on $[0, 1]$ and $\gamma'' > 0$ on $(1, \infty)$. Then

$$g(x) = \sum \gamma\left(\frac{\|x - x_i\|}{R_i}\right)$$

is a smooth function on $\mathbf{R}^k$, such that $g = 0$ on $\cap B_i$ and g has positive definite second order derivative on $\mathbf{R}^k - \cap\bar{B}_i$. Let $0 \leq \theta \leq 1$ be a compactly supported smooth function on $\mathbf{R}^k$, such that $\theta = 1$ on $\Omega^\epsilon \cap \Omega^{\epsilon',co}$. Then we consider the C^r-function $\tilde{f} = \theta f + Cg$, where $C > 0$ is a constant to be determined.

Since $g = 0$ on $\cap B_i \supset \Omega$, $\tilde{f}$ extends $f|_\Omega$.

To see $\tilde{f}$ has positive definite second order derivative on Ω', we make use of (9). On $\Omega^\epsilon \cap \Omega^{\epsilon',co}$, we have $\tilde{f} = f + Cg$. Since f has positive definite second order derivative on $U \supset \Omega^\epsilon$, and g always has semipositive definite second order derivative, $\tilde{f}$ has positive definite second order derivative.

By (10), $\mathbf{R}^k - \Omega^{\epsilon',co}$ is a closed subset of $\mathbf{R}^k - \cap\bar{B}_i$, on which g has positive definite second order derivative. Therefore on the compact subset $(\mathbf{R}^k - \Omega^{\epsilon',co}) \cap \mathrm{support}(\theta)$, the second order derivative of g has positive definite lower bound (the quadratic function is $\geq \delta||dx||^2$ for some $\delta > 0$). Therefore for sufficiently big C, $\tilde{f}$ has positive definite second order derivative.

$\square$

Theorem 6 suggests that $\Omega \cup (\mathbf{R}^k - \Omega^{co})$ is in some sense a "universal convex extendable region". However, the idea used in the proof of Theorem 6 can be adopted to produce other universal convex extendable regions not contained in $\Omega \cup (\mathbf{R}^k - \Omega^{co})$. For example, in the following picture, a function f with positive definite second order derivative on a neighborhood of Ω can be extended to a function f_1 of the same type in a neighborhood of Ω_1. Let A be the part of Ω_1 as described in the picture. Then the restriction $f_1|_A$ extends to a function $\tilde{f}_1$ of the same type on a region out of the convex hull of A but not contained in $\Omega \cup (\mathbf{R}^k - \Omega^{co})$. Being the same on the common part A of the respective domain, $\tilde{f}_1$ and f_1 combine together to form a function f_2 on a neighborhood of Ω_2. f_2 has positive definite second order derivative and extends $f|_\Omega$. The process can be repeated many times.

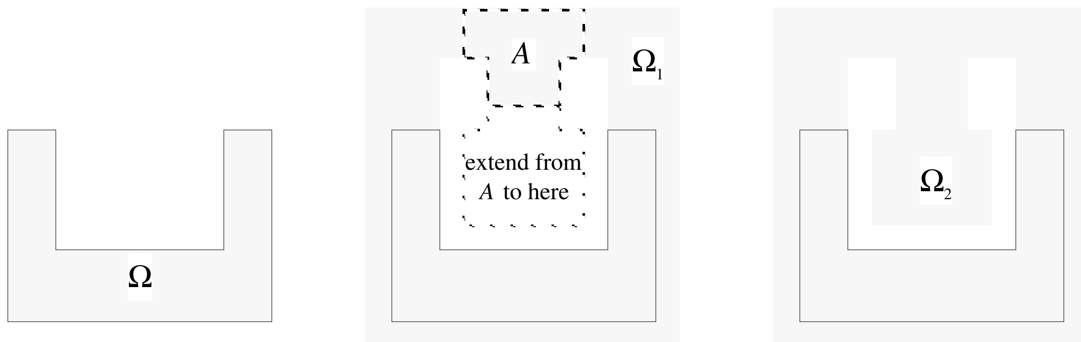

Fig.5 convex extendable region

3.3 Local Convexity vs. Global Convexity

On a convex domain local and global convexities are equivalent. The next result shows that, for open domain, the converse is almost true.

Theorem 7 *Suppose Ω is a bounded open subset of $\mathbf{R}^k$. If there is an open subset U of a codimension 1 hyperplane satisfying $U \subset \Omega^{co} - \Omega$, then there is a smooth function f on Ω, such that f has positive definite second order derivative, but f is not globally convex.*

Proof: Write points $x \in \mathbf{R}^k$ as $x = (t, w)$, $t \in \mathbf{R}$, $w \in \mathbf{R}^{k-1}$. By an affine transformation and a shrinking of U, we may assume $U = \{(0, w) : \|w\| < 1\}$ and $(1, 0) \in \Omega$, $(a, 0) \in \Omega$ for some $a < 0$.

Fix a smooth function ϕ on $\mathbf{R}$ such that $\phi(r) = 1$ for $r \leq 0$ and $\phi(r) = 0$ for $r \geq 1$. Then the function

$$f(t,w) = \begin{cases} t^2 - t\phi(\|w\|) + A\phi(\|w\|) + B\|w\|^2 & \text{for } t > 0 \text{ and } \|w\| < 1 \\ t^2 + B\|w\|^2 & \text{for } \|w\| \geq 1, \text{ or } t \leq 0 \text{ and } \|w\| < 1 \end{cases}$$

is smooth on $\mathbf{R}^k - U$. We expect the global convexity to be violated by (see (7))

$$\frac{f(1,0) - f(a,0)}{1-a} = \frac{A - a^2}{1-a} > 1 = \frac{\partial f}{\partial t}(1,0).$$

This can be achieved, for example, by choosing $A = (a-1)^2$ (any number bigger than $a^2 - a + 1$ will do). For $\|w\| \geq 1$, or $t < 0$ and $\|w\| < 1$, f has positive definite second order derivative as long as $B > 0$. For $t > 0$ and $\|w\| < 1$, the second order derivative in direction $\xi = (\tau, \omega)$ is

$$\begin{aligned} \frac{\partial^2 f}{\partial \xi^2} &= 2\tau^2 - 2\tau\frac{d\phi}{d\omega} + (A-t)\frac{d^2\phi}{d\omega^2} + B\frac{d^2(\|w\|^2)}{d\omega^2} \\ &= 2\tau^2 - 2\tau\|\omega\|\phi' \cos\theta + \|\omega\|^2((A-t)\phi'' \cos^2\theta + \phi' \sin^2\theta + 2B), \end{aligned}$$

where θ is the angle between w and ω. Therefore by boundedness of Ω and ϕ', we may find big enough $B > 0$ so that f has positive definite second order derivative throughout Ω.

$\square$

By the same construction, we note that it is really not necessary to assume Ω is bounded when $k = 2$. We expect the boundedness condition is redundant in general.

If $\Omega = \{2 \text{ points}\}$, then any function on Ω is globally convex. More generally, if $\Omega = (\Omega_1 \times 0) \cup (0 \times \Omega_2) \subset \mathbf{R}^{k_1} \times \mathbf{R}^{k_2}$, with Ω_i being a convex subset of $\mathbf{R}^{k_i}$ not containing 0, then any locally (strictly) convex function on Ω is globally (strictly) convex. Therefore it is necessary to require Ω to be open in Theorem 7.

Naively, one might suspect that the following statement more straightforward than Theorem 7 should be true: On any nonconvex open subset there is a locally convex but not globally convex function. The next result indicates this is not the case.

Theorem 8 *Suppose Ω is an open convex set, and $K \subset \Omega$ is a closed subset of codimension ≥ 2. Then any locally (strictly) convex function on $\Omega - K$ is globally (strictly) convex.*

Remark: The codimension ≥ 2 assumption is used to conclude that generic straight lines do not intersect the subset K. Here we are deliberately vague about the definition of dimension because there are many different definitions (applicable to different types of subsets) for which the conclusion is true. Important examples are polyhedra of codimension ≤ 2, unions of finitely many smooth submanifolds of codimension ≤ 2.

Proof: Given $x_1, x_2 \in \Omega - K$, and a convex combination $x = \lambda x_1 + (1-\lambda)x_2 \in \Omega - K$, we may choose sequences x_{1k} and x_{2k} in $\Omega - K$ such that

1. the line segment $[x_{1k}, x_{2k}]$ connecting x_{1k} and x_{2k} is contained in $\Omega - K$;

2. $\lim_{k\to\infty} x_{1k} = x_1$, $\lim_{k\to\infty} x_{2k} = x_2$.

We note that the codimension ≥ 2 is used for the first property. The local convexity and the first property imply that $f(\lambda_1 x_{1k} + \lambda_2 x_{2k}) \leq \lambda_1 f(x_{1k}) + \lambda_2 f(x_{2k})$. Taking limits, then the continuity of f (see Theorem 10.1 of [3], for example) implies that $f(x) \leq \lambda_1 f(x_1) + \lambda_2 f(x_2)$.

Now assume f is locally strictly convex. We already know that $f(x) \leq \lambda f(x_1) + (1-\lambda)f(x_2)$ for any convex combination $x = \lambda x_1 + (1-\lambda)x_2$ in $\Omega - K$. For $0 < \lambda < 1$, we may choose small $0 < \epsilon < \min\{\lambda, 1-\lambda\}$ so that the line segment connecting $x_+ = (\lambda+\epsilon)x_1 + (1-\lambda-\epsilon)x_2$ and $x_- = (\lambda-\epsilon)x_1 + (1-\lambda+\epsilon)x_2$ is contained in $\Omega - K$. Then we still have

$$f(x_+) \leq (\lambda+\epsilon)f(x_1) + (1-\lambda-\epsilon)f(x_2); \tag{11}$$

$$f(x_-) \leq (\lambda-\epsilon)f(x_1) + (1-\lambda+\epsilon)f(x_2). \tag{12}$$

Since f is locally strictly convex, and $[x_-, x_+] \subset \Omega - K$, we have

$$f(x) = f(\frac{x_+ + x_-}{2}) < \frac{f(x_+) + f(x_-)}{2}.$$

Combined with (11) and (12), we obtain $f(x) < \lambda f(x_1) + (1-\lambda)f(x_2)$. It then follows from Theorem 3 that f is globally strictly convex.

□

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