Singularity from Eulerian Viewpoint

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Contents

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Abstract

This is a survey on our work generalizing the classical Dehn-Sommerville equations (analogous to Poincaré duality, see [10]) for f-vectors of triangulations of manifolds without boundary to general polyhedra. Our key observation is that the exact data needed for the generalization is the classification of points of polyhedra by the Euler characteristics of their links. From this viewpoint, a point is regular if the Euler characteristic of the link is the same as the sphere of the appropriate dimension. For any polyhedron, the extent to which the classical Dehn-Sommerville equations are still valid is measured by the size of the singularities from Eulerian viewpoint.

1 Introduction

Manifolds are spaces with the same certain local properties as the Euclidean spaces. The choice of the local property depends on specific problems one is interested in. Thus smooth, PL, and topological manifolds are locally homeomorphic (in various senses) to Euclidean spaces. Homology manifolds have the same local homology as the Euclidean balls. Moreover, it is often possible to generalize the solution of some problems for manifolds to singular spaces, i.e., spaces that fail to satisfy the local property along some lower dimensional part.

In this survey, we are concerned with the combinatorial problem of linear conditions on the number of simplices at various dimensions for triangulations of a fixed polyhedron. It turns out that the exact data needed for determining these linear conditions is the Euler characteristic of the links of points in the polyhedron. In other words, if a polyhedron has no singularity from Eulerian viewpoint, then the linear conditions on the number of simplices for all triangulations of the polyhedron is the same as the ones for usual PL-manifolds.

Let X be an n-dimensional compact polyhedron. For any triangulation Δ of X, define the f-vector $f(X, \Delta) = (f_0, f_1, \cdots, f_n)$, where f_i is the number of i-simplices in Δ. In addition to the Euler equation $\chi(f) = f_0 - f_1 + \cdots + (-1)^n f_n = \chi(X)$, it has been known for a long time that the f-vector of a simplicial polytope (which is a special triangulation of sphere) also satisfies the Dehn-Sommerville equations (1) (see [2] [7] [15] [17], for examples). In [10], Klee showed that the equation is also valid for triangulations Δ with the property that the simplicial link of an s-dimensional simplex in Δ has the same Euler characteristic as S^{n-s}. Our Lemma 2.2.1 further implies that this is equivalent to the link of any point in the polyhedron has the same Euler characteristic as S^{n-1}. Following Klee, we call such polyhedra Eulerian manifolds.

The classical Dehn-Sommerville equations (1) is the combinatorial analogue of the Poincaré duality for the homology of boundaryless manifolds. One may think of the linear theory of f-vectors as the analogue of the homology theory (the generalized homology theory, or stable homotopy theory, is the additive part of homotopy theory). Although Poincaré duality spaces are not yet manifolds (a further surgery obstruction needs to vanish [13]), in our simpler combinatorial theory the classical Dehn-Sommerville equations indeed characterizes Eulerian manifolds (see Theorem 4.2.2).

The Poincaré duality (for homotopy invariant homology theories) fails for singular spaces. Moreover, the extent to which the Poincaré duality fails can be measured by the size of the singularities [6]. Analogously, our generalized Dehn-Sommerville equations (3) give explicit measurement on the extent to which the classical Dehn-Sommerville equations fail in terms of the size of singularities from Eulerian viewpoint.

One way to study the topology of singular spaces is to think of spaces as obtained by gluing manifold pieces together in a particularly nice way. Such stratified viewpoint can also be adapted to our study of linear conditions on f-vectors. Thus we introduce the notion of Eulerian stratified spaces and parallelly establish the theory of linear conditions on the system of f-vectors of individual Eulerian manifold pieces.

Additionally, there is a combinatorial structure inherent in any Eulerian stratification (Theorem 5.2.1). Such structure should underlie many linear combinatorial theories.

The classification of f-vectors for certain class of triangulations of certain class of polyhedra is an interesting and difficult problem. The historical highlights include the classifications for all triangulations of all compact polyhedra [9] [11], for all triangulations of all closed surfaces [8] [14], and for all simplicial polytopes [1] [12] [16]. The classification typically involves linear equalities and nonlinear inequalities. What we present here is the linear equalities. The nonlinear inequalities are much harder. We hope that our approach, especially the stratified one, may shed some light on the classification problem.

The survey is organized as follows. In section 2 we review the classical Dehn-Sommerville equations and its generalization to polyhedra. In section 3 we discuss the Dehn-Sommerville equations for manifolds with boundary and its Eulerian analogues. This lays down the foundation for understanding more complicated situations. In section 4 we present our main results for arbitrary polyhedra and more detailed results for simplest singular spaces. In section 5 we present our stratified theory.

Some detailed proofs are included to illustrate the main techniques used in developing our theories. More details can be found in [3] [4] [5].

Finally we remark that an f-vector over an n-dimensional polyhedron X is an $(n+1)$-dimensional vector. However, we find it often more convenient to think of the f-vector stably, i.e., by adding zeros at coordinates at dimensions $> n$, we may think of f as N-dimensional vector for any $N \geq n+1$.

2 Dehn-Sommerville Equations

2.1 Classical Theory

An n-dimensional *Eulerian manifold* is defined in [10] as a finite simplicial complex Δ such that for any simplex σ of dimension s, the *simplicial link*

$$\mathrm{lk}(\sigma, \Delta) = \{\tau \in \Delta : \sigma \text{ and } \tau \text{ are complementary faces in a simplex } \sigma * \tau \in \Delta\}$$

has the same Euler characteristic $1-(-1)^{n-s}$ as an $(n-s-1)$-sphere. As a consequence of this property, the f-vector of an Eulerian n-manifold satisfies the Dehn-Sommerville equations:

$$(1-(-1)^{n-i})f_i + \sum_{j=i+1}^{n} (-1)^{n-j-1} \binom{j+1}{i+1} f_j = 0, \quad 0 \leq i \leq n-1. \tag{1}$$

We call the coefficient matrix $D(n)$ on the left side the n-th *Dehn-Sommerville Matrix*. We have explicitly *for even n*:

$$D(n) = \begin{pmatrix} 0 & \binom{2}{1} & -\binom{3}{1} & \binom{4}{1} & -\binom{5}{1} & \cdots & \binom{n}{1} & -\binom{n+1}{1} \\ 0 & 2 & -\binom{3}{2} & \binom{4}{2} & -\binom{5}{2} & \cdots & \binom{n}{2} & -\binom{n+1}{2} \\ 0 & 0 & 0 & \binom{4}{3} & -\binom{5}{3} & \cdots & \binom{n}{3} & -\binom{n+1}{3} \\ 0 & 0 & 0 & 2 & -\binom{5}{4} & \cdots & \binom{n}{4} & -\binom{n+1}{4} \\ \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & \binom{n}{n-1} & -\binom{n+1}{n-1} \\ 0 & 0 & 0 & 0 & 0 & \cdots & 2 & -\binom{n+1}{n} \end{pmatrix}$$

for odd n:

$$D(n) = \begin{pmatrix} 2 & -\binom{2}{1} & \binom{3}{1} & -\binom{4}{1} & \binom{5}{1} & \cdots & \binom{n}{1} & -\binom{n+1}{1} \\ 0 & 0 & \binom{3}{2} & -\binom{4}{2} & \binom{5}{2} & \cdots & \binom{n}{2} & -\binom{n+1}{2} \\ 0 & 0 & 2 & -\binom{4}{3} & \binom{5}{3} & \cdots & \binom{n}{3} & -\binom{n+1}{3} \\ 0 & 0 & 0 & 0 & \binom{5}{4} & \cdots & \binom{n}{4} & -\binom{n+1}{4} \\ \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & \binom{n}{n-1} & -\binom{n+1}{n-1} \\ 0 & 0 & 0 & 0 & 0 & \cdots & 2 & -\binom{n+1}{n} \end{pmatrix}$$

Note that $\mathrm{rank} D(n) = \left[\frac{n+1}{2}\right]$. Moreover, if $m < n$ and m, n have the same parity, then there is a decomposition

$$D(n) = \begin{pmatrix} D(m) & F(m,n) \\ 0 & E(m,n) \end{pmatrix}. \tag{2}$$

This means that stably speaking, there are only two Dehn-Sommerville matrices: $D(even)$ and $D(odd)$. Thus for any vector $v \in \mathbf{R}^m$ and $n \geq m$, $D(n)v$ depends only on the parity of n.

2.2 Dehn-Sommerville Equations on Polyhedra

Our generalization of the classical Dehn-Sommerville equations to triangulations of arbitrary polyhedra is based on the following two observations:

1. The assumption about the Euler characteristics of the simplicial links is an intrinsic property of the geometrical realization of the simplicial complex. In other words, the property is independent of the choice of triangulations of a polyhedron;

2. Klee's proof of the Dehn-Sommerville equations can be generalized to arbitrary simplicial complexes.

A (locally compact) polyhedron may be characterized as a space X such that any point $x \in X$ has a cone neighborhood xL with x as the cone point. L is a compact polyhedron of lower dimension and is unique up to piecewise linear homeomorphism. L is the *link* of x in X and is denoted $\mathrm{lk}(x, X)$. Our first observation is summarized in the following lemma.

Lemma 2.2.1 *Suppose Δ is a triangulation of a polyhedron X. Suppose $\sigma^s \in \Delta$ is an s-dimensional simplex and $x \in \dot{\sigma}$ is an interior point of the simplex. Then*

$$\chi(\mathrm{lk}(\sigma, \Delta)) = 1 - (-1)^s + (-1)^s \chi(\mathrm{lk}(x, X)).$$

The lemma is a consequence of the homeomorphism $\mathrm{lk}(x, X) = \partial\sigma * |\mathrm{lk}(\sigma, \Delta)|$, where $|?|$ is the geometrical realization of simplicial complexes and $*$ is the join of spaces. The lemma shows that Klee's condition $\chi(\mathrm{lk}(\sigma, \Delta)) = 1-(-1)^{n-s}$ is equivalent to $\chi(\mathrm{lk}(x, X)) = 1 - (-1)^n = \chi(S^{n-1})$. This leads to the following more intrinsic definition of Eulerian manifolds.

Definition 2.2.2 *A locally compact polyhedron M is called an n-*dimensional Eulerian manifold (without boundary) *if the link of any point in M has the same Euler characteristic $1 - (-1)^n$ as an $(n-1)$-sphere.*

We remark that the Euler characteristic of links determines only the parity of n. However, if m is the topological dimension of M, then the link of a point in the interior of an m-dimensional simplex in a triangulation is an $(m-1)$-sphere. Therefore m and n have the same parity and we will usually take $n = m$.

Our second observation is based on the following reformulation of Klee's proof of (1):

$$\begin{aligned}
\sum_{\dim\sigma=i} \chi(\mathrm{lk}(\sigma, \Delta)) &= \sum_{\dim\sigma=i} \sum_{\tau\in\mathrm{lk}(\sigma,\Delta)} (-1)^{\dim\tau} \\
&= \sum_{\dim\sigma=i} \sum_{\substack{\rho\supset\sigma \\ \rho\neq\sigma}} (-1)^{\dim\rho-i-1} \\
&= \sum_{\dim\rho>i} \sum_{\substack{\sigma\subset\rho \\ \dim\sigma=i}} (-1)^{\dim\rho-i-1} \\
&= \sum_{\dim\rho>i} (-1)^{\dim\rho-i-1} \binom{\dim\rho+1}{i+1} \\
&= \sum_{j>i} (-1)^{j-i-1} \binom{j+1}{i+1} f_j(X, \Delta),
\end{aligned}$$

where $\tau \in \mathrm{lk}(\sigma, \Delta)$ is in one-to-one correspondence with $\rho = \sigma * \tau$. In case X is an Eulerian manifold, the left side is simply $(1 - (-1)^{n-i}) f_i(X, \Delta)$, and the Dehn-Sommerville equations then easily follows. For a general polyhedron, we have to classify simplices according to $\chi(\mathrm{lk}(\sigma, \Delta))$. In view of Lemma 2.2.1, we introduce

$$X_k = \{x \in X : \chi(\mathrm{lk}(x, X)) = k\}.$$

Then Lemma 2.2.1 implies

$$X_k = \bigcup\{\dot{\sigma} : \chi(\mathrm{lk}(\sigma, X)) = 1 - (-1)^{\dim\sigma} + (-1)^{\dim\sigma} k\},$$

so that X_k is a union of the interiors of some simplices in the triangulation Δ. In particular, we have the f-vectors $f(X_k, \Delta)$ that count the number of simplices whose interiors are in X_k and

$$\sum_{\dim\sigma=i} \chi(\mathrm{lk}(\sigma, \Delta)) = \sum_k (1 - (-1)^i + (-1)^i k) f_i(X_k, \Delta).$$

It is then not difficult to deduce the following generalized Dehn-Sommerville equations.

Theorem 2.2.3 *Suppose X^n is a compact polyhedron. Then for any triangulation Δ of X and any $m \geq n$,*

$$D(m)f(X, \Delta) = \sum_{k \neq 1-(-1)^m} (1 - (-1)^m + (-1)^m k) f(X_k, \Delta). \tag{3}$$

3 Eulerian Manifolds with Boundary

The generalized Dehn-Sommerville equations (3) enable us to study linear equations on more complicated polyhedra. In this section, we deduce properties on the Dehn-Sommerville matrix $D(n)$ by applying (3) to PL-manifolds with boundary. Then we make use of the properties of $D(n)$ to study the following analogue of PL-manifolds.

Definition 3.0.4 *A (locally compact) polyhedron M^n is an* Eulerian manifold *with a closed subpolyhedron ∂M as* boundary *if* $\dim \partial M = n - 1$ *and*

$$\chi(\mathrm{lk}(x, M)) = \begin{cases} 1 - (-1)^n & \text{if } x \in M - \partial M \\ 1 & \text{if } x \in \partial M \end{cases}$$

PL-manifolds with boundary are clearly Eulerian manifolds with boundary. For an Eulerian manifold M^n with boundary ∂M, the Dehn-Sommerville equation (3) becomes

$$D(m)f(M, \Delta) = \begin{cases} f(\partial M, \partial\Delta) & \text{if } m - n \text{ is even} \\ 2f(M, \Delta) - f(\partial M, \partial\Delta) & \text{if } m - n \text{ is odd.} \end{cases} \tag{4}$$

3.1 Linear Conditions over Manifolds with Boundary

It is well known that the only (rational) linear conditions on f-vectors of triangulations of simplicial polytopes are the Euler equation and the Dehn-Sommerville equations (see [2] [7] [17], for examples). This is the consequence of the following two facts:

1. The rank of combined Euler and Dehn-Sommerville equations for f-vectors of n-dimensional simplicial polytopes is $\left[\frac{n}{2}\right] + 1$;

2. There are n-dimensional simplicial polytopes δ_i^n, $0 \leq i \leq \left[\frac{n+1}{2}\right]$, with affinely independent f-vectors.

The first fact implies that the dimension of the affine span of the f-vectors of all simplicial polytopes is *at most* $n+1-\left[\frac{n}{2}\right]-1=\left[\frac{n+1}{2}\right]$. The second fact implies that the dimension of the affine span is *at least* $\left[\frac{n+1}{2}\right]$.

The argument can be easily adapted to compact PL-manifolds, or more generally compact Eulerian manifolds. The result is summarized in the following theorem.

Theorem 3.1.1 *Suppose M^n is a compact Eulerian manifold. Then the only linear conditions on the f-vectors of all triangulations of M are*

1. *the Euler equation $\chi(v)=\chi(M)$ and the Dehn-Sommerville equations $D(n)v=0$ in case $\partial M=\emptyset$;*

2. *the Euler equation $\chi(v)=\chi(M)$ in case $\partial M\neq\emptyset$.*

Proof: Suppose M has no boundary. Then the Euler and the Dehn-Sommerville equations are satisfied. This further implies that the affine span of the f-vectors of all triangulations of a given closed PL-manifold M has dimension $\leq\left[\frac{n+1}{2}\right]$. Now to find $\left[\frac{n+1}{2}\right]+1$ triangulations of the manifold with affinely independent f-vectors, we fix a triangulation Δ of M and construct "connected sums" $\Delta\#\delta_i^n$ (delete the interior of one n-dimensional simplex each from Δ and δ_i^n, and then glue them together along the boundaries of the simplices). The second fact implies that the f-vectors $f(M,\Delta\#\delta_i^n)$, $0\leq i\leq\left[\frac{n+1}{2}\right]$, are all affinely independent. The first part of the theorem then follows by dimension reason, as in the simplicial polytope case.

Suppose M has nonempty boundary. The Euler equation is still valid, and specifies an affine subspace of dimension n. Therefore to prove the second part of the theorem, it suffices to find $n+1$ triangulations of M with affinely independent f-vectors. We fix a triangulation Δ of M. By taking connected sums with δ_j^{n-1}, $0\leq j\leq\left[\frac{n}{2}\right]$, we obtain triangulations $\partial\Delta\#\delta_j^{n-1}$ of ∂M with affinely independent f-vectors. These triangulations may be extended to the interior of M to become triangulations $\Delta_0,\Delta_1,\cdots,\Delta_{\left[\frac{n}{2}\right]}$ of M such that $f(\partial M,\partial\Delta_i)$ are affinely independent. Now we take the connected sums of Δ_0 with δ_i^n, $0\leq i\leq\left[\frac{n+1}{2}\right]$, along an n-dimensional simplex of Δ_0 lying completely in the interior $M-\partial M$ (such a simplex always exists if we start with a fine enough triangulation Δ). The results are triangulations $\Delta_{00},\Delta_{01},\cdots,\Delta_{0\left[\frac{n+1}{2}\right]}$ with affinely independent f-vectors and the same restrictions $\partial\Delta_0$ on ∂M. Now consider the f-vectors of the triangulations

$$\Delta_{00},\Delta_{01},\cdots,\Delta_{0\left[\frac{n+1}{2}\right]};\Delta_1,\cdots,\Delta_{\left[\frac{n}{2}\right]}.$$

The number of triangulations is $1+\left[\frac{n+1}{2}\right]+\left[\frac{n}{2}\right]=n+1$. The reason for their f-vectors to be affinely independent is the following general fact: Suppose D is a linear transformation, and $u_1,\cdots,u_a,v_1,\cdots,v_b$ are vectors such that

1. $u_1,\cdots,u_a$ are affinely independent;

2. $Du_1=\cdots=Du_a$;

3. $Du_1, Dv_1, \cdots, Dv_b$ are affinely independent.

Then $u_1, \cdots, u_a, v_1, \cdots, v_b$ are affinely independent.

$\square$

3.2 Properties of Dehn-Sommerville Matrix and Eulerian Manifolds

Note that the Euler equation explicitly depends on the Euler characteristic. Moreover, for any $n \geq 0$, any positive integer (or for $n \geq 2$, any integer) can be the Euler characteristic of an n-dimensional PL-manifold with boundary. Thus a corollary of Theorem 3.1.1 is that there are no nontrivial linear equations for f-vectors of all triangulations of all compact PL-manifolds (with or without boundary). On the other hand, we have the following universal linear equations for any triangulation of any compact n-dimensional PL-manifold M:

$$D(n-1)D(n)f(M,\Delta) = D(n-1)f(\partial M, \partial\Delta) = f(\partial^2 M, \partial^2\Delta) = f(\emptyset, \partial^2\Delta) = 0;$$

$$\chi(D(n)f(M,\Delta)) = \chi(f(\partial M, \partial\Delta)) = \chi(\partial M) = (1-(-1)^n)\chi(M) = (1-(-1)^n)\chi(f(M,\Delta)).$$

Consequently we have the following properties of the Dehn-Sommerville matrix.

Lemma 3.2.1 *$D(n-1)D(n) = 0$, $\chi \circ D(n) = (1-(-1)^n)\chi$.*

The algebraic property of $D(n)$ derived from considering PL-manifolds can then be used to prove the following properties of Eulerian manifolds.

Corollary 3.2.2 *Suppose M^n is a compact Eulerian manifold with boundary. Then $\chi(\partial M) = (1-(-1)^n)\chi(M)$. In particular, if M is an odd dimensional compact Eulerian manifold without boundary, then $\chi(M) = 0$.*

In addition to the corollary, we have the following simple properties. They follow from easy computations on the Euler characteristics of links.

Proposition 3.2.3 *Suppose M_1 and M_2 are Eulerian manifolds with homeomorphic boundaries. Then the glue $M_1 \cup_\partial M_2$ along a homeomorphism on the boundaries is a boundaryless Eulerian manifold.*

Proposition 3.2.4 *Given polyhedra M_1, M_2 and closed subpolyhedra $\partial M_1 \subset M_1$, $\partial M_2 \subset M_2$ of codimension 1, the product $M_1 \times M_2$ is an Eulerian manifold with boundary $(M_1 \times \partial M_2) \cup (\partial M_1 \times M_2)$ if and only if M_1 and M_2 are Eulerian manifolds with boundaries ∂M_1 and ∂M_2.*

3.3 Dehn-Sommerville Homology and Eulerian Cobordism

The equality $D(n-1)D(n) = 0$ gives rise to the following *Dehn-Sommerville chain complex*:

$$DS_* : \cdots \longrightarrow \mathbf{Z}^{n+1} \xrightarrow{D(n)} \mathbf{Z}^n \longrightarrow \cdots \longrightarrow \mathbf{Z}^2 \xrightarrow{D(1)} \mathbf{Z},$$

where $DS_n = \mathbf{Z}^{n+1}$. Its homology H_*DS may be computed as follows. The decomposition (2) implies that the inclusion $v \in DS_n \mapsto (v, 0, 0) \in DS_{n+2}$ makes the "suspension" $\Sigma^2 DS_*$ into a subcomplex of DS_*. Then we have a short exact sequence $0 \to \Sigma^2 DS_* \to DS_* \to T_* \to 0$ of chain complexes, with

$$T_* : \cdots \longrightarrow \mathbf{Z}^2 \xrightarrow{E(n,n+2)} \cdots \longrightarrow \mathbf{Z}^2 \xrightarrow{E(1,3)} \mathbf{Z}^2 \xrightarrow{D(2)} \mathbf{Z}^2 \xrightarrow{D(1)} \mathbf{Z}.$$

It is easy to verify that the homology of T_* is trivial, which implies a "periodicity" $H_n DS \cong H_{n+2} DS$. An easy computation then shows that $H_0 DS = \mathbf{Z}_2$ and $H_1 DS = 0$. Therefore $H_n DS = \mathbf{Z}_2$ for even n and $H_n DS = 0$ for odd n. Moreover, it can be easily verified that the nontrival homology elements at even dimensions are detected by the modulo 2 Euler characteristic.

The Dehn-Sommerville chain complex DS_* may be compared with the "Eulerian chain complex"

$$\mathcal{E}^* : \cdots \longrightarrow \mathcal{E}^{n+1} \xrightarrow{\partial} \mathcal{E}^n \longrightarrow \cdots \longrightarrow \mathcal{E}^1 \xrightarrow{\partial} \mathcal{E}^0,$$

where $\mathcal{E}^n$ is the collection of compact n-dimensional triangulated Eulerian manifolds and ∂ is the operation of taking boundary. The equality $D(n)f(M, \Delta) = f(\partial M, \partial\Delta)$ means that the map

$$f\text{-vector} : \mathcal{E}^* \to DS_*$$

is a "chain homomorphism".

Define two compact boundaryless Eulerian manifolds M_1 and M_2 to be *cobordant* if $M_1 \coprod M_2$ is the boundary of some compact Eulerian manifold W. Denote by Ω_n^E the cobordism classes of n-dimensional boundaryless compact Eulerian manifolds. Since cobordant PL-manifolds have the same mod 2 Euler characteristic, which completely determines elements in the Dehn-Sommerville homology, the f-vector map induces a homomorphism $\Omega_n^E \to H_n DS$.

Suppose M is an odd dimensional compact Eulerian manifold without boundary. Then by Corollary 3.2.2, cone$M$ is an Eulerian manifold with boundary M. This shows that $\Omega_{odd}^E = 0$. Similarly, the cone construction on even dimensional boundaryless Eulerian manifolds with Euler characteristic 2 enables us to prove that two boundaryless Eulerian manifolds M_1 and M_2 are cobordant if and only if $\chi(M_1)$ and $\chi(M_2)$ have the same parity. This implies $\Omega_{even}^E = \mathbf{Z}_2$.

In summary, we conclude the following.

Theorem 3.3.1 *The f-vector and the modulo 2 Euler characteristic induce isomorphisms*

$$\Omega_n^E \cong H_n DS \cong \begin{cases} \mathbf{Z}_2 & \textit{for even } n \\ 0 & \textit{for odd } n \end{cases}$$

4 Linear Conditions over Polyhedra

Let X^n be any compact polyhedron. Denote

$$X_{sing} = \bigcup_{k\neq 1-(-1)^n} X_k = \{x : \chi(\mathrm{lk}(x, X)) \neq 1-(-1)^n\},$$

which is the *singularity of* X *from Eulerian viewpoint*. Then Theorem 2.2.3 implies that, for any triangulation Δ of X, the coordinates of the vector $D(n)f(X,\Delta)$ vanish at dimensions $> \dim X_{sing}$. This observation leads to the determination of rational linear conditions on $D(n)f(X,\Delta)$ for all triangulations Δ.

If the singularities of X is simple in the sense that X_k is empty for all but $k = 1-(-1)^n$ and $k = k_0 \neq 1-(-1)^n$. Then Theorem 2.2.3 also implies that $D(n)f(X,\Delta) = 0 \bmod (1-(-1)^n+(-1)^n k_0)$. This observation leads to the determination of all linear conditions with value in any abelian group for all triangulations of Eulerian 2-strata spaces.

4.1 Parity of Dimension

A technical result that plays a crucial role in the development of the linear conditions on f-vectors is the following.

Lemma 4.1.1 *For any polyhedron* X*,* $\dim X$ *and* $\dim X_{sing}$ *have different parity.*

Proof: Denote $n = \dim X$ and $s = \dim X_{sing}$. For a triangulation of X there is a simplex σ^s such that $\dot\sigma \subset X_k$ for some $k \neq 1-(-1)^n$. Fix a point $x \in \dot\sigma$. Then by dimension reason we have a link pair $(L, K) \subset (X, X_{sing})$ very close to x such that the open cone $(\dot{x}L, \dot{x}K) = (xL - L, xK - K)$ is a neighborhood of x in (X, X_{sing}). Moreover, points in $\dot{x}L - \dot{x}K$ are Eulerian regular (i.e., inside $X_{1-(-1)^n}$), and points in $\dot{x}K$ are in X_k.

If we delete x from $(\dot{x}L, \dot{x}K)$, then we obtain $(L, K) \times \mathbf{R}$, satisfying:

1. L is a compact polyhedron and K is a closed subpolyhedron;
2. $\chi(\mathrm{lk}((y,t), L\times\mathbf{R})) = 1-(-1)^n$ for $y \in L - K$ and $\chi(\mathrm{lk}((y,t), L\times\mathbf{R})) = k$ for $y \in K$.

The second condition is equivalent to $\chi(\mathrm{lk}(y, L)) = 1-(-1)^{n-1}$ for $y \in L-K$ and $\chi(\mathrm{lk}(y, L)) = 2-k$ for $y \in K$. By Theorem 2.2.3, we have

$$\begin{aligned} D(n-1)f(L,\Delta) &= (1-(-1)^{n-1}+(-1)^{n-1}(2-k))f(K,\Delta|_K) \\ &= (1-(-1)^n+(-1)^n k)f(K,\Delta|_K) \end{aligned}$$

for any triangulation Δ of (L, K). By Lemma 3.2.1, we then have

$$0 = D(n-2)D(n-1)f(L,\Delta) = (1-(-1)^n+(-1)^n k)D(n-2)f(K,\Delta|_K).$$

Since $1-(-1)^n+(-1)^n k \neq 0$, we conclude that $D(n-2)f(K,\Delta|_K) = 0$.

Since $\dot\sigma$ is a neighborhood of x in X_{sing} and $K = \mathrm{lk}(x, X_{sing})$, we have $K \cong S^{s-1}$. Therefore by (4) and $D(n-2)f(K,\Delta|_K) = 0$ we conclude that $s-1$ and $n-2$ have the same parity.

□

4.2 Rational Linear Conditions over Polyhedra

The Dehn-Sommerville equations (3) imply that the coordinates of $D(n)f(X,\Delta)$ at dimensions $> s = \dim X_{sing}$ vanish. By Lemma 4.1.1, we know n and $s+1$ have the same parity. In view of the decomposition (2), we see that the coordinates of $D(n)f(X,\Delta)$ at dimensions $> s$ is exactly $(0, E(s+1,n))f(X,\Delta)$, where 0 occupies the first $s+2$ columns.

Thus for any fixed compact polyhedron X^n, the f-vector of any triangulation satisfies $\chi(v) = \chi(X)$ and $(0, E(s+1,n))v = 0$. The rank of these two equations is $\frac{n-s+1}{2}$. Then a slight modification of the proof of Theorem 3.1.1 shows that there are $\frac{n+s+1}{2}+1$ triangulations of X with affinely independent f-vectors. This implies the following result.

Theorem 4.2.1 *Suppose X^n is a compact polyhedron. Let s be the dimension of the part of X where the Euler characteristic of links are not $1-(-1)^n$. Then the only rational linear conditions on the f-vectors of all triangulations of X are the Euler equation $\chi(v) = \chi(X)$ and the partial Dehn-Sommerville equations $(0, E(s+1,n))v = 0$.*

In case the Eulerian singular part of X is empty, $(0, E(s+1,n))v = 0$ should be replaced by the full Dehn-Sommerville equations $D(n)v = 0$. Conversely, if the Eulerian singular part is not empty, then $(0, E(s+1,n))v = 0$ is never the full Dehn-Sommerville equations. Therefore we conclude that the full Dehn-Sommerville equations characterize boundaryless Eulerian manifolds.

Theorem 4.2.2 *A compact polyhedron X is a boundaryless Eulerian manifold if and only if there is $m \geq \dim X$ such that $D(m)f(X,\Delta) = 0$ for any triangulation Δ of X.*

4.3 Eulerian 2-Strata Spaces

We consider the special case that there are two types of points in a polyhedron from the viewpoint of Euler characteristic.

Definition 4.3.1 A locally compact polyhedron X^n is an *Eulerian 2-strata space* if there is $k \neq 1-(-1)^n$ and a closed subpolyhedron $Y \subset X$, such that

$$\chi(\mathrm{lk}(x,X)) = \begin{cases} 1-(-1)^n & \text{if } x \in X - Y \\ k & \text{if } x \in Y \end{cases}$$

We call Y the *lower stratum* and $X - Y$ the *upper stratum* of X.

In case $k = 1$ and $\dim Y = n-1$, X is an Eulerian manifold with boundary Y.

Another special case is when X is a disjoint union of Eulerian manifolds (possibly of different dimensions) without boundary. Those Eulerian manifolds in X with dimensions of the same parity as $\dim X$ form the upper stratum. The lower stratum

Y consists of the ones with dimensions of different parity from $\dim X$. Thus $k = 1 + (-1)^{\dim X}$.

For a compact Eulerian 2-strata space (X^n, Y), the generalized Dehn-Sommerville equations (3) become

$$D(n)f(X, \Delta) = (1 - (-1)^n + (-1)^n k) f(Y, \Delta|_Y). \tag{5}$$

In particular, this means that the f-vector of the lower stratum may be expressed in terms of the f-vector of the whole polyhedron.

A prototypical example of Eulerian 2-strata spaces is PL 2-strata spaces widely studied in topology: (X, Y) is a PL 2-strata space if Y is a closed and connected subpolyhedron of X, $X - Y$ and Y are PL-manifolds, and a neighborhood of Y in X is the mapping cylinder of a PL-block bundle map $f : L \to Y$ with homeomorphic fibre over different components of Y. Any two points $y_1, y_2 \in Y$ in such a PL 2-strata space have homeomorphic neighborhoods in X. In particular, PL 2-strata spaces are Eulerian 2-strata spaces.

In [4], we proved the following result, which indicates that Eulerian 2-strata spaces have similar structures as PL 2-strata spaces.

Theorem 4.3.2 *A polyhedron X^n with a closed subpolyhedron Y is an Eulerian 2-strata space with $k \neq 1 - (-1)^n$ if and only if*

1. *$X - Y$ and Y are Eulerian manifolds without boundary;*
2. *A neighborhood of Y in X is the mapping cylinder of a PL-map $f : L \to Y$ such that $\chi(f^{-1}(y)) = 1 + (-1)^n - (-1)^n k$ for any $y \in Y$.*

A 1-dimensional Eulerian 2-strata space is obviously a graph (loops allowed) such that each vertex is the meeting point of equal number of edges.

In [4], we found that compact 2-dimensional Eulerian 2-strata spaces are always constructed in the following way: Start with a compact surface S with boundary and a map $f : \partial S \to Z$, where Z is a disjoint union of isolated points and circles. f must satisfy the condition that for any z in $Y = Z -$ isolated points $=$ circles in Z, $f^{-1}(z)$ consists of $2 - k$ points. If we glue the boundary ∂S to Z along f, then the polyhedron $X = S \cup_f Z$ is an Eulerian 2-strata space with k and lower stratum Y .

We also have an explicit description of the maps f between circles in [4].

4.4 Integral Linear Conditions over Eulerian 2-Strata Spaces

For Eulerian 2-strata spaces, we can say more about the linear conditions on the f-vectors. Specifically, we are able to obtain all linear conditions with values in any abelian group on f-vectors of all triangulations of any compact Eulerian 2-strata space. This involves the study of torsion linear conditions in addition to all the rational linear conditions.

An obvious torsion linear condition one can draw from the Dehn-Sommerville equations (5) for Eulerian 2-strata spaces is that $D(n)f(X, \Delta) = 0 \bmod (1 - (-1)^n + (-1)^n k)$. In [4], we proved that this is the only torsion linear condition.

Theorem 4.4.1 *Suppose (X^n, Y^s) is a compact Eulerian 2-strata space. Then the only integral linear conditions on the f-vectors $f(X, \Delta)$ for all triangulations Δ of X are the Euler equation $\chi(v) = \chi(X)$, the partial Dehn-Sommerville equations $(0, E(s+1, n))v = 0$, and the torsion Dehn-Sommerville equations $D(n)v = 0$ mod $(1-(-1)^n+(-1)^n k)$. Moreover, any linear condition with value in an abelian group factors through these equations.*

If $k = 1$ or $k = 1-(-1)^n 2$, then $1-(-1)^n+(-1)^n k = \pm 1$, so that the torsion Dehn-Sommerville equations become trivial. Consequently, the integral linear conditions in the theorem become $\chi(v) = \chi(X)$ and $(0, E(s+1, n))v = 0$.

The proof that the list is all the integral conditions is based on the fact that the special triangulations δ_i^n used in the proof of Theorems 3.1.1 and 4.2.1 can be carefully chosen so that the integral affine lattice generated by their f-vectors is an "affine direct summand" of $\mathbf{Z}^{n+1}$. The proof about linear conditions with value in an abelian group is based on the fact that the list of equations is a "direct summand" of some sort.

We may also study the relative f-vector

$$f(X, Y, \Delta) = f(X, \Delta) - f(Y, \Delta|_Y).$$

By Lemma 4.1.1 and Theorem 4.3.2, Y is a boundaryless Eulerian manifold with dimension of different parity from n. Then (4) and (5) imply

$$D(n)f(X, Y, \Delta) = (-1-(-1)^n+(-1)^n k)f(Y, \Delta|_Y). \tag{6}$$

The following similar result is also proved in [4] by the same method.

Theorem 4.4.2 *Suppose (X^n, Y^s) is a compact Eulerian 2-strata space. Then the only integral linear conditions on the relative f-vectors $f(X, Y, \Delta)$ for all triangulations Δ of X are the Euler equation $\chi(v) = \chi(X) - \chi(Y)$, the partial Dehn-Sommerville equations $(0, E(s+1, n))v = 0$, and the torsion Dehn-Sommerville equations $D(n)v = 0$ mod $(-1-(-1)^n+(-1)^n k)$. Moreover, any linear condition with value in an abelian group factors through these equations.*

Note that it is possible that $k = 1+(-1)^n$. In this case, $-1-(-1)^n+(-1)^n k = 0$ and the integral linear conditions in Theorem 4.4.2 become $\chi(v) = \chi(X) - \chi(Y)$ and $D(n)v = 0$. Such situation never occurs in Theorem 4.4.1.

On the other hand, if $k = 1$ or $k = 1+(-1)^n 2$, then $-1-(-1)^n+(-1)^n k = \pm 1$, so that there is no torsion equations.

5 Eulerian Stratification

In this section, we generalize the notion of Eulerian 2-strata spaces to polyhedra with more than 2 strata.

Definition 5.0.3 A *stratification* of a compact polyhedron X is a collection $\{\bar{X}_a : a \in P\}$ of closed subpolyhedra indexed by a finite partially ordered set P, such that

$$X = \bigcup \bar{X}_a, \quad \bar{X}_a \bigcap \bar{X}_b = \bigcup_{c \le a, c \le b} \bar{X}_c. \tag{7}$$

A triangulation Δ of X is called *stratified* if each $\bar{X}_a$ is a subcomplex.

Denote

$$\bar{X}_{<a} = \bigcup_{b<a} \bar{X}_b, \quad X_a = \bar{X}_a - \bar{X}_{<a}.$$

We call X_a *strata* and $\bar{X}_a$ *closed strata*. The two conditions (7) for stratification may be rephrased in terms of strata:

$$X = \bigsqcup_{a \in P} X_a, \quad \text{closure}(X_a) = \bigsqcup_{b \le a} X_b, \tag{8}$$

where the closure of X_a is simply $\bar{X}_a$.

Definition 5.0.4 An *Eulerian stratification* on a compact polyhedron X is a stratification such that for $x \in X_a$ and $a \le b$,

$$\bar{\chi}(a, b) = \chi(\text{lk}(x, \bar{X}_b))$$

is independent of the choice of x.

Denote $d(a) = \dim X_a$. For a point x in the interior of a $d(a)$-dimensional simplex in a triangulation of X_a, we have

$$\chi(a, a) = \chi(\text{lk}(x, \bar{X}_a)) = \chi(S^{d(a)-1}) = 1 - (-1)^{d(a)}. \tag{9}$$

Then by the definition of Eulerian stratification, we have $\chi(\text{lk}(x, \bar{X}_a)) = 1 - (-1)^{d(a)}$ for any $x \in X_a$. Since X_a is an open subset of $\bar{X}_a$, we have $\text{lk}(x, \bar{X}_a) = \text{lk}(x, X_a)$ for $x \in X_a$. Therefore X_a is an Eulerian manifold, so that an Eulerian stratified polyhedron is obtained by gluing Eulerian manifolds in "Eulerian fashion". In fact, it is possible to generalize the structure Theorem 4.3.2 of Eulerian 2-strata spaces to general Eulerian stratified polyhedra.

A stratified triangulation of X gives rise to a system of f-vectors of individual stratum. In [5], the Dehn-Sommerville equations and the theory of linear conditions were generalized to such systems. Moreover, we showed that the stratifications themselves have interesting combinatorial structures.

5.1 Weighted f-vectors and Linear Conditions

For $x \in X_a$ and $a \le b$, let

$$\chi(a, b) = \chi(\text{lk}(x, X_b)) = \chi(\text{lk}(x, \bar{X}_b)) - \chi(\text{lk}(x, \bar{X}_{<b})).$$

Then from (8) we have

$$\bar{\chi}(a,b) = \sum_{a \le c \le b} \chi(a,c). \tag{10}$$

This is equivalent to

$$\chi(a,b) = \sum_{a \le c \le b} \bar{\chi}(a,c)\mu_P(c,b), \tag{11}$$

where μ_P is the *Möbius function* on the partially ordered set P. (8) and (10) implies that the systems $\{\bar{\chi}(a,b)\}$ and $\{\chi(a,b)\}$ determine each other. In particular, $\chi(a,b)$ is independent of the choice of $x \in X_a$. We call $\chi(a,b)$ the *relative Euler characteristic* between strata.

The *Euler characteristic* of the stratum X_a (which is usually not compact) is $\chi(a) = \chi(X_a) = \chi(\bar{X}_a) - \chi(\bar{X}_{<a})$.

Let Δ be a stratified triangulation of a stratified polyhedron X. Then each stratum X_a is a union of interiors of some simplices in Δ. Let Δ_a be the collection of such simplices. Then we have the f-vector $f(X_a, \Delta_a)$ of the a-th stratum. Clearly, $\chi(f(X_a, \Delta_a)) = \chi(a)$.

To study the linear conditions over the system $\{f(X_a, \Delta_a)\}$ of f-vectors, we consider a function ω on P and the *ω-weighted f-vector*

$$f(X, \Delta, \omega) = \sum_{a \in P} f(X_a, \Delta_a)\omega(a).$$

It satisfies the weighted Euler equation

$$\chi(f(X, \Delta, \omega)) = \chi(X, \omega) = \sum_{a \in P} \chi(a)\omega(a).$$

By choosing different weights, we may recover interesting f-vectors. For example, if $\omega(a) = 1$ for all index a, then the ω-weighted f-vector is the usual f-vector $f(X, \Delta)$. Moreover, if we take ω to be 1 at a and vanish at other indices, then we recover $f(X_a, \Delta_a)$.

Note that the "effective dimension" of an ω-weighted f-vector is the *dimension of weight*

$$d(\omega) = \max\{d(a) : \omega(a) \neq 0\}.$$

In fact, for any stratified triangulation, the ω-weighted f-vector of X is the same as the ω-weighted f-vector of the union of strata of X of dimension $\le d(\omega)$.

The proof of Theorem 2.2.3 can be easily adapted to weighted f-vectors. A key step is the analogue of Lemma 2.2.1 that computes the Euler characteristic of the simplicial link $\mathrm{lk}(\sigma, \Delta_b)$ of a simplex $\sigma^s \in \Delta_a$ in strata X_b, $a \le b$:

$$\chi(\mathrm{lk}(\sigma, \Delta_b)) = (1 - (-1)^s)\delta(a,b) + (-1)^s \chi(\mathrm{lk}(x, X)),$$

where x is any interior point of σ, and $\delta(a,b) = 1$ for $a = b$, $\delta(a,b) = 0$ for $a \neq b$. From this it is easy to deduce the following Dehn-Sommerville equations for weighted f-vectors.

Theorem 5.1.1 *Suppose Δ is a stratified triangulation of an Eulerian stratified polyhedron X. Suppose ω is a weight function on the index set P. Then for any $n \geq \dim X$ we have*

$$D(n)f(X,\Delta,\omega) = f(X,\Delta,\partial_n\omega),$$

where $\partial_n\omega$ is another weight function given by

$$\partial_n\omega(a) = (1-(-1)^n))\omega(a) + (-1)^n \sum_{a\leq b} \chi(a,b)\omega(b).$$

Therefore the application of the Dehn-Sommerville matrix to weighted f-vectors is interpreted as a boundary operation on the weight. For example, the Dehn-Sommerville equations (5) for an Eulerian 2-strata space (X^n, Y) is interpreted as sending $\omega = (1,1)$ (values at upper and lower strata) to $\partial_n\omega = (0, 1-(-1)^n+(-1)^n k)$. Note that since Y is an Eulerian manifold without boundary and with dimension of different parity from n, we also have $\partial_n(0,1) = (0,2)$. Therefore we have $\partial_n(\lambda,\mu) = (0, (1-(-1)^n+(-1)^n k)\lambda + 2(\mu-\lambda)) = (0, (-1-(-1)^n+(-1)^n k)\lambda + 2\mu)$.

The correspondence between $D(n)$ and ∂_n enables us to translate properties of $D(n)$ (Lemma 3.2.1 in particular) to properties of ∂_n. Thus the equality $\chi \circ D(n) = (1-(-1)^n)\chi$ becomes

$$\sum_{a\leq b} \chi(a)\chi(a,b) = 0, \quad \text{for any fixed } b. \tag{12}$$

The equality $D(n-1)D(n) = 0$ becomes $\partial_{n-1}\partial_n = 0$. If we further use the explicit formula for ∂_* in Theorem 5.1.1, the equality $\partial_{n-1}\partial_n = 0$ means exactly

$$\sum_{a\leq c\leq b} \chi(a,c)\chi(a,b) = 2\chi(a,b), \quad \text{for any fixed } a,\ b. \tag{13}$$

The analogue of Lemma 4.1.1 on the parity of dimensions is that $r = d(\omega)$ and $s = d(\partial_r\omega)$ should have different parity (recall that r is the "effective dimension" of weighted f-vectors). To see this is indeed the case, we observe from the definition of ∂_n that if $\theta \neq 0$ and $\partial_n\theta = 0$. Then for an index a with $d(a) = d(\omega)$ and $\omega(a) \neq 0$ we have $0 = \partial_n\theta(a) = (1-(-1)^{n+d(a)})\omega(a)$ from the definition of ∂_* in Theorem 5.1.1. Since $\omega(a) \neq 0$, we conclude that n and $d(a) = d(\omega)$ have the same parity. Now by $\partial_{r-1}\partial_r = 0$, we may apply this to $\theta = \partial_r\omega$ and $n = r-1$. Therefore we conclude that either $\partial_r = 0$ or $r-1$ and s have the same parity.

Once the analogue of Lemma 4.1.1 is established, the proof of Theorems 3.1.1 and 4.2.1 can be applied to obtain all the linear conditions on ω-weighted f-vectors for a fixed weight.

Theorem 5.1.2 *Suppose X is an Eulerian stratified polyhedron. Suppose ω is a (rationally valued) weight function. Let $r = d(\omega)$ and $s = d(\partial_r\omega)$. Then the only rational linear conditions on the ω-weighted f-vectors for all stratified triangulations of X are the Euler equation $\chi(v) = \chi(X,\omega)$ and the partial Dehn-Sommerville equations $(0, E(s+1,r))v = 0$.*

When $\partial_r \omega = 0$, the partial Dehn-Sommerville equations should be replaced by the whole $D(r)v = 0$.

As a consequence of the theorem, we can also determine all the linear conditions of all weighted f-vectors for all weights and all stratified triangulations.

Theorem 5.1.3 *Suppose X is an Eulerian stratified polyhedron. Let $n = \dim X$ and $m = \max\{d(a) : n \text{ and } d(a) \text{ have different parity}\}$. Then the only rational linear conditions on the weighted f-vectors for all weights and all stratified triangulations of X are*

1. *the Euler equation $\chi(v) = 0$ and the partial Dehn-Sommerville equations $(0, E(m+1, n))v = 0$ in case all $\chi(a) = 0$;*

2. *the partial Dehn-Sommerville equations $(0, E(m+1, n))v = 0$ in case some $\chi(a) \neq 0$.*

Again in case all $d(a)$ have the same parity as n (so that m is not defined), the partial Dehn-Sommerville equations should be replaced by the whole $D(n)v = 0$.

5.2 Realization Theorem

The Euler characteristic $\chi(a)$ and the relative Euler characteristic $\chi(a, b)$ for an Eulerian stratified polyhedron must satisfy (9), (12), (13). In [5], we proved that this is all the relations between these numbers.

Theorem 5.2.1 *Suppose P is a finite partially ordered set, and $d : P \to \mathbf{N}$ is a function such that $d(b) - 2 \geq d(a) \geq 1$ for $a < b$. Suppose $\chi(a)$ is a collection of integers for $a \in P$, and $\chi(a, b)$ is another collection of integers for $a \leq b$ in P. Then there exists an Eulerian stratified polyhedron X indexed by P with dimension function $d(a)$, Euler characteristic function $\chi(a)$, and relative Euler characteristic function $\chi(a, b)$ if and only if* (9), (12), (13) *are satisfied.*

The construction of the Eulerian stratified polyhedron X with prescribed combinatorial data is modeled on the structure of geometrically stratified spaces studied by topologists. In 2-strata case, the structure Theorem 4.3.2 suggests that we need to construct an Eulerian manifold Y_{lower} without boundary as the lower stratum, a PL-map $f : L \to Y_{\text{lower}}$ such that $\chi(f^{-1}(y))$ is a prescribed constant for any $y \in Y_{\text{lower}}$, and an Eulerian manifold Y_{upper} with L as boundary as the "closed interior" of the upper stratum. Then $X = Y_{\text{upper}} \cup L \times [0, 1] \cup_f Y_{\text{lower}}$. If we choose $L = Y_{\text{lower}} \times F$ and f to be the projection, then the construction boils down to finding Eulerian manifolds Y_{lower}, Y_{upper}, F, such that

1. Y_{lower} is a boundaryless $d(\text{lower})$-dimensional Eulerian manifold with Euler characteristic $\chi(\text{lower})$;

2. F is a boundaryless $(d(\text{upper}) - d(\text{lower}) - 1)$-dimensional Eulerian manifold with Euler characteristic $(-1)^{d(\text{upper})-1}\chi(\text{lower}, \text{upper})$;

3. Y_{upper} is a $d(\text{upper})$-dimensional Eulerian manifold with Euler characteristic $(-1)^{d(\text{upper})}\chi(\text{upper})$ and boundary $Y_{\text{lower}} \times F$.

Note that for any even $n \geq 2$, any integer can be realized as the Euler characteristic of some n-dimensional boundaryless Eulerian manifold (in fact by PL-manifold). Therefore by the computation of the cobordism group Ω_*^E in Theorem 3.3.1, the only condition for constructing Y_{lower}, Y_{upper}, F comes from the equality $\chi(\partial M) = (1-(-1)^{\dim M})\chi(M)$ for Eulerian manifolds M. The condition (9) means the relevant pieces are Eulerian manifolds. The conditions (12) and (13) are exactly the equality $\chi(\partial M) = (1-(-1)^{\dim M})\chi(M)$ for M to be one of the three pieces.

The general situation is more complicated only in bookkeeping. The key observation that (12) and (13) are equivalent to the equality $\chi(\partial M) = (1-(-1)^{\dim M})\chi(M)$ for all the pieces is still valid in general. The realization Theorem 5.2.1 then follows.

The theorem suggests a combinatorial structure intrinsic to Eulerian stratifications. The structure consists of a finite partially ordered set equipped with dimension function $d(a)$, the (Euler characteristic) function $\chi(a)$, and the relative (Euler characteristic) function $\chi(a, b)$, satisfying (9), (12), (13). We expect a linear combinatorial theory can be established over such structure, and such theory may include many classical linear combinatorial theories.

A key feature in such a theory would be the boundary operator ∂_n on the functions over P introduced in Theorem 5.1.1. The property $\partial_{n-1}\partial_n = 0$ and the fact that ∂_n depends only on the parity of n induces two homologies (at even and odd dimensions). The homology should play a role in such problems as torsion linear conditions on weighted f-vectors.

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