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ABSTRACT

The classical surgery theory as developped by Browder, Kervaire, Milnor, Novikov, Sullivan, Wall, etc., computes the structure set $S(M, \text{rel } \partial)$ of homeomorphism classes of manifolds simple homotopy equivalent to M rel ∂M. Siebenmann showed that the set is 4-periodic: $S(M, \text{rel } \partial) \cong S(M \times D^4, \text{rel } \partial)$ for topological manifolds with boundary[KS]. This paper attempts to establish a surgery theory for manifolds with certain group actions and generalize the periodicity phenomenon in the new setting.

Let G be a finite group and $S_G(M, \text{rel } \partial M)$ be the homeomorphism classes of homotopically stratified G-manifolds isovariantly homotopically transverse equivalent to M rel ∂M in the sense of Quinn [Q5]. Let $S_G^{-\infty}(M, \text{rel } \partial M)$ be its stablized version. Then we proved that with small gap hypothesis $S_G^{-\infty}(M, \text{rel } \partial) \cong S_G^{-\infty}(M \times V, \text{rel } \partial)$ for the unit disc V of a periodicity representation, among them the $4l$-fold regular representatinon of any odd order group. We also proved that with slightly bigger gap, there is a periodicity map $S_G(M, \text{rel } \partial) \to S_G(M \times V, \text{rel } \partial)$ and we identified the extent to which this fails to be isomorphic.

CHAPTER 1

INTRODUCTION

Let M be a closed topological manifold. The structure set $S(M)$ is the set of simple homotopy equivalences $f : N \to M$ from a closed topological manifold N, modulo the relation that f and f' represent the same "structure" if there exists a homeomorphism $\phi : N \cong N'$ such that $f'\phi \simeq f$.

If M has a boundary, then $S(M, \mathrm{rel}\ \partial)$ may be defined in a similar way, with the additional requirement that the restrictions of all the maps on $\partial N \to \partial M$ are homeomorphisms, and the homotopies are rel ∂.

In [KS], Siebenmann proved the 4 fold periodicity of S, i.e., there exists a natural 1-1 correspondence

$$S(M, \mathrm{rel}\ \partial) \cong S(M \times D^4, \mathrm{rel}\ \partial).$$

Note that the map cannot be easily defined as crossing with D^4 since this will not produce homeomorphisms on the boundaries. In fact, Cappell and Weinberger [CW] found a delicate geometric construction by making use of the codimension≥ 3 embedding theory and the Edward's result on recognizing topological manifolds. On the other hand, the periodicity fails in the PL and smooth category. This is one of the several facts that mildly indicate that the topological category is the easier one for geometric topology, once the technical difficulty of the existence of handle structures and the K-theory of topological manifolds are settled in [KS]. Moreover, the periodicity has some interesting applications. For example, the periodicity may be used to define the functoriality of the structure set S (see [We]), which for instance implies that $S(M, \mathrm{rel}\ \partial)$ only depends on the homotopy type of M. Finally, the periodicity is related to other problems in the geometric topology, among them the Borel conjecture.

This paper attempts to generalize the periodicity phenomenon to manifolds with group actions. We try to generalize Siebenmann's original approach.

Briefly speaking, Siebenmann's proof is as follows. Following Quinn's [Q1] spacified version of Wall's [Wa] surgery theory, there is a *surgery fibration*

$$S(M, \text{rel } \partial M) \to N(M, \text{rel } \partial M) \to L(M)$$

where

1. $S(M, \text{rel } \partial M)$ is the *structure space* whose 0-th homotopy group is the strucutre set (note that we abused notation by using S for both the set and the space);
2. $N(M, \text{rel } \partial M)$ is the space of *normal invariants* and is homotopy equivalent to the space $Maps(M/\partial M, F/Top)$ of maps from $M/\partial M$ to a universal space F/Top;
3. $L(M)$ is the space of *surgery obstructions* which only depends on $\pi_1(M)$ and the first Stiefel-Whitney class $\omega_M : \pi_1 M \to \{\pm 1\}$ of M.

Then we may think of $N(M, \text{rel } \partial M)$ and $L(M)$ as functors of M, and apply them to the following operations

$$M \xrightarrow{\times \mathbf{CP}^2} M \times \mathbf{CP}^2 \xleftarrow{\text{incl}} M \times D^4,$$

where D^4 may be taken as a small neighborhood of a point in $\mathbf{CP}^2$. By the classical periodicity in the chapter 9 of [Wa], and the fact that L only depends on the fundamental group and the first Stiefel-Whitney class, we see that both operations induce equivalences and then the 4 fold periodicity for L:

$$L(M) \overset{\simeq}{\to} L(M \times \mathbf{CP}^2) \overset{\simeq}{\leftarrow} L(M \times D^4) = \Omega^4 L(M)$$

Moreover, Siebenmann proved that $F/Top \times \mathbf{Z} \simeq L(pt.)$ by the Poincaré conjecture and his own work on the topological manifolds. Hence F/Top is also 4 fold periodic up to a factor of $\mathbf{Z}$, which produces the periodicity for $N(M, \text{rel } \partial M)$. The two periodicities are compatible in the surgery fibration and therefore induce a homotopy equivalence of the fibres

$$S(M, \text{rel } \partial) \simeq S(M \times D^4, \text{rel } \partial).$$

By the way, since $F/CAT \times \mathbf{Z} \not\simeq L(pt.)$ for $CAT = PL$ or O, there is no such periodicity in the PL and smooth categories.

In the equivariant case, Weinberger [We] developped the homotopically stratified surgery theory, which is applicable to *homotopically stratified group actions* and *homotopically transverse maps.* For any homotopically stratified topological manifold M, this involves a *stablized surgery fibration*

$$S_G^{-\infty}(M, \mathrm{rel}\ \partial M) \to H(M/G; L^{-\infty}(locM/G)) \to L_G^{-\infty}(M),$$

and a *destablization fibration*

$$S_G(M, \mathrm{rel}\ \partial M) \to S_G^{-\infty}(M, \mathrm{rel}\ \partial M) \to \hat{H}(\mathbf{Z}_2; Wh_G^{Top,\leq 0}(M)).$$

The stablized surgery fibration may be constructed inductively from the nonequivariant case, and the spaces $S_G^{-\infty}$, $L_G^{-\infty}$ have similar properties enjoyed by their nonequivariant counterparts. However, this only produces stable structures. We need the second fibration to relate the stable structure $S_G^{-\infty}$ to the unstable structure S_G, whose 0th homotopy group is the set of homotopically stratified G-manifolds simple homotopically transversely equivalent to M rel ∂M.

Then following Siebenmann's approach, we think of $H(?; L^{-\infty}(loc?))$, $L_G^{-\infty}$ as functors and apply them to the operations

$$M \xrightarrow{\times P} M \times P \xleftarrow{\text{incl}} M \times V$$

where P is a *periodicity manifold* (see the Definition 4.4.1), V is the tangent space of P at a fixed point, and the inclusion is by viewing V as a representation neighborhood. The result is the following.

Theorem 5.1 *Let P be a periodicity manifold and V be the tangential G-representation of P at a fixed point. Suppose that M is a homotopically stratified topological G-manifold that has the codimension ≥ 3 gap, and M, $M \times P$ have the same isotropy everywhere. Then there exists a natural homotopy equivalence of structure spaces:*

$$\Pi^{-\infty} : S_G^{-\infty}(M, \mathrm{rel}\ \partial) \simeq S_G^{-\infty}(M \times V, \mathrm{rel}\ \partial).$$

As for examples of periodicity manifolds, we have

$$P = \underbrace{\mathbf{CP}^2 \times \cdots \times \mathbf{CP}^2}_{S}$$

where G acts on a finite set S in such a way that the orbits of S/G are of odd orders. The corresponding periodicity representation is $\mathbf{R}^4 \times \mathbf{R}S$. Moreover the complex projective space may be replaced by the quaternionic projective spaces or more general ones.

Then we try to destablize the periodicity of $S_G^{-\infty}$ to obtain the periodicity of S_G. We however have only limited success in doing so. In fact, we are able to construct a periodicity map which we are not sure is an equivalence or not.

Theorem 8.1 *Let P be a periodicity manifold and V be the tangential G-representation of P at a fixed point. Suppose that M is a homotopically stratified topological G-manifold that has the codimension ≥ 5 gap, P^H is 3-connected for any $H \in iso(X)$, and M, $M \times P$ have the same isotropy everywhere. Then there is a natural periodicity map*

$$\Pi : S_G(M, \mathrm{rel}\ \partial) \to S_G(M \times V, \mathrm{rel}\ \partial),$$

whose homotopy fibre is the loop of the homotopy fibre of the map

$$\times P : \hat{H}(\mathbf{Z}_2; Wh_G^{Top,\leq 0}(M)) \to \hat{H}(\mathbf{Z}_2; Wh_G^{Top,\leq 0}(M \times P)).$$

There are many cases that $Wh^{Top,\leq 1}$ is known to be contractible. Hence the destablization is successful in such cases, and we have the unstable periodicity.

The condition is not satisfied by the permutation product of $\mathbf{CP}^2$'s. However, it is satisfied by the product of $\mathbf{HP}^2$'s. Note that the corresponding periodicity representation is $\mathbf{R}^8 \times \mathbf{R}S$.

The paper basically consists of two parts. The chapters 2 and 3 are the development of basic machinaries, including the homology, the stablized surgery, and the destablization. The chapters 4, 5 and 6 are the proofs of the periodicity results.

In the chapter 2, we first define the notion of homotopically stratified spaces and briefly discuss the facts about such spaces that are most useful for our work.

Then we establish the homology theory over such spaces, so that the notations like $H(X; L^{-\infty}(locX))$ in the stablized surgery fibration make sense.

In the chapter 3, we first study the $\mathbf{R}^k$-controlled geometrically stratified spaces, which will serve as the stablized version of homotopically stratified spaces. Then we establish in detail the surgery fibration for such spaces and the simple homotopically transverse maps between such spaces. The section is concluded with a discussion about the destablization process.

The chapter 4 defines the notion of periodicity manifolds and proves that such manifolds produce the periodicity of the surgery obstruction L and its stablization $L^{-\infty}$. The section is ended with some examples of periodicity manifolds for odd order group actions.

The chapter 5 is the proof of the Theorem 5.1 on the periodicity of stablized surgery theory. Also proved is a naturality property enjoyed by the stablized periodicity map. The property will be needed in destablizing the periodicity.

The chapter 6 is the proof of the Theorem 8.1, which establishes the unstable periodicity map. Then we point out the difficulty in proving the map being an equivalence.

We conclude the introduction with some words on the *spacification.* Often the geometrically defined sets may be identified with the homotopy groups of certain spaces or spectra. We will mostly work with spaces, with basic computation done on sets. We will think of the isomorphisms of sets as parallel to the homotopy equivalences of spaces, and long exact sequences as the the homotopy exact sequences of fibrations.

CHAPTER 2

STRATIFIED SPACES

This chapter explains the works of Quinn on homotopically stratified spaces and extends his homology theory to such spaces. The basic references are [Q2][Q4][Q5].

Our primary concern is manifolds with group actions. The approach we are going to take is to consider the quotient spaces, called *orbifolds*. The points in the orbifolds may be classified by their isotropy groups. This makes the orbifolds into stratified spaces.

As suggested by Quinn in [Q5], the right condition one should impose on an equivariant topological manifold is that its orbifold should be homotopically stratified. Such a condition is so weak that it is satisfied by many interesting group actions. On the other hand, after much work, we find that the condition is strong enough for extending many nice classical theories.

2.1. Homotopically Stratified Spaces

A *stratified space* is a space X with a filtration of closed subspaces

$$X = X_n \supset X_{n-1} \supset \cdots \supset X_1 \supset X_0.$$

The subspaces X_i are called *skeleta*, and the differences $X_i - X_{i-1}$ are called *strata*. Then we have the category of stratified spaces in which we require that the morphisms map strata to strata (perhaps with some change on the indices).

The stratified space X is *geometrically stratified* if for any $k > i$, a neighborhood of $X_i - X_{i-1}$ in $(X_i - X_{i-1}) \cup (X_k - X_{k-1})$ is a bundle of some kind, and there are compatibility conditions on the bundles at places where more than two strata meet together. A stratified map between geometrically stratified spaces is *geometrically transverse* if its restriction on the bundle neighborhoods are bundle maps. With such maps, we get smooth, PL, or topological stratified spaces depending on the types of strata and bundles used. The details on the geometrically stratified spaces may be found in [BQ].

Our paper is however about the surgery on homotopically stratified spaces. The foundations of homotopically stratified spaces has been developped in detail in [Q5]. And a brief outline of the basic results is in order.

Let (A, B) and (X, Y) be pairs of spaces. A map $(A, B) \to (X, Y)$ is *strict* if it maps $A - B$ into $X - Y$. Let Y be a closed subset of X. Then the *homotopy link*

$$hlk(X, Y) = Maps_s([0, 1], 0; X, Y) \tag{2.1}$$

is the space of paths in X that starts in Y and immediately leaves Y (the subscript "s" stands for strictness).

The motivation for introducing the homotopy links is that it characterizes the property that a neighborhood of Y in X homotopically looks like a mapping cylinder. We say that Y is *tame* in X if Y has a neighborhood N that strictly deformation retracts in X to Y, where the strictness of the deformation means that it keeps mapping $N - Y$ into $X - Y$ until the last moment, when the whole N is necessarily mapped into Y. The tameness is equivalent to the existence of a strict homotopy equivalence $(Z, Y) \simeq_s (X, Y)$ rel Y for some space Z in which Y has a mapping cylinder neighborhood (Lemma 2.4 of [Q5]). In fact, we may take $Z = Y \times 0 \cup hlk(X, Y) \times [0, 1] \cap Z \times 1$ as the homotopy pushout of

$$Y \xleftarrow{p} hlk(X, Y) \xrightarrow{q} X - Y \tag{2.2}$$

where p and q are the end points of the paths. In particular, if a neighborhood of Y is the mapping cylinder of a map $f : E \to Y$, then f is homotopic to $p : hlk(X, Y) \to Y$.

A stratified space X is *homotopically stratified* if for any $k > i$, the stratum $X_i - X_{i-1}$ is tame in $(X_i - X_{i-1}) \cup (X_k - X_{k-1})$, and the homotopy link $hlk((X_i - X_{i-1}) \cup (X_k - X_{k-1}), X_i - X_{i-1})$ is a fibration over $X_i - X_{i-1}$. We emphasis that these fibrations fit together without any compatibility assumption. The key is the Proposition 2.9 of [Q5], which discusses the relation between various homotopy links arising from three spaces $X \supset Y \supset Z$. The only condition in the Proposition is that a strict deformation retraction $r : N \times [0, 1] \to Y$ of a neighborhood

N of Z in Y has the restriction $(N - Z) \times (0, 1] \to Y - Z$ covered by a fibre map $hlk(X - Z, Y - Z) \times (0, 1] \to hlk(X - Z, Y - Z)$. And this follows from the assumption that all the homotopy links are fibrations. In short, the special properties of homotopy links and the fact that homotopically stratified spaces are defined via homotopy links render the compatibility condition redundant.

A map between homotopically stratified spaces is *homotopically transverse* (as opposed to geomotrically transverse maps between geometrically stratified spaces) if it induces fibrewise homotopy equivalences on the homotopy link fibrations over the strata. Thus we obtain the category of homotopically stratified spaces.

A subspace Y of X is transverse (to the strata of X) if Y is homotopically stratified with respect to the induced stratification and the inclusion map is homotopically stratified. Equivalently, the homotopy links in Y are fibrations over the strata of Y and fibrewise homotopy equivalent to the restrictions of the homotopy links in X.

X is a *manifold homotopically stratified space* if each stratum is a topological manifold. $\partial X \subset X$ is the boundary of X if $\partial X \cap (X_i - X_{i-1})$ is the boundary of $(X_i - X_{i-1})$, and ∂X is closed and homotopically transverse to the stratification of X.

Among important examples of stratified spaces are those induced from equivariant spaces.

Let Z be a G-space such that the set $iso(Z)$ of isotropy subgroups is finite. Then its orbifold Z/G decomposes into finitely many disjoint pieces $(Z^H - Z^{>H})/WH$, where H is an isotropy subgroup and $WH = NH/H$ is its Weyl group. There is a map $i : iso(Z) \to \mathbf{Z}$ such that $H \subset K$ and $H \neq K$ imply that $i(H) < i(K)$. For example, one may define $i(H) = 0$ for maximal isotropy subgroups H of Z, define $i(K) = 1$ for maximal isotropy subgroups K of $Z - \cup_{i(H)=0} Z^H$, and proceed by induction. Now let $(Z/G)_i = \cup_{i(H)\leq i}(Z^H - Z^{>H})/WH$. Then Z/G becomes a stratified space. If Z/G is homotopically stratified, then we say that the action is homotopically stratified. If Z is a topological manifold, then each of the disjoint pieces is a topological manifold and therefore Z/G is a manifold stratified space. The following provides a sufficient condition for Z/G to be homotopically stratified (see the Corollary 1.9 of [Q5])

Lemma 2.1.1 *Suppose that G is a finite group that acts on a topological manifold M, such that if $H \subset K$, then M^K is locally flat and has no codimension 2 components in M^H. Then the action is homotopically stratified.*

Remark 2.1.2 The indices $\{i\}$ for a stratified space is not an important factor. One may actually use any partially ordered finite set as the index set. For example, the set $iso(Z)$ is a natural index set for Z/G.

2.2. Homology on Homotopically Stratified Spaces

We are interested in the homologies of the form $H(X; J(p))$, where X is a stratified space, J is a spectrum valued (homotopically transversely) homotopy invariant functor on homotopically stratified spaces, and $p : E \to X$ is a stratified system of fibrations with homotopically stratified fibre. With some further construction, we have for any homotopically stratified space X the homology $H(X; J(locX))$. This is the homology we will mostly use.

The homology is a slight variant of the one constructed by Quinn in the chapter 8 of [Q2] and the chapter 2 of [Q4]. The key is that Quinn's construction works out nicely for stratified systems of fibrations with fibres being homotopically stratified spaces (In Quinn's construction, the fibres are usual spaces). This produces a homology theory which we denote by $H(X; J(p))$ after Quinn. Then one more construction produces the homology $H(X; J(locX))$ from $H(X; J(p))$.

Before the construction of the homology theory, we note that a spectrum valued functor J consists of a sequence $\{J_k : k \in \mathbf{Z}\}$ of based space valued functors related by the suspension transformations $J_k \to \Omega J_{k+1}$. The homology space may be constructed for each functor and the resulting sequence of spaces form a spectrum.

For a reference on spectra see [Sw].

Let X be a simplicial complex. Let E be a homotopically stratified space and $p : E \to X$ be a transverse map in the sense that for any simplex σ of X, $E_\sigma = p^{-1}\sigma$ is a homotopically transverse subspace of E. Then for any face τ of σ the inclusion $E_\tau \subset E_\sigma$ is homotopically transverse. If we view X as a category in which the objects are simplices and the morphisms are inclusions of faces, then $\sigma \mapsto J(E_\sigma)$ is a functor. The homology will be defined as the direct limit of the functor, which may be constructed as a super homotopy pushout based on the barycentric subdivision X' of X. Specifically, a simplex σ of X' is an ascending sequence $\sigma_0 \subset \sigma_1 \subset \cdots \subset \sigma_n$ of simplices of X, with σ_0 being the smallest one. Then define

$$J(p) = \coprod_{\sigma \in X'} J(p^{-1}\sigma_0) \times \sigma / \sim, \tag{2.3}$$

with the relation given by the inclusion of faces in X'.

Clearly, there is a projection from $J(p)$ to $\coprod_{\sigma \in X'} \sigma / \sim$, which is the geometric realization of X' and is therefore equal to the space X. On the other hand, since the values of J are based, we have the inclusion $X = \coprod_{\sigma \in X'} \sigma / \sim \subset J(p)$ such that $X \to J(p) \to X$ is identity. Then define

$$H(X; J(p)) = J(p)/X. \tag{2.4}$$

More generally, if Y is a subcomplex of X, then we have $J(p|_Y) \subset J(p)$. Define the relative homology

$$H(X, Y; J(p)) = J(p)/(X \cup J(p|_Y)). \tag{2.5}$$

The next step is to extend the construction over more general spaces. This may be done by taking homotopy inverse limits. The process is similar to extending a generalized homology theory over finite complexes to more general spaces.

Thus, in the general situation of a map p from a homotopically stratified space to a space X, we consider the simplicial complex $S(X)$ of singular simplices σ of X homotopically transverse to the stratified structure of E, i.e., the restrictions of $E \to X$ on the simplices are homotopically transverse subspaces of E. There induces a homotopically stratified space $p^* : E^* \to S(X)$ over the singular complex. Let U be a neighborhood of the diagonal of $X \times X$. Denote by $S_U(X)$ the subcomplex of $S(X)$ consisting of singular simplices σ such that $\sigma \times \sigma \subset U$. Then we have the restriction $p^*_U : E^*_U \to S_U(X)$. Corresponding to the system of the neighborhoods of the diagonal of $X \times X$, we obtain an inverse system $\{p^*_U\}$ of transverse maps from homotopically stratified spaces to simplicial complexes. The homology of p is then defined as the homotopy inverse limit of the homology of the system:

$$H(X; J(p)) = \lim_{\leftarrow} H(S_U(X); J(p)). \tag{2.6}$$

The final step is to produce a *locally finite homology.* If X is locally compact, then the homology is the homotopy direct limit

$$H^{lf}(X; J(p)) = \lim_{\rightarrow} H(X, Y; J(p)), \tag{2.7}$$

where Y ranges over the subspaces of X such that the complements $X - Y$ have compact closures. In what follows, the homologies will always be the locally finite one, and we will suppress the decoration lf.

Homology in such generality does not have many of the usual properties. Therefore we need to restrict the class of maps $p : E \to X$. The kind of maps we will consider are stratified systems of fibrations.

Let X be a space, E be a homotopically stratified space, and $p : E \to X$ be a map. A closed subspace $Y \subset X$ is a $p - NDR$ (neighborhood deformation retract) if there is a neighborhood U of Y and a deformation retraction $u : U \times [0, 1] \to X$ of U to Y in X, such that u lifts to a homotopically transverse deformation retraction $\hat{u} : p^{-1}U \times [0, 1] \to E$ of $p^{-1}U$ to $p^{-1}Y$ in E. p is a *stratified system of fibrations* if there is a stratification $X = X_n \supset X_{n-1} \supset \cdots \supset X_0$ such that each X_i is a $p - NDR$ and each $p^{-1}(X_i - X_{i-1})$ is a fibration over $X_i - X_{i-1}$ (with homotopically stratified fibre).

X is called *ANR-stratified* if each strata $X_i - X_{i-1}$ is a locally compact and σ-compact ANR (absolute neighborhood retract). If X is a simplicial complex, then X is *simplicially stratified* if the skeleta are the subcomplexes of X.

Theorem 2.2.1 *Let J be a covariant, spectrum valued (homotopically transversely) homotopy invariant functor on the category of homotopically stratified spaces. Then there is an induced functor $H(X; J(p))$ on the category of ANR-stratified systems of fibrations $p : E \to X$ with the following properties.*

1. *(homotopy)* $H(X; J(p))$ *is invariant under homotopically transverse homotopy equivalences of stratified systems of fibrations.*

2. *(exactness) If* Y *is a closed* $p-NDR$ *subspace of* X*, then there is the restriction* $H(X; J(p)) \to H(X-Y; J(p))$ *such that the sequence*

$$H(Y; J(p)) \xrightarrow{\text{incl}} H(X; J(p)) \xrightarrow{\text{rest}} H(X-Y; J(p)) \tag{2.8}$$

is homotopically a fibration.

3. *(Meyer-Vietoris) If* $X = Y \cup Z$ *is a union of closed* $p-NDR$ *subspaces, then the diagram*

$$\begin{array}{ccc} H(Y \cap Z; J(p)) & \to & H(Y; J(p)) \\ \downarrow & & \downarrow \\ H(Z; J(p)) & \to & H(X; J(p)) \end{array} \tag{2.9}$$

is a homotopy pushout.

4. *(local finiteness) The restrictions define a homotopy equivalence*

$$H(X; J(p)) \simeq \lim_{\leftarrow} H(X-Y; J(p)), \tag{2.10}$$

where Y *runs through closed* $p-NDR$ *subspaces of* X *such that* $X-Y$ *have compact closures.*

Remark 2.2.2 The properties listed in the Theorem are the ones one would expect from a locally finite homology theory. Thus we may call the properties the *homology axioms*. And the functors that satisfy the homology axioms are called the *homology theories*.

Now we discuss the significance of each of the homology axioms.

The homotopy property follows from the homotopy invariance of J. One necessarily needs this for extending the homology via homotopy limits. This is also necessary for practical computation.

The local finiteness follows from the very construction. Combined with the homotopy invariance, the property is used in reducing the homology to simplicially stratified systems of fibrations. The key is that ANRs are homotopy equivalent to inverse limits of simplicial complexes. The homotopy invariance and the local finiteness properties then imply that the homology on ANRs is determined by the homology on finite simplicial complexes.

The exactness and the Meyer-Vietoris properties also follow from the construction. They are homotopy properties preserved under homotopy limits, which reduces the axioms to the case of simplicially stratified systems of fibrations. Then the axioms are the consequences of the initial constructions.

On the other hand, the exactness and the Meyer-Vietoris properties may be viewed as the axioms that make the the homology theory unique. Specifically, the homology on simplicially stratified systems of fibrations $p : E \to X$ is determined by the homology of the trivial projections $F \to pt.$. In fact, as in the case of usual generalized homology theories, the Meyer-Vietoris is equivalent to the suspension equivalence: $H(X; J(p)) \simeq \Omega^k H(X \times \mathbf{R}^k; J(p \times id))$ (see the proof of the Lemma 2.3 in [Q4]). Hence the homologies of the projections $F \times \mathbf{R}^k \to \mathbf{R}^k$ are also determined. Assume for the finite simplicial complex X that the homology $H(X^{k-1}; J(p))$ over the $(k-1)$-skeleton has been determined. Then the exactness property produces the fibration

$$H(X^{k-1}; J(p)) \to H(X^k; J(p)) \to \prod_{\sigma^k} H(int\sigma^k; J(p)).$$

Note that over the interior of the k-simplex σ^k the projection p is homotopy equivalent to $F \times \mathbf{R}^k \to \mathbf{R}^k \cong int\sigma^k$, whose homology has been determined. Hence the homology over the k-skeleton is determined.

The discussion may be summerized in the following Lemma.

Lemma 2.2.3 *Suppose that there is a natural transformation between two homology theories over ANR-stratified systems of fibrations. If the transformation is a homotopy equivalence for the systems $F \to pt.$ over a point, then the transformation is a homotopy equivalence of the two functors.*

The uniqueness implied by the homology axioms may be viewed from another perspective.

We observe that like the integration of functions, the construction of the homology $H(X;J(p))$ is literally adding pieces $J(p^{-1}\sigma)$ of the functor J together. Each piece may be mapped into $J(E)$ via the inclusion. If τ is a face of σ, then the map $J(p^{-1}\tau) \to J(E)$ is equal to the composition $J(p^{-1}\tau) \to J(p^{-1}\sigma) \to J(E)$. Therefore we obtain a map (called the *assembly map*):

$$\alpha : H(X;J(p)) \to J(E) \tag{2.11}$$

from the maps $J(p^{-1}\sigma_0) \times \sigma \to J(p^{-1}\sigma_0) \to J(E)$. Then we may go through the simplicial approximations and the local finiteness manipulation to define the assembly map for ANR-stratified systems of fibrations. We may view the assembly map as a natural transformation of spectrum-valued functors.

Note that the homology $H(pt.;J(F \to pt.))$ is simply $J(F)$. Hence the assembly is a homotopy equivalence for systems over a point. As a consequence of the Lemma 2.5, we have the following result. In [Q2], this is called the *characterization theorem.*

Lemma 2.2.4 *Let J be a homotopy invariant functor over the category of homotopically stratified spaces. Then there is a natural transformation $\alpha : H(X;J(p)) \to J(E)$. If J satisfies the properties similar to the homology axioms (see the remark), then α is a homotopy equivalence.*

Remark 2.2.5 By the properties similar to the homology axioms we means the axioms obtained after the following change:

1. stratified systems replaced by homotopically stratified spaces;

2. $p - NDR$ subspaces replaced by homotopically transverse subspaces.

We certainly may also call these the homology axioms, although for different kind of functors.

For the proof of the Lemma we note that J may be viewed as a homology theory over stratified systems by composing J with the forgetful functor $(p : E \to X) \mapsto E$.

As far as the computation of the homology is concerned, we define

$$H_k(X; J(p)) = \pi_k H(X; J(p)). \tag{2.12}$$

Let X_{n-1} be the top but one skeleton in the stratified system. Then we have a fibration

$$H(X_{n-1}; J(p)) \to H(X; J(p)) \to H(X - X_{n-1}; J(p)). \tag{2.13}$$

This produces a long exact sequence of groups

$$\cdots \to H_k(X_{n-1}; J(p)) \to H_k(X; J(p)) \to$$

$$\to H_k(X - X_{n-1}; J(p)) \to H_{k-1}(X_{n-1}; J(p)) \to \cdots. \tag{2.14}$$

Because X_{n-1} has fewer strata than X, the homology is simpler. Thus by induction, the computation is reduced to the homologies over $X - X_{n-1}$, over which p is a fibration.

Let $p : E \to X$ be a fibration of homotopically stratified spaces. Then as in the usual homology theory, the homology $H_*(X; J(p))$ may be computed by a spectral sequence

$$E^2_{ij} = H_i(X; \pi_j J(F)) \Rightarrow H_{i+j}(X; J(p)), \tag{2.15}$$

where F is the fibre of p.

Suppose that Y is a subspace of X such that the inclusion is k-connected. Then $H_i(X, Y; \pi_j J(F)) = 0$ for $i \leq k$. Assume that there is a fixed integer j_0 such that $\pi_j J(F) = 0$ for any $j < j_0$. Then in case $i + j \leq k + j_0$, we have either $i \leq k$ or $j < j_0$. Hence the E^2-term of the spectral sequence for $H_*(X, Y; J(p))$ vanishes at dimensions $\leq k + j_0$. Consequently,

$$H_i(X, Y; J(p)) = 0 \quad \text{for } i \leq k + j_0.$$

Lemma 2.2.6 *Suppose that $p : E \to X$ is a stratified system of fibrations with fibre being homotopically stratified spaces. Suppose that the stratification of X is homotopically stratified and Y is a closed homotopically transverse subspace such that the inclusions $Y_i - Y_{i-1} \subset X_i - X_{i-1}$ of strata are all k-connected. Suppose that J is* *a homotopically invariant functor such that* $\pi_j J(F) = 0$ *for any* $j < j_0$ *and the fibre* F *of* p*. Then*

$$H_i(Y; J(p)) \to H_i(X; J(p))$$

is isomorphic for $i < k + j_0$ *and epimorphic for* $i = k + j_0$*.*

Remark 2.2.7 In [St], the condition on $J(F)$ in the Theorem is called the *vanishing condition* at dimensions $\geq 1 - j_0$.

Proof: We have seen the nonstratified case. The stratified case is by induction. By the exact sequence (14), it suffices to show that $H(Y - Y_{n-1}; J(p)) \to H(X - X_{n-1}; J(p))$ is $(k + j_0)$-connected. Let $L = hkl(X, X_{n-1})$. Then L is a stratified system of fibrations over X_{n-1} and after a homotopy we may assume that X_{n-1} has a neighborhood homeomorphic to the mapping cylinder of $L \to X_{n-1}$ (the Lemma 4.3 of [Q5]). Let $\bar{X}$ be the closure of the complement of the neighborhood. Then $H_*(X - X_{n-1}; J(p)) = H_*(\bar{X}, L; J(p))$ and similarly $H_*(Y - Y_{n-1}; J(p)) = H_*(\bar{Y}, L_Y; J(p))$ for Y. L_Y used to be the homotopy link of Y_{n-1} in Y. By the homotopy transversality of Y in X, we may take L_Y to be the restriction of $L \to X_{n-1}$ over Y_{n-1}. Since p is a fibration over $\bar{X}$, all we need is that $\bar{Y} \subset \bar{X}$ and $L_Y \subset L$ are k-connected. The first is the assumption. It remains to show that $L_Y \subset L$ is k-connected.

L is obtained by glueing fibrations over the strata in X_{n-1} together. Therefore the k-connectivity of $L_Y \to L$ is a consequence of the following two assertions.

1. If $p : E \to X$ is a fibration and the inclusion $Y \subset X$ is k-connected, then the inclusion $p^{-1}Y \subset E$ is k-connected;

2. If $X = X_1 \cup X_2$ is a union of closed subspaces, and Y is a subspace such that $Y \cap X_1 \subset X_1$, $Y \cap X_2 \subset X_2$, and $Y \cap X_1 \cap X_2 \subset X_1 \cap X_2$ are all k-connected, then $Y \subset X$ is k-connected.

The first assertion follows from the exact sequence of homotopy groups. In the second situation, one may use Van Kampen to take care of the fundamental groups. Then the homotopy connectivity is equivalent to the homological connectivity. The assertion thus follows from the Meyer-Vietoris sequence.

If we only require that the maps in the definition of stratified systems of fibrations to be stratified instead of homotopically transverse, then we have the definition for the maps $q : R \to X$ of stratified spaces to be stratified systems of fibrations. Let E be a homotopically stratified space and $p : E \to R$ be a stratified system of fibrations in our usual sense. Then the restriction p_F of p on any fibre F of q is a stratified system of fibrations. In fact with fancier language, we may call $p : E \to R$ a stratified system of fibrations with fibres being (usual) stratified systems of fibrations.

The next result is called Fubini Theorem since it is similar to the Fubini Theorem in the calculus.

Lemma 2.2.8 *Suppose that X and R are ANR-stratified spaces, and E is a homotopically stratified space. Suppose that $p : E \to R$ and $q : R \to X$ are stratified systems of fibrations in the respective category (see the explanation before). Then*

$$H(R; J(p)) \simeq H(X; H(F; J(p|_F))), \tag{2.16}$$

where F is the fibre of q.

The proof follows from the construction. In case that the spaces X and J are simplicial and the restriction of q on strata are block bundles, the super pushout construction for $J(p)$ may be done first along the fibres of q and then assembled along the base X. Thus we get $J(p) = J(q_* : J(p|_F) \to X)$. This implies that $H(R; J(p)) \simeq H(X; H(F; J(p|_F)))$ in the simplicial case. The general case follows from the extension via the homotopy and local finiteness axioms.

The behavior of the assembly map under the Fubini Theorem is as expected. We first have the assembly

$$\alpha_F : H(F; J(p|_F)) \to J(p^{-1}F) \tag{2.17}$$

along the fibres. We may view α_F as a natural transformation of homotopy invariant functors and apply the homology functor to it. The result is called the *partial assembly map* (of the homology of p)

$$\beta = \alpha_{F*} : H(X; H(F; J(p|_F))) \to H(X; J(qp)) \tag{2.18}$$

where we use the fact that $p^{-1}F$ is the fibre of qp and therefore $H(X; J(p^{-1}F)) = H(X; J(qp))$.

Now the claim is that under the Fubini Theorem, the total assembly

$$H(E; J(p)) \to J(E) \tag{2.19}$$

is equal to the composition of the partial assembly β and the assembly

$$H(X; J(qp)) \to J(E) \tag{2.20}$$

for the homology of the stratified system $qp : E \to X$ of fibrations. Again this follows from the very construction.

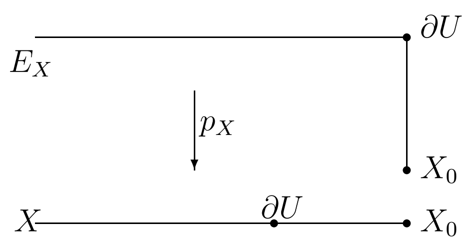
Finally, we may view $H(X; J(p))$ as a functor of J and obtain

Lemma 2.2.9 *If $J' \to J \to J''$ are natural transformations of homotopy invariant functors over homotopically stratified spaces, such that for any F, $J'(F) \to J(F) \to J''(F)$ is a fibration. Then the induced $H(X; J'(p)) \to H(X; J(p)) \to H(X; J''(p))$ is a fibration.*

The reason is that homotopy direct limits preserve fibrations.

We conclude this section by defining the homology $H(X; J(locX))$ we will mostly use.

Let X be a homotopically stratified space. Then we may associate to X a stratified system of fibrations $p_X : E_X \to X$. The idea is to take $E_X = X$ and define p_X as the map that shrinks the mapping cylinder like neighborhoods of strata to the strata. In the case of two strata, this is illustrated by the following picture.



With the stratification of X, p_X is a stratified system of fibrations with fibre $p_X^{-1}(x) =$ the local homotopically stratified structure of X at x.

Strictly speaking, the map p_X may be constructed only if we have true mapping cylinder neighborhoods. Hence we quote the Pullback Lemma 3.7 of [Q5], or rather the proof of it, to find a homotopically stratified space Y and a homotopically transverse equivalence $Y \to X$, such that Y_i has a neighborhood U_i in Y_{i+1} that is homeomorphic to the mapping cylinder of a stratified system $\partial U_i \to Y_i$ of fibrations. In particular, we may write $Y = Y_{n-1} \cup \partial U_{n-1} \times [0,1] \cup \bar{Y}$. By induction, we define $p_0 : Y_0 \to Y_0$ to be the identity. Assume that at the last but one stage, a shrinking deformation $p_{n-1}^t : Y_{n-1} \to Y_{n-1}$ ($t \in [0,1]$) has been defined, with the initial case to be the trivial deformation. Then the deformation is lifted to a deformation $p_{n-1}^{*t} : \partial U_{n-1} \to \partial U_{n-1}$. The new shrinking deformation $p_n^t : Y_n \to Y_n$ is then defined to be the combination of

1. the identity on $\bar{Y}$;

2. the deformation $(y, s) \mapsto (p_{n-1}^{*(1-s)t}(y), \theta^t(s))$ on $\partial U_{n-1} \times [0,1]$, where θ^t is affine on the intervals $[0, 1/2]$ and $[1/2, 1]$ and shrinks the former to 0 while expands the later to $[0,1]$;

3. the deformation p_{n-1}^t on Y_{n-1}.

Now we are able to define $p_Y : E_Y \to Y$ as the map $p_n^1 : Y_n \to Y_n$.

The homology $H(X; J(locX))$ is then defined as the homology $H(Y; J(p_Y))$. Since the space Y may be explicitly constructed from X, with the choices involved also being naturally explicit, this is well defined. The assembly map takes the form

$$\alpha : H(X; J(locX)) = H(X; J(p_X)) \to J(E_X) = J(X).$$

CHAPTER 3

HOMOTOPICALLY STRATIFIED SURGERY

This chapter is an explanation of the work of Weinberger on the surgery theory of homotopically stratified spaces. The theory has been outlined in the expository paper [We]. One may find many related topics there. For our purpose, we only try to outline those results essential to the proof of the periodicity. And in case of surgery theory, we will be as detailed as possible.

Surgery is the technique that computes the homeomorphism classes of manifolds that are homotopically equivalent to a given one. The conditions satisfied by homotopically stratified spaces are not strong enough for a surgery theory. However, by crossing with $\mathbf{R}^k$ for large enough k, a homotopically stratified space becomes a controlled geometrically stratified space, which has a nicer neighborhood that enables us to construct a surgery theory. The information about the original space is then recovered from a destablization process.

3.1. Controlled Geometrically Stratified Spaces

A space X is $\mathbf{R}^k$-*controlled* if it is equipped with a map $\phi : X \to \mathbf{R}^k$. A map $f : X \to Y$ of $\mathbf{R}^k$-controlled spaces is bounded if there is a constant B such that

$$|\phi_Y f(x) - \phi_X x| < B.$$

Then one may define controlled homotopies and do some algebraic topology. There is also a bounded K-theory [AM]. Therefore it makes sense to talk about controlled simple homotopy equivalences.

If X is a simplicial complex, then the notion of $\mathbf{R}^k$-*controlled block bundle* over X may be defined in a way similar to [RS], with additional $\mathbf{R}^k$-control imposed on everything. Specifically, there is an $\mathbf{R}^k$-controlled map $p : E \to X$ and an $\mathbf{R}^k$-controlled space F such that for each simplex σ of X, the *block*

$$E_\sigma = p^{-1}\sigma \cong_b F \times \sigma,$$

where $\cong_b$ means $\mathbf{R}^k$-bounded homeomorphism. F is called the *fibre* of the bundle.

An $\mathbf{R}^k$-controlled stratified space X is *$\mathbf{R}^k$-controlled geometrically stratified* if each strata $X_i - X_{i-1}$ has a neighborhood U_i that is the mapping cylinder of an $\mathbf{R}^k$-controlled block bundle, and the bundles are compatible where several strata meet together. One may simply copy the definition in [BQ], with $\mathbf{R}^k$-control imposed on everything. If the strata $X_i - X_{i-1}$ are controlled topological manifolds, then we call X a *manifold controlled geometrically stratified space.*

The kind of maps between controlled geometrically stratified spaces we are interested in are not the one as defined in [BQ]. Instead of stratified maps that restricts to blockwise bounded homeomorphisms on the controlled block bundle neighborhoods, we only require the restrictions to be bounded simple homotopy equivalences. We call such maps *(controlled) simple homotopically transverse.*

The motivation for introducing the above notions is the following result.

Lemma 3.1.1 *Let X be a manifold homotopically stratified space. Then for big enough k that only depends on the dimensions of the strata of X, $X \times \mathbf{R}^k$ is a manifold $\mathbf{R}^k$-controlled geometrically stratified space, with the control being the projection on $\mathbf{R}^k$. Moreover, homotopically transverse maps become simple homotopically transverse maps.*

This is a consequence of the Theorem 2.2.4 of [Q5]. The key is that the existence of block bundle neighborhoods for strata is obstructed by a bunch of K-theoretical invariants. These may be killed by crossing X with a torus T^k with large enough k. By taking the universal covering of the torus, we obtain the $\mathbf{R}^k$-controlled block bundle structure.

We note that in a homotopically stratified space. The strata are only assumed to be topological manifolds. Strictly speaking, therefore, the blocks we are talking about should be replaced with topological handle structures. However, we could be content with the arguement in which we pretend to have simplicial structures.

We may summerize our strategy by distinguishing three categories:

1. manifold homotopically stratified spaces with homotopically transverse maps (which means fibrewise homotopy equivalence on neighborhoods of strata);
2. manifold $\mathbf{R}^k$-controlled geometrically stratified spaces with $\mathbf{R}^k$-bounded simple homotopically stratified maps (which means blockwise controlled simple homotopy equivalence on neighborhoods of strata);
3. manifold geometrically stratified spaces with geometrically stratified maps (which means blockwise homeomorphism on neighborhoods of strata).

The Lemma 3.1 provides a functor (called *stablization*) from the first category to the second for large enough k. The third category is the subject of [BQ], where a surgery theory has been established. It turns out that a surgery theory on the second category also exists, and there is a *destablization* process that recovers the first category from the second.

3.2. Induction on Stratified Spaces

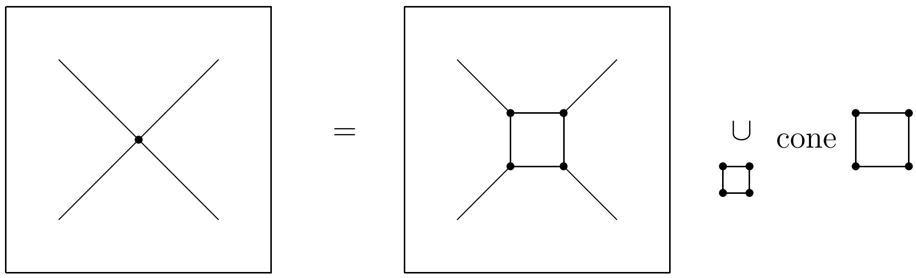
Stratified spaces may be manipulated inductively, similar to the way geometrically stratified spaces are handled in [BQ].

In fact, in the context of geometrically stratified spaces, there are two ways of doing the induction. First, consider the smallest skeleton X_0. It has a neighborhood U in X that is a the mapping cylinder of a block bundle $\partial U \to X_0$. Moreover, there is a filtration

$$\partial U \supset \partial U \cap X_{n-1} \supset \cdots \supset \partial U \cap X_1 \tag{3.1}$$

of block subbundles of $\partial U \to X_0$. The stratified structure near X_0 is the mapping cylinder of the filtration. Let F be the fibre of $\partial U \to X_0$. Then the filtration turns F

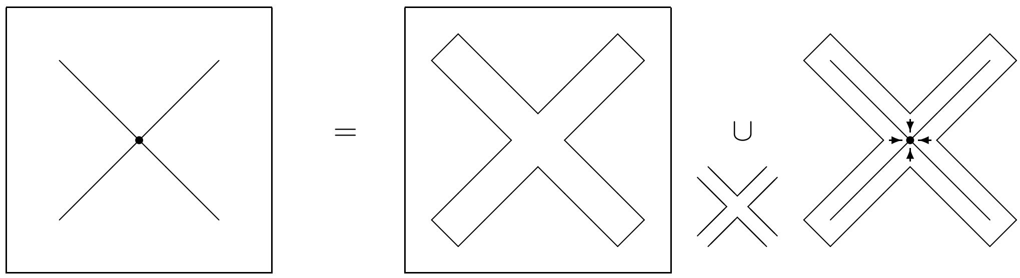
into a geometrically stratified space with $(n-1)$ strata and therefore is simpler than X. And the neighborhood of X_0 is a block bundle with fibre $coneF$.



Consider the case that X is a manifold geometrically stratified space. Let $\bar{X} = X - intU$. Then $\bar{X}$ is a manifold geometrically stratified space with boundary ∂U and fewer strata than X. And $X = \bar{X} \cup U$ with $\bar{X} \cap U = \partial U$. The inductive point is that $(\bar{X}, \partial U)$ is simpler than X and U is the mapping cylinder of a blocked bundle $\partial U \to X_0$ with fibre simpler than X.

The decomposition may be also done word for word in the case of $\mathbf{R}^k$-controlled geometrically stratified spaces. In case of homotopically stratified spaces, the decomposition is valid with the following change: X is homotopically transversely equivalent to $\bar{X} \cap U$ rather than equal to it; $\partial U \to X_0$ is a fibration with fibre being a homotopically stratified space simpler than X. In fact we may take $\partial U = hlk(X, X_0)$.

The other way of doing induction is to consider the top but one skeleton X_{n-1}, which is a geometically stratified space with $(n-1)$ strata. It however does not have a block bundle neighborhood in X, since over different strata of X_{n-1} the fibre may be different. Instead we have a *geometrically stratified system of bundles* (as apposed to stratified system of fibrations) $\partial U \to X_{n-1}$, and a neighborhood of X_{n-1} in X is the mapping cylinder of the system of bundles. More specifically, the system consists of block bundles $\partial U_i \to X_i - X_{i-1}$ over the strata of X_{n-1} that are compatible where several strata meet. The whole bundle is then the union of the block bundles along block bundles (=the intersection of block bundles).



Again let X be a manifold geometrically stratified space. Let $\bar{X} = X - intU$. Then $\bar{X}$ is a topological manifold with boundary ∂U, and ∂U is a geometrically stratified system of block bundles over X_{n-1}, and U is the mapping cone of the system. Finally, $X = \bar{X} \cup U$ and $\bar{X} \cap U = \partial U$. The inductive point is that X_{n-1} is a manifold geometrically stratified space simpler than X.

The process can be generalized to the case of $\mathbf{R}^k$-controlled geometrically stratified spaces, simply by putting the control on everything. It can also be applied to homotopically stratified spaces up to homotopy. Thus we may take $\partial U = hlk(X, X_{n-1})$, which is a stratified system of fibrations over X_{n-1}, and X is homotopically transversely equivalent to $\bar{X} \cup U$.

3.3. Surgery on Controlled Geometrically Stratified Spaces

The surgery theory for manifold $\mathbf{R}^k$-controlled geometrically stratified spaces will be constructed inductively over strata. Before we do the construction, we outline the classical surgery theory for compact manifolds.

Let X be a manifold. To compute the structure set $S(X, \text{rel } \partial X)$ of the homeomorphism classes of manifolds simple homotopy equivalent to X relative to the boundary of X, we construct the space of *simple structures* of X rel ∂X, still denoted $S(X, \text{rel } \partial X)$, as the simplicial complex in which an n-simplex is a simple homotopy equivalence $f : Y \to (X, \partial X) \times (\Delta^n; \partial_i \Delta^n, 0 \leq i \leq n)$ of $(n+2)$-ads that restricts to a homeomorphism of $(n+2)$-ads on $\partial X \times \Delta^n$. The boundaries of the simplex is just the restriction. By the s-cobordism theorem, a manifold simple homotopy equivalent to $(Y \times [0,1]; Y \times 0, Y \times 1)$ is homeomorphic to a product with $[0,1]$. This shows that the structure set is the 0-th homotopy group of the structure space.

The computation of the structure space is accomplished by viewing it as the homotopy fibre of the surgery obstruction map from the space of normal invariants to the space of surgery obstructions. These gadgets are defined after surgery problems.

A surgery problem to the manifold X is a bundle map

$$\begin{array}{ccc} \nu_Y & \xrightarrow{b} & \xi \\ \downarrow & & \downarrow \\ Y & \xrightarrow{f} & X \end{array} \tag{3.2}$$

in which Y is a manifold, f is a map of degree 1, ν_Y is the normal bundle of Y, and ξ is a bundle over X. The type of bundles is determined by the type of manifolds under consideration. The problem is relative to the boundary of the manifolds if the restriction of f on the boundary is a homeomorphism (of that type).

Now the space $N(X, \mathrm{rel}\ \partial X)$ of normal invariants is a simplicial complex in which an n-simplex is a surgery problem $(Y, \partial Y) \to (X, \partial X) \times (\Delta^n; \partial_i \Delta^n, 0 \le i \le n)$ of $(n+2)$-ads of pairs relative to $\partial X \times \Delta^n$. An extraordinary fact about the normal invariants is that it may be identifed with the space $Maps(X/\partial X, F/Top)$ of maps from X to a universal space F/Top that is trivial on the boundary. A reference for this assertion is the chapter 1 of [B].

The space $L(X, \mathrm{rel}\ \partial X)$ of surgery obstructions is constructed in a similar way. It is a simplicial complex in which an n-simplex is a surgery problem $(Z, \partial Z) \to (Y, \partial Y)$ of pairs of $(n+2)$-ads with a reference map $Y \to X \times (\Delta^n; \partial_i \Delta^n, 0 \le i \le n)$ such that $\partial Z \to \partial Y$ is a simple homotopy equivalence and the reference map preserves the orientation class $\pi_1 \to \{\pm 1\}$ (=first Stiefel-Whitney class).

Clearly, the definition of L can also be made with any space X equipped with an orientation class $\pi_1 X \to \{\pm\}$. However, in case X is a manifold, it is implicit in the definition of $L(X)$ that an n-simplex is a surgery problem of dimension $n + dimX$ with a reference to X. In case X is any space, the dimension is ambigious. If we ask the dimension of the n-simplices in $L(X)$ to be of dimension $n+k$, then the resulting surgery obstruction space is denoted $L_k(X)$. In particular, $L(X) = L_{dimX}(X)$ for a manifold X. In case the dimensions involved are clear from the context, we will simply use $L(X)$ without dimensional subscript.

A less obvious fact about L is that homotopy class of the space of surgery obstructions only depends on the fundamental group of X and the associated orientation class $\omega : \pi_1 X \to \{\pm 1\}$.

According to classical periodicity in the chapter 9 of [Wa], the map $\times \mathbf{CP}^2 : L_n(X) \to L_{n+4}(X \times \mathbf{CP}^2)$ obtained by crossing surgery problems with $\mathbf{CP}^2$ is a homotopy equivalence. As in [KS], this implies that $L_n(X) \simeq \Omega^4 L_n(X)$. Hence the surgery obstruction spaces fit into a connective Ω-spectrum. The spectrum is still denoted L.

The definitions may be extended to pairs of manifolds and more generally n-ads of manifolds.

The surgery theory claims that if $dimX \geq 5$, then the forgetful maps induce a fibration (called the *surgery fibration*)

$$S(X, \mathrm{rel}\ \partial X) \to N(X, \mathrm{rel}\ \partial X) \to L(X) \tag{3.3}$$

up to homotopy. The fibration also holds in the general case of n-ads, provided that the smallest component in the n-ad has dimension ≥ 5. The key to the proof is the π-π theorem (the Theorem 3.3 of [Wa]). It fills the gap between the structures (which are basically homotopy equivalences) and the surgery problems (which are basically cobordism equivalences). The proof that the π-π theorem implies the fibration is essentially in the chapter 9 of [Wa]. The translation into the spacified setting was provided by Quinn in [Q1].

If we apply the surgery fibration to the sphere S^n, then the solution to the high dimensional Poincaré conjecture implies that F/Top and $L(pt.)$ have the same high dimensional (meaning ≥ 5) homotopy group. Then Siebenmann showed in [KS] that in fact $L(pt.) \simeq F/Top \times \mathbf{Z}$. Note that this remarkable fact is not true if Top (representing the topological category) is replaced by PL or O (representing the PL or smooth category). We will see this has a fundamental impact on our work.

The equivalence enables us to replace F/Top by $L(pt.)$ in the normal invariants $N(X, \text{rel } \partial X) = Maps(X/\partial X, F/Top)$ up to a factor of $\mathbf{Z}$. Hence the normal invariants may be viewed as the cohomology of the pair $(X, \partial X)$ with the coefficient $L(pt.)$. On the other hand, Ranicki [R] showed that $L(pt.)$ is a module over the topological Thom spectrum $MTop$. Since any topological manifold is automatically $MTop$-oriented, it is also $L(pt.)$-oriented. Thus we have the Poincaré duality for $L(pt.)$. By viewing the normal invariants as the cohomology with coefficient $L(pt.)$, we may convert the normal invariants into a homology

$$N(X, \text{rel } \partial X) = H(X; L(pt.)). \tag{3.4}$$

and obtain the surgery fibration for topological manifolds

$$S(X, \text{rel } \partial X) \to H(X; L(pt.)) \xrightarrow{\alpha} L(X, \text{rel } \partial X.) \tag{3.5}$$

It turns out that α is the assembly map. The reference for the orientability and the Poincaré duality in the general setting is the chapter 14 of [Sw].

Before we construct the surgery theory, we note that the classical surgery theory may be generalized to $\mathbf{R}^k$-controlled manifolds. A reference on controlled surgery is [FP].

Let X be a manifold $\mathbf{R}^k$-controlled geometrically stratified space. A simple structure of X is a simple homotopically stratified homotopy equivalence $f : Y \to X$ of manifold $\mathbf{R}^k$-controlled geometrically stratified spaces. Then as in the classial case we may define the space $S(X)$ of simple structures, and in case X has a boundary ∂X, the space $S(X, \text{rel } \partial X)$ of simple structures rel boundary. The 0th fundamental group of the space consists of homeomorphism classes of simple structures of X.

Then we need to define the surgery obstruction space L. It turns out that the surgery obstruction space of Browder and Quinn is sufficient for our purpose.

In [BQ], the geometrically stratified surgery problems in which maps are geometrically transverse are studied. We recall that a map of geometrically stratified spaces is geometrically transverse if it restricts to blockwise homeomorphisms on the block bundle neighborhoods of strata, as opposed to a simple homotopically transverse map in which only blockwise simple homotopy equivalence is required. Then for a geometrically stratified space X equipped with compatible orientation classes of strata we may construct the surgery obstruction space $L(X)$ using such surgery problems.

Lemma 3.3.1 *Let Y be a closed union of strata of X. Then the inclusion and the restriction induce a genuine fibration*

$$L(X-Y) \xrightarrow{\text{incl}} L(X) \xrightarrow{\text{rest}} L(Y) \tag{3.6}$$

We note that by $L(X-Y)$, we actually mean the surgery obstruction space of the complement of an open bundle neighborhood of Y. The complement is compact if X is.

In short, the Lemma says that the surgery obstruction space decomposes along the strata. In the proof of the periodicity of L, the fact that this is a genuine fibration is used.

The proof of the Lemma uses the pullback of surgery problems. Consider a classical surgery problem:

$$\begin{array}{ccc} \nu_Y & \xrightarrow{b} & \xi \\ \downarrow & & \downarrow \\ Y & \xrightarrow{f} & X. \end{array}$$

in which Y and X are manifolds. Let $p : E \to X$ be a block bundle with manifold fibre. Then we may pull f back to form a map $f^* : E' = f^*E \to E$ of degree 1, and the pull b back to form the bundle map $b^* : \nu_Y^* = (f^*p)^*\nu_Y \to \xi^* = p^*\xi$, where $f^*p : E' \to Y$ is the pullback of the projection map p along f. However, (f^*, b^*) does not form a surgery problem since ν_Y^* is not the normal bundle of the G-manifold E'. In fact, they differ by a *topological connection*. There is a unique bundle ν_p on E constructed by assembling the normal bundles of the fibres of $p : E \to X$ together, such that there is an isotopically unique bundle isomorphism $\nu_E \cong \nu_X^* \oplus \nu_p$. The construction is natural to the pullback and we have a bundle map $c : \nu_{f^*p} \to \nu_p$. Now

$$\begin{array}{ccc} \nu_Y^* \oplus \nu_{f^*p} & \xrightarrow{b^*\oplus c} & \xi \oplus \nu_p \\ \downarrow & & \downarrow \\ E' & \xrightarrow{f^*} & E \end{array}$$

is a surgery problem with the correct bundles. This is then defined as the pullback of the surgery problem. Consequently, we have the pullback

$$p_* : L(X) \to L(E),$$

called the *transfer* of surgery obstructions.

The transfer operation is natural.

Lemma 3.3.2 *Suppose that $p : E \to X$ and $q : K \to E$ are block bundles with manifold fibres. Then the transfer $(pq)_* : L(X) \to L(K)$ is the composition of the transfers $p_* : L(X) \to L(E)$ and $q_* : L(E) \to L(K)$.*

Since the pullback of bundles along a composition is the composition of pullbacks, the Lemma is clear at the level of maps of manifolds. Then we need to consider the covering map of normal bundles. This follows from $\nu_{pq} = \nu_q \oplus \nu_p^*$, where ν_p^* is the pullback of ν_p along q.

We may extend the transfer to the case that the fibre of p is a manifold geometrically stratified space. The construction is virtually word for word, with a little bit more care to the normal data.

Even more generally, suppose that X is a geometrically stratified space, and $p : E \to X$ is a compatible system of block bundles with fibres being manifold geometrically stratified spaces. Then E is a manifold geometrically stratified space and we have the transfer $p_* : L(X) \to L(E)$. The construction is made on each strata and then glued together. The glueing is possible by the Lemma 3.3, or rather the proof of it.

Proof of the Lemma 3.2: We need to consider the diagram

$$\begin{array}{ccc} K & \xrightarrow{\lambda} & L(X) \\ \downarrow i_0 & & \downarrow \\ K \times [0,1] & \xrightarrow{\mu} & L(Y) \end{array}$$

for which we would like to find a lifting $K \times [0,1] \to L(X)$ of μ that extends λ. In what follows, we will omit the normal data in surgery problems, which should be clear from the context and the transfer construction.

First we look at the special situation that K is a point. Then λ is a surgery problem $Z \to W$ with a reference $W \to X$, and μ is a surgery problem $P \to Q$ of triads with a reference $(Q; Q_0, Q_1) \to Y \times ([0,1]; 0, 1)$. The commutativity of the diagram means that $P_0 \to Q_0 \to Y$ is equal to the restriction $Z_Y \to W_Y \to Y$. A neighborhood of Y in X is the mapping cylinder of a (geometrically) stratified system of block bundles $p : Y^* \to Y$. If we always use $*$ to denote the pullbacks, then the surgery problem λ decomposes into

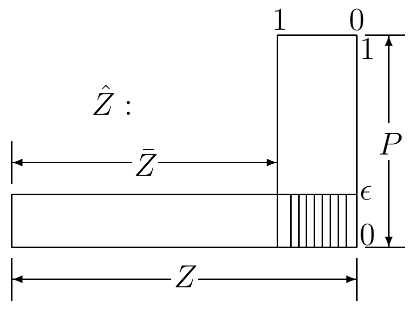
$$Z = \bar{Z} \cup Z_Y^* \times [0,1] \cup Z_Y$$

$$\to W = \bar{W} \cup W_Y^* \times [0,1] \cup W_Y.$$

Then we may assume that $P \to Q$ is a product $(P_0 \to Q_0) \times [0, \epsilon]$ over $[0, \epsilon]$ and construct a surgery problem

$$\hat{Z} = Z \times [0, \epsilon] \cup (P^* \times [0,1] \cup P)$$

$$\to \hat{W} = W \times [0, \epsilon] \cup (Q^* \times [0,1] \cup Q),$$



where the stripped area is the identification of $(Z_Y^* \times [0,1] \cup Z_Y) \times [0, \epsilon]$ in $Z \times [0, \epsilon]$ is with $(P_0^* \times [0,1] \cup P_0) \times [0, \epsilon]$ in $P^* \times [0,1] \cup P$, and similarly for W and Q. As for the reference map, we may manage to map $W \times 0$ to $X \times 0$ as the origional $W \to X$, map Q to $Y \times [0,1]$ as the origional $Q \to Y \times [0,1]$ and extend this to a map $\hat{W} \to X \times [0,1]$ in such a way that $(\bar{W} \times \epsilon) \cup (Q^* - Q_0^* \times [0, \epsilon)) \times 1$ is mapped to $X \times 1$. The map may be derived from the origional $\bar{W} \to \bar{X}$. The surgery problem combined with the reference is a lifting of μ that extends λ. This completes the construction in case $A = pt.$.

In general, we have a compatible collections of surgery problems, one for each simplex of A. The construction for the case of one point may be done for each of them and they combined to produce the lifting.

A consequence of the Lemma 3.2 is that the operation of crossing with $\mathbf{CP}^2$ induces the 4 fold periodicity $L(X) \simeq \Omega^4 L(X)$ of the stratified surgery obstruction spaces. The proof is an inductive 5-lemma type arguement (see [BQ]). Hence like the classical case, the surgery obstruction spaces fit into a connective Ω-spectrum. The spectrum is still denoted L. Moreover, the classical fact that L is an $MTop$-module is still valid in the stratified case. In particular, topological manifolds are L-oriented.

We are now in the position to write down the surgery fibration.

Theorem 3.3.3 *Suppose that X is a manifold $\mathbf{R}^k$-controlled geometrically stratified space whose smallest strata has dimension ≥ 5. Then there is a natural fibration*

$$S(X; \text{rel } \partial X) \to H(X; L(locX)) \xrightarrow{\alpha} L(X). \tag{3.7}$$

Proof: We prove this by induction. The case of only one stratum is the classical controlled surgery theory. Since the local structure of a manifold is always a point, the fibration is the usual surgery fibration.

Now suppose we have the surgery fibration as stated in the Theorem for controlled geometrically stratified spaces with $(n-1)$ strata. Let X have n strata. Then we adapt the notations of the first decomposition in the section 3.2.

We construct the following diagram

$$\begin{array}{ccccc} S(X; \text{rel } X_0) & \to & S(X) & \to & S(X_0) \\ \downarrow & & \downarrow & & \downarrow \\ H(X; L(loc(X - X_0))) & \to & H(X; L(locX)) & \to & H(X_0; L(locX_0)) \\ \downarrow & & \downarrow & & \downarrow \\ L(X - X_0) & \to & L(X) & \to & L(X_0) \end{array} \tag{3.8}$$

In the upper row, the restriction map $S(X) \to S(X_0)$ is a fibration, and the fibre consists of simple structures that is homeomorphic on X_0. In the middle row, we have a fibration $L(loc(X - X_0)) \to L(locX) \to L(locX_0)$ by the Lemma 3.2. By the Theorem 2.11 we have a fibration $H(X; L(loc(X - X_0))) \to H(X; L(locX)) \to H(X; L(locX_0))$. Since $L(locX_0)$ is concentrated over X_0, the excision shows that $H(X; L(locX_0)) = H(X_0; L(locX_0))$. Hence the middle row is also a fibration. Again by the Lemma 3.2, the last row is a fibration.

The right column is the surgery fibration for the manifold X_0. Hence to show that the middle column is a fibration, as asserted by the Theorem, it suffices to show that the left column is a fibration.

With the notation in the section 3.2, $(\bar{X}, \partial U)$ is a manifold controlled geometrically stratified space. By the very meaning of the structure space, we have the pullback diagram

$$\begin{array}{ccc} S(X, \text{rel } X_0) & \to & S(\bar{X}, \partial U) \\ \downarrow & & \downarrow \\ S(\partial U \to X_0, \text{rel } X_0) & \to & S(\partial U) \end{array} \tag{3.9}$$

where $S(\partial U \to X_0, \text{rel } X_0)$ consists of the block bundles $p : E \to X_0$ that are blockwise simple homotopy equivalent to $q : \partial U \to X_0$ over id_{X_0}. The fibre F of q is a manifold controlled geometrically stratified space with $(n-1)$ strata and satisfies the conditions on the dimensions. Therefore the inductive hypothesis may be applied to produce the surgery fibration for F. Moreover, the inductive hypothesis also applies to $(\bar{X}, \partial U)$ and ∂U.

For any simplex σ of X_0, the simple homotopy equivalence $p^{-1}\sigma \simeq q^{-1}\sigma \cong F \times \sigma$ is a simplex in the structure space of F. These simplices are compatible and produces a simplicial map $X_0 \to S(F)$. Conversely, any simplicial map $X_0 \to S(F)$ may be assembled to produce a bundle $p : E \to X_0$ blockwise simple homotopy equivalent to $q : \partial U \to X_0$ over id. Consequently,

$$S(\partial U \to X_0, \text{rel } X_0) = Maps(X_0, S(F)). \tag{3.10}$$

Now consider the diagram

$$\begin{array}{ccccc} Maps(X_0, S(F)) & \to & Maps(X_0, H(F; L(locF))) & \to & Maps(X_0, L(F)) \\ \downarrow & & \downarrow & & \downarrow \\ S(\partial U) & \to & H(\partial U; L(loc(\partial U)) & \to & L(\partial U) \end{array} \tag{3.11}$$

The upper row is a homotopy fibration because of the surgery fibration $S(F) \to H(F; L(locF)) \to L(F)$. The lower row is the surgery fibration for the manifold homotopically stratified space ∂U.

Since $L(locF)$ is a module over $MTop$, $H(F; L(locF))$ is also a module over $MTop$ (the homology is basically a super homotopy pushout, which preserves the module structure). Since X_0 is the smallest stratum, it is a genuine topological manifold and is therefore $MTop$-oriented. Consequently we have the Poincaré duality $Maps(X_0, H(F; L(locF))) \simeq H(X_0; H(F; L(locF)))$. On the other hand, it follows from the Theorem 2.10 that $H(X_0; H(F; L(locF))) \simeq H(\partial U; L(loc(\partial U))$. Hence we see that the middle map is a homotopy equivalence. It follows that if D is the homotopy fibre of the right side, then we have the fibrations:

$$D \to Maps(X_0, L(F)) \to L(\partial U); \tag{3.12}$$

$$Maps(X_0, S(F)) \to S(\partial U) \to D. \tag{3.13}$$

Again since $L(F)$ is a module over $MTop$, we see that the topological manifold X_0 is $L(F)$-oriented. Consequently, we have the Poincaré duality $Maps(X_0, L(F)) = H(X_0; L(F))$ and the fibration (32) may be rewritten as

$$D \to H(X_0; L(F)) \to L(\partial U). \tag{3.14}$$

On the other hand, we have the fibration

$$S(X, \text{rel } X_0) \to S(\bar{X}, \partial U) \to D \tag{3.15}$$

by combining the pullback (29), the identity (30), and the fibration (33).

Now construct another diagram

$$\begin{array}{ccccc} S(X, \text{rel } X_0) & \to & S(\bar{X}, \partial U) & \to & D \\ \downarrow & & \downarrow & & \downarrow \\ H(X; L(loc(X - X_0))) & \to & H(X, X_0; L(loc(X - X_0)) & \to & H(X_0; L(F)) \\ \downarrow & & \downarrow & & \downarrow \\ L(\bar{X}) & \to & L(\bar{X}, \partial U) & \to & L(\partial U) \end{array} \tag{3.16}$$

We have already shown that the right column and the upper row are homotopy fibrations. By excision, $H(X, X_0; L(loc(X - X_0)) = H(\bar{X}, \partial U; L(loc\bar{X}))$. The column is then the surgery fibration for $(\bar{X}, \partial U)$, which has $(n-1)$ strata. Finally, the bottom row is a homotopy fibration by the Lemma 3.2. Recall that our purpose is to show that the left column is a homotopy fibration. This is equivalent to showing that the middle row is a homotopy fibration.

X locally is $cone(F) \times V$ at $x \in X_0$, where V is a neighborhood of x in X_0. Hence $loc(X - X_0)$ is homotopic to F along X_0. Thus we see that the middle row is the exactness fibration for the homology of the pair (X, X_0) with coefficient $L(loc(X - X_0))$.

This completes the proof of the surgery fibration.

We conclude by pointing out that there is certain delicacy involved in the use of the surgery obstructions of Browder and Quinn in the surgery theory for simple homotopically transverse equivalences. This may cause some problem in defining the map $S(X) \to H(X; L(locX))$ in the surgery fibration. However, one may define L by using simple homotopically transverse map instead of geometrically transverse maps. With such L, one may define the map from the structure to the homology of L. Then it remains to see that our new L is equivalent to Browder and Quinn's L. For more details see [We]. On the other hand, one may rearrange the diagram (28) so that the middle column is viewed as the homotopy fibre of the map from the right column to the suspension of the left column, which is constructed from the proof of the Theorem. Then one needs to check the map is a map of fibrations. This can be done by the geometric interpretation of the maps involved.

3.4. Destablization

The stablization Lemma 3.1 is proved by killing the K-theoretical obtructions. Hence we would expect the destablization process to be a K-theoretical problem. In this section, we list the formulas that exactly describe the obstructions to such a problem.

We will certainly use the $\mathbf{R}^k$-controlled version of the K-theory. For details the readers are referred to for instance [AM].

Let X be a manifold homotopically stratified space. We have, from the Lemma 3.1, that $X \times \mathbf{R}^k$ is a manifold $\mathbf{R}^k$-controlled geometrically stratified space for large enough k. Then we apply the Theorem 3.4 to obtain the following homotopy fibration

$$S_b(X \times \mathbf{R}^k, \text{rel } \partial X) \to H(X; L_b(locX \times \mathbf{R}^k)) \to L_b(X \times \mathbf{R}^k), \tag{3.17}$$

where the subscript b means $\mathbf{R}^k$-controlled simple homotopically stratified maps.

Since the dimension k is irrelevant to our problems, we may simply take the limit and end up with stablized surgery fibration

$$S^{-\infty}(X, \text{rel } \partial X) \to H(X; L^{-\infty}(locX)) \to L^{-\infty}(X), \tag{3.18}$$

where

$$S^{-\infty}(X, \text{rel } \partial X) = \lim_{\to} S_b(X \times \mathbf{R}^k, \text{rel } \partial X); \tag{3.19}$$

$$L^{-\infty}(X) = \lim_{\to} L_b(X \times \mathbf{R}^k). \tag{3.20}$$

The stablization map is by crossing with $\mathbf{R}$.

By the Lemma 3.2, the surgery obstructions have decompositions

$$L_b((X - X_i) \times \mathbf{R}^k) \to L_b(X \times \mathbf{R}^k) \to L_b(X_i \times \mathbf{R}^k). \tag{3.21}$$

This reduces the problem to the computation of $L_b(X \times \mathbf{R}^k)$ for a pure space X without stratification. This may be done by an inductive relation which is basically the bounded version of the Rothenberg's exact sequence (see [Sh] for the classical situation without bound and [WW] for the spacified version).

$$L_b(X \times \mathbf{R}^k) \to L_b(X \times \mathbf{R}^{k+1}) \to \hat{H}(\mathbf{Z_2}; \pi_{1-k}K(X)) \tag{3.22}$$

where $K(X)$ is a spectrum we will describe in more detail later, and $\hat{H}(\mathbf{Z_2}; A)$ is the Tate cohomology space defined for abelian groups A with dualities. The homotopy groups of the Tate cohomology space are the Tate cohomology groups.

$$\pi_i \hat{H}(\mathbf{Z_2}, A) = \hat{H}^i(\mathbf{Z_2}, A) = \{a : a^* = (-1)^i a\}/\{a + (-1)^i a^*\}. \tag{3.23}$$

These are 2-torsion groups. The fibrations (41) for various k may be stacked together to produce a fibration

$$L(X) \to L^{-\infty}(X) \to \hat{H}(\mathbf{Z_2}; K^{\leq 1}(X)) \tag{3.24}$$

where $K^{\leq 1}$ is obtained from K by killing homotopy groups at dimensions > 1.

On the other hand, the bounded structures may be related inductively to the simple structure by the fibrations

$$S_b(X \times \mathbf{R}^k) \to S_b(X \times \mathbf{R}^{k+1}) \to \hat{H}(\mathbf{Z_2}; \pi_{1-k} Wh^{Top}(X)) \tag{3.25}$$

which is the bounded version of Rothenberg's exact sequence for the structures (again see [Sh] for the classical situation). $Wh^{Top}(X)$ is a spectrum related to $K(X)$. As in the case for L, we may stack the fibrations (45) together to produce a fibration

$$S(X, \mathrm{rel}\ \partial X) \to S^{-\infty}(X, \mathrm{rel}\ \partial X) \to \hat{H}(\mathbf{Z_2}; Wh^{Top, \leq 0}(X)) \tag{3.26}$$

where $Wh^{Top,\leq 0}$ is similar to $K^{\leq 1}$.

The Tate cohomology spectrum $\hat{H}(\mathbf{Z_2}; K)$ may be defined for any $\mathbf{Z}_2$-spectrum K, and preserves fibrations. Moreover, its homotopy groups may be computed from a spectral sequence

$$E^2_{ij} = \hat{H}^i(\mathbf{Z_2}; \pi_j K) \Rightarrow \pi_{i+j} \hat{H}(\mathbf{Z_2}; K). \tag{3.27}$$

A reference for the Tate cohomology spectrum and its relation to the surgery obstructions is [WW].

It remains to explain the spectra $K(X)$ and $Wh^{Top}(X)$. There are several ways of defining them. And different ways may have different high homotopy groups. However, in the destablization fibrations, the homotopy groups of dimensions > 1 are killed for K, and those of dimensions > 0 are killed for Wh^{Top}. So the difference does not matter here.

Let X be a manifold homotopically stratified space. It is implicit in [Q5] and [St] that the two functors are related by a fibration

$$Wh^{Top}(X) \to H(X; K(locX)) \to K(X). \tag{3.28}$$

Note that this is very similar to the surgery fibration. In fact, we may view the stable structure space $S^{-\infty}$ as the fibre of the assembly for the homotopy functor $L^{-\infty}$. Similarly, the topological Whitehead spectrum Wh^{Top} may be viewed as the fibre of the assembly for the homotopy functor K. In other words, $S^{-\infty}$ and Wh^{Top} respectively measure the extent to which the functors $L^{-\infty}$ and K fail to be additive.

Anyhow, it remains to analyse K.

For a construction of K and its application in geometric problems, one may look at [Q2], in which our K is denoted $\mathcal{S}$. One may also look at [St][W]. Note that the group $Wh^{Top,iso}$ studied in [St] corresponds to the 0th homotopy group of our Wh^{Top}.

The homotopy groups of K up to dimension 1 are certain geometric obstruction groups.

$$\pi_i K(X) = \begin{cases} Wh(X) & i = 1 \\ \tilde{K}_0(X) & i = 0 \\ K_i(X) & i < 0 \end{cases} \tag{3.29}$$

The group $Wh(X)$ is the usual Whitehead group for measuring simple homotopies of smooth or PL manifolds. It is different from the topological $\pi_0 Wh^{Top}(X)$. The group $\tilde{K}_0(X)$ is the finiteness obstruction group. The groups $K_i(X)$ for $i < 0$ are the so called negative K-groups.

The analogy between the pairs $(S^{-\infty}, L^{-\infty})$ and (Wh^{Top}, K) makes us wonder whether K and L have similar properties. Since the higher homotopies of K are different for different models and are irrelevant to our problem, we should only look at the low dimensional homotopy groups. There is a *split exact* sequence

$$0 \to Wh(X - X_i) \to Wh(X) \to Wh(X_i) \to 0 \tag{3.30}$$

However, the splitting tends to be artificial and does not preserve the duality. Hence we will try not to use it. Similar sequences for the other $\pi_i K$ for $i \leq 1$ are also split exact.

On the other hand, The lower K-groups are all determined by the Whitehead group. Any finite cover of the torus T^k induces a transfer $tr : Wh(X \times T^k) \to Wh(X \times T^k)$. Then an elaboration of Bass-Heller-Swan formula asserts that those elements in $Wh(X \times T^k)$ that are invariant under all transfer maps form a subgroup isomorphic to $\pi_{1-k}K(X)$. Therefore, problems can often be solved by looking at the Whitehead group first.

CHAPTER 4

PERIODICITY OF SURGERY OBSTRUCTIONS

G is always a finite group.

Let P be a closed G-manifold. Then for any homotopically stratified G-space X, there induces a map

$$\times P : L(X/G) \to L((X \times P)/G). \tag{4.1}$$

In case G is trivial, Wall [Wa] showed that this is an isomorphism if P is connected, simply connected, with dimension divisible by 4 and the signature of P being 1. It turns out that in the equivariant situation, one needs the equivariant version of the signature condition and also needs to take care of the relation between the various strata.

4.1. The Case of Free Actions

Let $\omega : G \to \{\pm 1\}$ be a homomorphism. Let V be a finitely generated torsionfree left $\mathbf{Z}G$-module. Then the dual $V^* = \mathrm{Hom}_{\mathbf{Z}}(V, \mathbf{Z})$ is also a left $\mathbf{Z}G$-module with $(g\phi)(v) = \omega(g)\phi(g^{-1}v)$. As expected, $V \cong V^{**}$ as $\mathbf{Z}G$-modules.

There are two ways of thinking of $\mathbf{Z}G$-bilinear forms on V. It is either a bilinear form $\lambda : V \otimes_{\mathbf{Z}} V \to \mathbf{Z}$ that satisfies $\lambda(gu, gv) = \omega(g)\lambda(u, v)$, or a $\mathbf{Z}G$-homomorphism $ad(\lambda) : V \to V^*$. The form is nonsingular if $ad(\lambda)$ is isomorphic. The form is symmetric if $\lambda(u, v) = \lambda(v, u)$. The form is hyperbolic if $ad(\lambda)$ is isomorphic to $H \oplus H^* \xrightarrow{id} H \oplus H^* \stackrel{t}{=} (H \oplus H^*)^*$ for some H.

Consider the trivial $\mathbf{Z}G$-module $\mathbf{Z}$ and the multiplication of integers:

$$\text{mult.} : \mathbf{Z} \otimes \mathbf{Z} \to \mathbf{Z}. \tag{4.2}$$

We say that a nonsingular symmetric $\mathbf{Z}G$-form has *G-signature* 1 if it is isomorphic to (52) up to taking direct sums with hyperbolic forms.

Let G act on a simply connected closed manifold P of dimension 4p. Then there is a homomorphism $G \to \{\pm 1\}$ that measures the change of orientation under the group action. The intersection form at the middle dimension

$$(,) : H_{2p}(P, \mathbf{Z})/\text{torsion} \otimes H_{2p}(P, \mathbf{Z})/\text{torsion} \to \mathbf{Z}$$

is a nonsingular symmetric $\mathbf{Z}G$-bilinear form. Hence we may talk about the G-signature of P.

Yosida proved the following result about the classical surgery obstruction group [Y].

Lemma 4.1.1 *Suppose G acts freely on X. Suppose P is a connected, simply connected, closed G-manifold. If the dimension of P is divisible by 4 and the G-signature of P is 1, then*

$$\times P : L(X/G) \simeq L((X \times P)/G). \tag{4.3}$$

Actually one would expect this to induce homotopy equivalences on the obstructions to other types of surgery problems, including the stablized surgery obstructions. In fact, from the fibration (44) which relates L and $L^{-\infty}$ we only need to show

$$\times P : \hat{H}(\mathbf{Z_2}; K^{\leq 1}(X/G)) \to \hat{H}(\mathbf{Z_2}; K^{\leq 1}((X \times P)/G)) \tag{4.4}$$

With the spectral sequence (47), this is a consequence of the following result.

Lemma 4.1.2 *In addition to the conditions in the Lemma 4.1, assume that the homologies of P are torsionfree. Then*

$$\times P : \hat{H}^i(\mathbf{Z_2}; \pi_j K(X/G)) \to \hat{H}^i(\mathbf{Z_2}; \pi_j K((X \times P)/G)) \tag{4.5}$$

is isomorphic for any $j \leq 1$.

Proof: Let $dim P = 2l$. Consider the fibre bundle

$$P \to (X \times P)/G \xrightarrow{p} X/G.$$

The fundamental group $\pi_1(X/G)$ acts on the fibre via the homomorphism to G. Since the homologies of P are assumed torsionfree, we may apply Anderson's result [A] to compute the composition

$$Wh(X/G) \xrightarrow{\times P} Wh((X \times P)/G) \xrightarrow{p_*} Wh(X/G).$$

By Anderson's formula, if $\tau \in Wh(X/G)$ is represented by an automorphism $f : \Pi \to \Pi$ of a finitely generated projective $\mathbf{Z}\pi_1(X/G)$-module, then

$$p_*(\times P)(\tau) = \sum_i \sigma_i(\tau),$$

where $\sigma_i(\tau) = [f \otimes id : \Pi \otimes H_iP \to \Pi \otimes H_iP]$ and the action of $\pi_1(X/G)$ on $\Pi \otimes H_iP$ is diagonal. By the Poincaré duality, $H_iP \cong (H_{2l-i}P)^*$ as $\mathbf{Z}G$-modules. Now suppose τ represents an element in the Tate cohomology group $\hat{H}^0$. Then f is equivalent to f^* in the Whitehead group. Hence by $(A \otimes B)^* = A^* \otimes B^*$ and $(f \otimes g)^* = f^* \otimes g^*$ with respect to the diagonal actions we have

$$\begin{aligned}
& \sigma_i(\tau) \\
= \ & [f \otimes id : \Pi \otimes H_iP \to \Pi \otimes H_iP] \\
= \ & [f^* \otimes id : \Pi^* \otimes (H_{2l-i}P)^* \to \Pi^* \otimes (H_{2l-i}P)^*] \\
= \ & [f \otimes id : \Pi \otimes H_{2l-i}P \to \Pi \otimes H_{2l-i}P]^* \\
= \ & \sigma_{2l-i}(\tau)^*.
\end{aligned}$$

Moreover, the condition on the intersection form of P implies that there is a $\mathbf{Z}G$-module isomorphism $H_lP \oplus H_1 \oplus H_1^* \cong \mathbf{Z} \oplus H_2 \oplus H_2^*$, where $\mathbf{Z}$ is the trivial $\mathbf{Z}G$-module. Hence if we denote by $\sigma^1(\tau)$ and $\sigma^2(\tau)$ the torsions defined similarly as $\sigma_i(\tau)$ but with H_1 and H_2 instead of H_iP, then we have

$$\sigma_p(\tau) = \tau - \sigma^1(\tau) - \sigma^{1*}(\tau) + \sigma^2(\tau) + \sigma^{2*}(\tau).$$

Now we are able to couclude that

$$\begin{aligned}
& p_*(\times P)(\tau) \\
= \ & \textstyle\sum_{i<l}(\sigma_i(\tau) + \sigma_i(\tau)^*) - \sigma^1(\tau) - \sigma^{1*}(\tau) + \sigma^2(\tau) + \sigma^{2*}(\tau) + \tau \\
= \ & \sigma(\tau) + \sigma(\tau)^* + \tau.
\end{aligned}$$

This shows that $p_*(\times P)$ is isomorphic on the 0th Tate cohomology of the Whitehead group. Since P is connected and simply connected, p induces an isomorphism on the fundamental groups and therefore p_* is isomorphic. Consequently, $(\times P)$ is isomorphic on the 0th Tate cohomology. The situation with the Tate cohomology at the other dimensions is similar.

This only proves the isomorphism for $\pi_1 K = Wh$. For the other homotopy groups below the dimension 1. We use the transfer characterization of the groups in terms of Wh. As pointed out at the end of the section 3.4, the various finite coverings of the torus T^k over itself define transfer maps $tr : Wh(X \times T^k) \to Wh(X \times T^k)$ and the subgroup of those elements in $Wh(X \times T^k)$ that are invariant under transfers form exactly $\pi_{1-k}K(X)$. It is clear from Anderson's formula that the transfer commutes with the map $\times P$ (which is another transfer). Hence the computation for the general case may be obtained by restricting the computation for Wh to the transfer invariant elements.

Now we are able to conclude the periodicity of the stablized surgery obstructions.

Theorem 4.1.3 *Suppose that G acts on X freely. Suppose that P is a connected, simply connected, closed G-manifold. If P has torsionfree homologies, with dimension divisible by 4 and the G-signature 1. Then*

$$\times P : L^{-\infty}(X/G) \simeq L^{-\infty}((X \times P)/G). \tag{4.6}$$

4.2. Isovariance vs. Equivariance

Before we prove the periodicity for the non-free case, we investigate in detail the role played by the small gap hypothesis in the relation between isovariance and equivariance. The motivation for such discussion is that in many cases, we would like to replace the computation on isovariant objects by the easier computation on the equivariant objects. Moreover, certain notations developped in the discussion will be extensively used later.

Let X be a homotopically stratified G-space. Let $\mathcal{H}$ be the set of maximal isotropy subgroups of X. Then $\mathcal{H}$ is closed under the conjugate action by G, and for $H \neq K$ in $\mathcal{H}$, $X^H \cap X^K = \emptyset$. Hence if $\mathcal{I} \subset \mathcal{H}$ is a set of representatives of $\mathcal{H}/G$, then

$$(\coprod_{H \in \mathcal{H}} X^H)/G = \coprod_{H \in \mathcal{I}} X^H/WH, \tag{4.7}$$

where the Weyl group $WH = NH/H$ acts on X^H freely. On the other hand, $X - \coprod_{H \in \mathcal{H}} X^H$ is a G-space with fewer isotropy groups than X.

By repeating the above decomposition inductively, we may decompose X into disjoint pieces with free actions by the Weyl groups of the isotropy subgroups. These pieces are called *isovariant components*. Similarly, the connected components of the fixed point subsets X^H for generic subgroups H are called *equivariant components*. The isotropy group of an equivariant component is the intersection of the isotropy groups of its points.

Definition 4.2.1 A homotopically stratified G-space X has *codimension* $\geq k$ *(homotopy) gap* if for any subgroups $K \subset H$, the fibre of the homotopy link fibration of $X^H - X^{>H}$ in X^K is $(k-2)$-connected.

It might happen that the fibre is empty over some components. This is the case that the component is a component for a smaller subgroup.

Note that if X is a manifold and the action is locally flat, then the fixed point subsets are submanifolds and the fibres of the homotopy links are spheres of dimension $(codim - 1)$. Therefore the definition is compatible with the usual notion for submanifolds.

Also note that there is an H-action on the fibre. However, the action is ignored in the definition of the gap condition.

Remark 4.2.2 If X is a manifold, then the homotopy gap is the same as the geometric gap as defined before the surgery fibration theorem.

Lemma 4.2.3 *Suppose that X is a homotopically stratified G-space. Consider the inclusion from the isovariant components to the equivariant components.*

1. *If X has codimension ≥ 2 gap, then the inclusion is a 1-1 correspondence on the components;*

2. *If X has codimension ≥ 3 gap, then the inclusion induces isomorphisms on the fundamental groups of components.*

3. *If X has codimension $\geq k$ gap, then the inclusion induces isomorphisms on the homologies of components up to dimension $k-2$.*

Proof: By induction on $iso(X)$. Using the notations introduced at the begining of the section, we note that the conclusion trivially holds for those components with maximal isotropies since they are also equivariant components. It remains to consider those components whose isotropies are not in $\mathcal{H}$. Observe that by the inductive hypothesis, the conclusion holds for $X - \coprod X^H$. Moreover, the inclusion $X - \coprod X^H \to X$ preserves the isovariant components whose isotropies are not in $\mathcal{H}$. Therefore it suffices to show that the map induces a 1-1 correspondence between equivariant components with isotropies not in $\mathcal{H}$ and similar assertions with bigger gap conditions.

Let X^K_α be an equivariant component of X with the isotropy group $K \notin \mathcal{H}$. The problem is to show that $X^K_\alpha - \coprod X^H$ is connected and therefore X^K_α corresponds to only one equivariant component of $X - \coprod X^H \to X$. Any two points in $X^K_\alpha - \coprod X^H$ may be connected by a path ω in X^K_α. All we need to do is to modify ω away from $\coprod X^H$.

Suppose a segment $\omega[s,t]$ is inside a neighborhood U of a component X^H_β with the isotropy $H \in \mathcal{H}$, such that $\omega(s)$ and $\omega(t)$ are not in X^H_β. Up to transverse homotopy equivalence we may assume that U is the mapping cylinder of the homotopy link fibration $p : \partial U \to X^H_\beta$. Now we lift the path $p\omega[s,t]$ to a path ω' in $U - X^H_\beta$ such that $\omega'(s) = \omega(s)$ by using the fibration $p \circ \mathrm{proj} : U - X^H_\beta = \partial U \times [0,1) \to X^H_\beta$. Then $\omega'(t)$ and $\omega(t)$ are in the same fibre of $p \circ \mathrm{proj}$, which is homotopy equivalent to the fibre of p. With the codimension ≥ 2 condition, the fibre is connected. Therefore we may connect $\omega'(t)$ and $\omega(t)$ by a path ρ in the fibre of $p \circ \mathrm{proj}$. Now replacing the segment $\omega[s,t]$ by $\omega' * \rho$ (ω' followed by ρ), we modify the segment away from X^H_β.

Now we prove the assertion about the fundamental group. Again it suffices to show that

$$\pi_1(X^K_\alpha - \coprod X^H) \to \pi_1(X^K_\alpha)$$

is an isomorphism. As in the proof of the connectivity, up to transverse homotopy equivalence we have a mapping cylinder neighborhood U_β for each component X^H_β in X^K_α. We may assume that such neighborhoods are disjoint. Then the inclusion $X^K_\alpha - \coprod intU_\beta \to X^K_\alpha - \coprod X^H$ is an isovariant homotopy equivalence. Moreover,

$$X^K_\alpha = (X^K_\alpha - \coprod intU_\beta) \cup_{\coprod \partial U_\beta} \coprod intU_\beta.$$

Hence by Van Kampen, to show the fundamental groups are the same, it suffices to show that the inclusion $\partial U_\beta \to U_\beta$ induces an isomorphism on the fundamental groups. Note that the inclusion is homotopy equivalent to the projection $p_\beta : \partial U_\beta \to X^H_\beta$. With the codimension$\geq 3$ gap, the fibre of p_β is connected and simply connected. This implies that p_β induces an isomorphism on the fundamental groups.

Finally, the isomorphism on homology with bigger gaps. We simply apply the Meyer-Vietoris to the decomposition $(X^K_\alpha - \coprod intU_\beta) \cup_{\coprod \partial U_\beta} \coprod intU_\beta$. Then the problem becomes showing that $\partial U_\beta \to U_\beta$ induces isomorphisms on homologies at dimensions up to $k-2$. Equivalently, we may look at the projection $p_\beta : \partial U_\beta \to X^H_\beta$. Since the fibre of p_β is $(k-2)$-connected, its reduced homology vanishes up to dimension $k-2$. A spectral sequence arguement then finishes the proof.

4.3. The Isovariant Correspondence by $\times P$

In the periodicity problem, we consider the operation $\times P$ and try to prove that it induces an equivalence on the obstruction space. If we want to follow the inductive arguement, $\times P$ must induce a 1-1 correspondence between the isovariant components. With codimension≥ 2 gap, this is equivalent to the situation described in the following definition.

Definition 4.3.1 Denote by $iso(X)$ the set of isotropy subgroups of X. X is said to have the *same isotropy everywhere* as $X \times P$ if for any $x \in X$, there is an equivariant neighborhood U such that $iso(U) = iso(U \times P)$.

To get a feeling about the condition $iso(U) = iso(U \times P)$, let us look at two extreme situations:

1. If G acts on P freely, then G acts on $U \times P$ also freely. Therefore we require G acts on U freely.

2. If all subgroups of G are isotropies of P, then we need $iso(U)$ to be closed, i.e., if $H \subset K \in iso(U)$, then $H \in iso(U)$.

The isovariant components of $X \times P$ are quite complicated. On the other hand, the equivariant components are quite easy because $(X \times P)^H = X^H \times P^H$. In view of the Lemma 4.6, the following result is likely to be useful.

Lemma 4.3.2 *Suppose that X have the same isotropy everywhere as $X \times P$. If X has codimension $\geq k$ gap, so does $X \times P$.*

Proof: The gap condition is a local condition. Let $x \in X$ have isotropy group H. Let U be an H-invariant neighborhood of X such that $iso(U) = iso(U \times P)$ and $gU \cap U = \emptyset$ for $g \notin H$. Suppose that $K \subset H$ is in $iso(U \times P)$ and $(U \times P)^K \neq (U \times P)^H$. Then $K \neq H$ in $iso(U \times P) = iso(U)$. Consequently, $U^K \neq U^H$. Hence the fibre of the homotopy link of U^H in U^K is a nonempty space F. Let c be the codimension of P^H in P^K. Then the fibre of the homotopy link of $(U \times P)^H$ in $(U \times P)^K$ is the join $F * S^{c-1}$. If F is $(k-2)$-connected, then $F * S^{c-1}$ is $(k+c-2)$-connected.

Theorem 4.3.3 *Suppose that X has the same isotropy everywhere as $X \times P$. If X has codimension ≥ 3 gap, then the inclusion*

$$(X - \coprod_{H \in \mathcal{H}} X^H) \times P \to (X \times P - \coprod_{H \in \mathcal{H}} X^H \times P^H) \tag{4.8}$$

induces a 1-1 correspondence on the isovariant components and isomorphisms on the fundamental groups of the strata of the orbifolds.

Proof: We first look at the equivariant components. Observe that by the Lemma 4.8, the spaces $X \times P$ and $(X - \coprod X^H) \times P$ also have codimension ≥ 3 gap. Hence by the Lemma 4.6, in both spaces, the isovariant and equivariant components are in 1-1 correspondence. Hence as far as the correspondence of components are concerned, we do not need to distinguish isovariance and equivariance.

The same isotropy everywhere condition implies that the equivariant components of $X \times P$ are $X_\alpha^K \times P^K$, whose isotropy group K is determined by the first factor. It follows that the components of $X \times P$ corresponds to the components of X. Consequently, by cutting the isotropies in $\mathcal{H}$, the components of $X \times P - \coprod X^H \times P^H$ are in 1-1 correspondence with the components of $X - \coprod X^H$.

If we replace X by $X - \coprod X^H$ in the arguement, then we see that the components of $(X - \coprod X^H) \times P$ are also in 1-1 correspondence with the components of $X - \coprod X^H$. Thus we conclude that the inclusion induces a 1-1 correspondence on the components.

It remains to prove the isomorphism on the fundamental groups.

Adopting the notations in the proof of the Lemma 4.6, we let U_β be the disjoint mapping cylinder neighborhoods of the components X_β^H with $H \in \mathcal{H}$. The maps $p_\beta : \partial U_\beta \to X_\beta^H$ are fibrations and, by the gap condition, the fibres are simply connected. The inclusions

$$(X - \coprod intU_\beta) \times P \to (X - \coprod X^H) \times P$$

$$X \times P - \coprod intU_\beta \times P^H \to X \times P - \coprod X^H \times P^H$$

are isovariant homotopy equivalences. Therefore we only need to look at the inclusion of the spaces on the left side.

To compute the fundamental groups, let X_α^K be an equivariant component of X with $K \notin \mathcal{H}$. Then the corresponding equivariant components are

$$(X_\alpha^K - \coprod X^H) \times P^K \to X_\alpha^K \times P^K - \coprod X^H \times P^H.$$

Adopting the notations in the proof of the Lemma 4.6, we let V_β be the disjoint mapping cylinder neighborhoods of the components X_β^H in X_α^K with $H \in \mathcal{H}$. The maps $p_\beta : \partial V_\beta \to X_\beta^H$ are fibrations and, by the gap condition, the fibres are simply connected. Then the inclusion map is isovariantly homotopy equivalent to

$$(X_\alpha^K - \coprod intV_\beta) \times P^K \to X_\alpha^K \times P^K - \coprod intV_\beta \times P^H.$$

Note that

$$X_\alpha^K \times P^K - \coprod intV_\beta \times P^H = (X_\alpha^K - \coprod intV_\beta) \times P^K \cup_{\coprod \partial V_\beta \times (P-P^H)} \coprod V_\beta \times (P - P^H).$$

Hence by Van Kampen, it boils down to proving that the inclusions $\partial V_\beta \times (P-P^H) \to V_\beta \times (P-P^H)$ induce isomorphisms on the fundamental groups. Since this is homotopy equivalent to $p_\beta \times id : \partial V_\beta \times P^H \to X_\beta^H \times P^H$, and the fibre of p_β is simply connected, it indeed induces an isomorphism on the fundamental groups.

Now we use the codimension ≥ 3 condition and the Lemma 4.6 again to conclude that the inclusion also induces isomorphisms on the fundamental groups of isovariant components. Consequently, the induced map on the quotient is also isomorphic on the fundamental groups. In fact, suppose $f : X \to Y$ is an isovariant map that induces isomorphisms on the fundamental groups of isovariant components, then for the corresponding map $f_\alpha : X_\alpha^K \to Y_\alpha^K$ of isovariant components, the exact sequence $1 \to \pi_1(X_\alpha^K) \to \pi_1(X_\alpha^K/WK) \to WK \to 1$ and a five lemma arguement shows that

$$\bar{f}_\alpha : \pi_1(X_\alpha^K/WK) \cong \pi_1(Y_\alpha^K/WK).$$

4.4. Periodicity Manifolds

Definition 4.4.1 A closed G-manifold P is called a *periodicity manifold* if for any subgroup H of G, P^H is connected, simply connected, with torsionfree homologies, and the WH-signature being 1.

One may compare the definition with Dovermann and Schultz's in [DS]. They require that the homologies should be permutation representations of G. This does not seem to be necessary for the periodicity of surgery obstructions.

Theorem 4.4.2 *Suppose P is a periodic manifold and X is a homotopically stratified space. If X and $X \times P$ have the same isotropy everywhere, and X has the the codimension ≥ 3 gap, then*

$$\times P : L^{-\infty}(X/G) \simeq L^{-\infty}((X \times P)/G). \tag{4.9}$$

Proof: The proof is by induction on the isotropy groups. Using the notations introduced in the section 4.2, we construct the diagram

$$\begin{array}{ccc} L^{-\infty}((X-\coprod X^H)/G) & \stackrel{(\times P)^f}{\longrightarrow} & L^{-\infty}((X\times P-\coprod X^H\times P^H)/G) \\ \downarrow & & \downarrow \\ L^{-\infty}(X/G) & \stackrel{\times P}{\longrightarrow} & L^{-\infty}((X\times P)/G) \\ \downarrow & & \downarrow \\ L^{-\infty}((\coprod X^H)/G) & \stackrel{\coprod \times P^H}{\longrightarrow} & L^{-\infty}((\coprod X^H\times P^H)/G) \end{array} \tag{4.10}$$

where H runs through the set $\mathcal{H}$ of maximal isotropies of X. Both columns are genuine fibrations, and $(\times P)^f$ is the map induced on the fibre. Observe that the bottom map is

$$\times_{H\in\mathcal{I}}(\times P^H) : \times_{H\in\mathcal{I}}L^{-\infty}(X^H/WH) \to \times_{H\in\mathcal{I}}L^{-\infty}((X^H\times P^H)/WH),$$

where $\mathcal{I}\subset\mathcal{H}$ is a set of representatives of $\mathcal{H}/G$. Since WH acts freely on X^H, and the equivariant WH-signature of P^H is 1, it follows from the Theorem 4.3 that each $\times P^H$ induces a homotopy equivalence on $L^{-\infty}$.

Therefore to prove the middle map is a homotopy equivalence, it suffices to show that $(\times P)^f$ is a homotopy equivalence. An n-simplex in $L^{-\infty}((X-\coprod X^H)/G)$ is represented by a surgery problem (we omitted the $\mathbf{R}^k$ factor and the bundle data)

$$(N,\partial N)/G \to (M,\partial M)/G \to \Delta^n\times X/G$$

of pairs of $(n+2)$-ads such that the restriction on ∂ is a simple homotopy equivalence, and $M^H = N^H = \emptyset$ for any $H\in\mathcal{H}$. After crossing with P, we have the surgery problem

$$(N\times P,\partial N\times P)/G \to (M\times P,\partial M\times P)/G \to \Delta^n\times X\times P/G.$$

Since $N^H = M^H = \emptyset$ for any $H\in\mathcal{H}$, the reference map factors through $((X-\coprod X^H)\times P)/G$. In other words, $(\times P)^f$ is the composition of

$$\times P : L^{-\infty}((X-\coprod X^H)/G) \to L^{-\infty}(((X-\coprod X^H)\times P)/G) \tag{4.11}$$

which is a homotopy equivalence by the inductive hypothesis, and

$$\text{incl.} : L^{-\infty}((X - \coprod X^H) \times P)/G \to L^{-\infty}(X \times P - \coprod X^H \times P^H)/G. \qquad (4.12)$$

Now the Theorem 4.9 concludes that the inclusion induces a 1-1 correspondence on the isovariant components and induces isomorphisms on the fundamental groups of the strata of the orbifolds. Since $L^{-\infty}$ only depends on the fundamental groups and the first Stiefel-Whitney class, the inclusion induces a homotopy equivalence on the surgery obstruction spaces. This proves that $(\times P)^f$ is indeed a homotopy equivalence.

Remark 4.4.3 We do not necessarily need the periodicity condition hold for P for any subgroup of G if isotropy subgroups of X are controlled. In fact one may consider a class $\mathcal{G}$ of subgroups of G and define $\mathcal{G}$-periodicity manifolds as those who satisfy the condition for $H \in \mathcal{G}$. Then the Theorem 4.11 holds for those X with $iso(X) \subset \mathcal{G}$.

Definition 4.4.4 Suppose P is a periodicity manifold and $x \in P^G$. Then the unit disc V of the tangential G-representation T_xP is called a *periodicity representation* (related to P).

Note that V embeds in P isovariantly via the exponential map. With codimension ≥ 3 gap, the map induces homotopy equivalences

$$\text{incl} : L^{-\infty}((X \times V)/G) \simeq L^{-\infty}((X \times P)/G) \qquad (4.13)$$

since the embedding induces isomorphisms on the fundamental groups of the isovariant components. Combined with the periodicity of P, we obtain a natural homotopy equivalence

$$\text{incl}^{-1} \circ \times P : L^{-\infty}(X/G) \simeq L^{-\infty}((X \times V)/G). \qquad (4.14)$$

In the non-equivariant case, the unit disc of $T_*\mathbf{CP}^2$ is D^4 and Siebenmann makes use of the equivalence to prove the 4 fold periodicity of the nonequivariant surgery theory [KS]. We are going to do the same in the equivariant case, with D^4 replaced by V.

4.5. An Example

To construct examples of periodic manifolds, let G act on a set S and

$$P = \underbrace{\mathbf{CP}^2 \times \cdots \times \mathbf{CP}^2}_{S} \tag{4.15}$$

be the product of S copies of the complex projective space $\mathbf{CP}^2$, with G acting as permutations. For any subgroup H of G, P^H is the product of S/H copies of $\mathbf{CP}^2$, embedded in P via a product of diagonal maps. In particular, P^H is simply connected and has $\mathbf{Z}$-free homologies and has signature 1. However, the equivariant WH-signature may not be 1.

The following result has been proved in [DS]. The proof presented here is more elementary in language.

Theorem 4.5.1 *Suppose that G acts on a finite set S in such a way that the orbits are of odd order. Then the product of S copies of $\mathbf{CP}^2$ under the permutation action of G is a periodicity manifold.*

Proof: Suppose that H is a subgroup of G. Then as a WH-manifold, P^H is the product of S/H copies of the complex projective spaces, with the action induced from the action of WH on S/H. Clearly, the orbits of the action is still of odd orders. Consequently, the general study of the WH-signature of P^H is similar to the study of the action of G on P. Hence we only need to show that the G-signature of P is 1.

Let $q = 2s = dimP/2$. H_qP has an obvious $\mathbf{Z}$-base in a 1-1 correspondence to the set B of decompositions $S = S_0 \coprod S_2 \coprod S_4$ satisfying $|S_0| = |S_4|$. In terms of the decomposition, the action of G on the base elements is the action on the set $gS = gS_0 \coprod gS_2 \coprod gS_4$, and the duality action is $S_0^* = S_4, S_2^* = S_2, S_4^* = S_0$. If $gs = s^*$ for some $s \in S$, then we have for some decomposition that $gS_0 = S_4$, $gS_4 = S_0$. Consequently, $g^{even}S_0 = S_0$ and $g^{odd}S_0 = S_4$. Since the orders of the orbits of the action are odd, the later case can never happen, unless $S_0 = S_4 = \emptyset$, which corresponds to the only base element u_0 fixed by G. Consequently, $*$ acts freely on $(B - B^G)/G$. This produces a G-invariant decomposition

$$H_qP = H \oplus H^* \oplus \mathbf{Z}u_0$$

in which $u_0^* = u_0$, H is based on a G-subset C of B such that $C \cup C^* = B - B^G$ and $C \cap C^* = \emptyset$, H^* is the dual of H under the Poincaré duality, and the intersection takes the form

$$\begin{pmatrix} 0 & I & 0 \\ I & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Moreover, the self intersection is 0 on H and H^*. This proves that the form is hyperbolic on $H \oplus H^*$. By deleting the hyperbolic part, we see that the G-signature of P is 1.

In contrast, it is shown in [DS] that our example does not produce a periodicity manifold for the even order group $\mathbf{Z}/2$.

Note that the tangential representation of S copies of $\mathbf{CP}^2$ is $\mathbf{R}^4 \times \mathbf{R}S$. Hence by the Theorem 2, we have

Lemma 4.5.2 *Suppose that G acts on a finite set S in such a way that the orbits are of odd order. Then*

$$V = \mathbf{R}^4 \times \mathbf{R}S \tag{4.16}$$

is a periodicity representation.

Remark 4.5.3 We may replace $\mathbf{CP}^2$ with the quaternionic projective space $\mathbf{HP}^2$, or more generally, any $4l$-dimensional manifold Q that is $2l$-connected, with $H_{2l}(Q) = \mathbf{Z}$ and signature 1. Then we get the periodicity representation $\mathbf{R}^{4l} \times \mathbf{R}S$, which is l times what we have in the Lemma but related to a highly connected manifold.

CHAPTER 5

PERIODICITY OF STABLIZED SURGERY

This chapter proves that the periodicity of the stablized surgery theory is a formal consequence of the periodicity of the surgery obstructions. An explicit geometric construction of the periodicity at the structure set level seems unlikely for general representations (for the trivial representation, a variant of [CW] works). Instead, a property of the periodicity map is presented.

Again G is always a finite group.

5.1. The Statement and the Proof

In this section we make use of the stablized surgery fibration to prove the periodicity of the stable structure spaces of homotopically stratified equivariant topological manifolds.

Theorem 5.1.1 *Let P be a periodicity manifold and V be the tangential G-representation of P at a fixed point. Suppose that M is a homotopically stratified topological G-manifold that has the dimension ≥ 5, has the codimension≥ 6 gap, and M, $M \times P$ have the same isotropy everywhere. Then there exists a natural homotopy equivalence of structure spaces:*

$$\Pi^{-\infty} : S_G^{-\infty}(M, \mathrm{rel}\ \partial) \simeq S_G^{-\infty}(M \times V, \mathrm{rel}\ \partial). \tag{5.1}$$

Proof: As in Siebenmann's proof in the non-equivariant case. We compare the surgery fibrations for M and $M\times V$. In fact we may embed the assembly maps in the following diagram.

$$
\begin{tikzcd}
H(M\times V/G; L^{-\infty}(locM\times V/G)) \arrow[r, "\alpha_2"] \arrow[d, "\simeq"', "\text{Fubini}"] & L^{-\infty}(M\times V/G) \arrow[ddd, "\simeq"', "\text{incl}"] \\
H(M/G; H(V/G_x; L^{-\infty}(locM\times V/G))) \arrow[d, "\simeq"', "\alpha"] & \\
H(M/G; L^{-\infty}((locM)\times V/G)) \arrow[uur, "\beta_1"] \arrow[d, "\simeq"', "\text{incl}"] & \\
H(M/G; L^{-\infty}((locM)\times P/G)) \arrow[r, "\beta_2"] & L^{-\infty}(M\times P/G) \\
H(M/G; L^{-\infty}(locM/G)) \arrow[r, "\alpha_1"] \arrow[u, "\simeq"', "\times P"] & L^{-\infty}(M/G) \arrow[u, "\simeq"', "\times P"]
\end{tikzcd}
\tag{5.2}
$$

where

1. α_i, $i=1,2$ are the assemblies of $L^{-\infty}$ for M, $M\times V$ respectively;

2. α is the partial assembly along the V-direction, obtained by applying the homology to the assembly of the functor $L^{-\infty}(locM\times ?)$ for V/G_x;

3. β_i, $i=1,2$ are the assemblies for the blocked surgery problems to $M\times V/G\to M/G$ and $M\times P/G\to M/G$ respectively.

Note that by $locM\times V$ we mean the local G-structure of the product space $M\times V$, while $(locM)\times V$ means the product of the local G-structure of M with the whole G-space V.

To show that the partial assembly α is a homotopy equivalence, we note that the stratified space V/G_x is the cone on the space $X=S(V)/G_x$ for the unit sphere $S(V)$ of V. And for any homotopy invariant spectrum valued functor S, the assembly $H(coneX; S(loc(coneX))) \to S(coneX)$ is a homotopy equivalence. In fact, the inclusion of the cone point $*$ induces a fibration

$$\begin{array}{c} H(*; S(loc(coneX))) \\ \downarrow \\ H(coneX; S(loc(coneX))) \\ \downarrow \\ H(coneX - *; S(loc(coneX))). \end{array}$$

Let $cone^{\epsilon}X$ be the subcone of radius ϵ. Then $coneX - cone^{\epsilon}X$ form a cofinal system of open subsets of $coneX - *$ with compact closure. By the local finiteness property of homologies (see the Theorem 2.3), we have $H(coneX - *; S(loc(coneX))) = \lim_{\leftarrow} H(coneX, cone^{\epsilon}X; S(loc(coneX)))$. By excision and the homotopy invariance, $H(coneX, cone^{\epsilon}X; S(loc(coneX))) = H(X \times [\epsilon, 1], X \times \epsilon; S(loc(coneX)))$ is contractible, since $S(loc(coneX))$ is constant away fron the cone point. Thus $H(coneX - *; S(loc(coneX)))$ is contractible, and the map $H(*; S(loc(coneX))) \to H(coneX; S(loc(coneX)))$ is a homotopy equivalence. Note that the local structure of $coneX$ at the cone point is $coneX$, therefore $H(*; S(loc(coneX)))$ is simply $S(coneX)$. This is the homotopy inverse of the assembly. If we take $S = L^{-\infty}(locM \times ?)$, then we see that α is a homotopy equivalence.

By the Lemma 4.6, the inclusions $M \times V \to M \times P$ and $(locM) \times V \to (locM) \times P$ induce 1-1 correspondences of the isovariant components and isomorphisms of the fundamental groups of orbits. Since all the equivariant components of V and P are simply connected, the inclusion also preserves the first Stiefel-Whitney class. Because L only depends on the fundamental group and the first Stiefel-Whitney class, the inclusion induces a homotopy equivalence.

The map $\times P$ is homotopy equivalent by the Theorem 4.11.

The equivalence of Fubini type follows from the Lemma 2.10.

The diagram is commutative. The commutativity of the top triangle follows from the discussion after the Fubini Theorem (Lemma 2.10). The commutativity of the rest of the diagram follows from the naturality of the geometric definitions.

The surgery theory asserts that the fibre of α_1 is $S_G^{-\infty}(M, \mathrm{rel}\ \partial)$, and the fibre of α_2 is $S_G^{-\infty}(M \times V, \mathrm{rel}\ \partial)$. Therefore the the homotopy equivalences of the columns induce a homotopy equivalence of the fibres

$$\Pi^{-\infty} : S_G^{-\infty}(M, \mathrm{rel}\ \partial) \simeq S_G^{-\infty}(M \times V, \mathrm{rel}\ \partial).$$

This is the stable periodicity we are searching for.

5.2. A Naturality Relation

The periodicity is proved by the observation that the wrong direction inclusion map in $M \to M \times P \leftarrow M \times V$ is homotopically invertible after applying the normal invariants and the surgery obstruction functors. However, the homotopy equivalence follows from an algebraic fact and there is no canonical way of defining the homotopy inverse map. Therefore the geometric construction for the periodicity at the structure space level is hard to see. On the other hand, if we do not invert the wrong direction map, then we should obtain two geometrically explicit constructions, and periodicity should be related to the constructions in the expected way.

We would like to see what the maps $M \to M \times P \leftarrow M \times V$ do to the simple structures of equivariant manifolds.

Let $(M, \partial M)$ be a G-manifold. Let $(N, \partial N)$ be a closed codimension 0 submanifold of X. Let $\partial_0 N = N \cap \overline{M - N}$. Then $M = N \cup_{\partial_0 N} \overline{M - N}$. Now for any simple homotopy equivalence $f : N' \to N$ rel ∂N, we have the simple homotopy equivalence $f \cup id : M' = N' \cup_{\partial_0 N} \overline{M - N} \to N \cup_{\partial_0 N} \overline{M - N} = M$ rel ∂M. This produces a map

$$\mathrm{incl} : S_G(N, \mathrm{rel}\ \partial N) \to S_G(M, \mathrm{rel}\ \partial M). \tag{5.3}$$

The similar constructions can be made for the normal invariants and the surgery obstructions. It is tedious, but not difficult to prove that the inclusion is a map of surgery fibrations.

Let $(M, \partial M)$ and P be G-manifolds, $\partial P = \emptyset$. For any simple homotopy equivalence $f : M' \to M$ rel ∂M, we have the simple homotopy equivalence $f \times id : M' \times P \to M \times P$ rel $\partial M \times P$. This produces a map

$$\times P : S_G(M, \text{rel } \partial M) \to S_G(M \times P, \text{rel } \partial M \times P). \tag{5.4}$$

The similar constructions can be made for the normal invariants and the surgery obstructions. Again it is tedious, but not difficult to prove that $\times P$ is a map of surgery fibrations.

The constructions may be done in various categories. This is the reason we do not specify the kind of G-manifolds under the consideration. In fact this can also be done to homotopically stratified spaces, with $\times P$ replaced with a stratified system of fibrations.

In our situation, the inclusion is $M \times V \to M \times P$ and $\times P$ is the product with a periodicity manifold.

Theorem 5.2.1 *The following diagram is homotopically commutative.*

$$\begin{tikzcd}
S_G^{-\infty}(M \times V, \text{rel } \partial) \arrow[dr, "\text{incl}"] & \\
& S_G^{-\infty}(M \times P, \text{rel } \partial) \\
S_G^{-\infty}(M, \text{rel } \partial) \arrow[uu, "\Pi^{-\infty}"'] \arrow[ur, "\times P"] &
\end{tikzcd} \tag{5.5}$$

Proof: This is done by comparing the stablized surgery exact sequences for M, $M \times P$, and $M \times V$ via the $\times P$ map and the inclusion map.

Hence we construct the following diagram (compare with the diagram (68) in the proof of the Theorem 5)

$$\begin{tikzcd}
S_G^{-\infty}(M\times V,\text{rel }\partial) \arrow[r] \arrow[dd, "\text{incl}"] & H(M\times V/G;L^{-\infty}(locM\times V/G)) \arrow[rr, "\alpha_2"] \arrow[dr, "\simeq\ \alpha_V\circ\text{Fubini}"] \arrow[dd, "\text{incl}"'] & & L^{-\infty}(M\times V/G) \arrow[dd, "\simeq\text{incl}"] \\
& & H(M/G;L^{-\infty}((locM)\times V/G)) \arrow[ur, "\beta_1"] \arrow[dd, "\simeq\text{incl}"] & \\
S_G^{-\infty}(M\times P,\text{rel }\partial) \arrow[r] & H(M\times P/G;L^{-\infty}(locM\times P/G)) \arrow[rr] \arrow[dr, "\alpha_P\circ\text{Fubini}"] & & L^{-\infty}(M\times P/G) \\
& & H(M/G;L^{-\infty}((locM)\times P/G)) \arrow[ur, "\beta_2"] & \\
S_G^{-\infty}(M,\text{rel }\partial) \arrow[r] \arrow[uu, "\times P"] & H(M/G;L^{-\infty}(locM/G)) \arrow[rr, "\alpha_1"] \arrow[uu, "\times P"'] \arrow[dr, equal] & & L^{-\infty}(M/G) \arrow[uu, "\simeq\times P"] \\
& & H(M/G;L^{-\infty}(locM/G)) \arrow[uu, "\simeq\times P"] \arrow[ur, "\alpha_1"] &
\end{tikzcd} \tag{5.6}$$

where the left column consists of the homotopy fibres of the maps from the second to the fourth column, and incl and $\times P$ are the maps induced on fibres. The diagram is homotopy commutative by the naturality of the constructions.

Note that the maps on the third and the fourth columns appear in the diagram (68) and are homotopy equivalences. Hence if (Fib=homotopy fibre)

$$F = \text{Fib}(\beta_2),$$

then the induced maps

$$\text{Fib}(\beta_1) \to F \leftarrow \text{Fib}(\alpha_1)$$

are homotopy equivalences. Since $\alpha_V \circ$Fubini is a homotopy equivalence (this is the main part of the proof of the stablized periodicity), the induced map

$$\lambda : S_G^{-\infty}(M \times V, \text{rel }\partial) \simeq \text{Fib}(\alpha_2) \to \text{Fib}(\beta_1) \to F$$

is a homotopy equivalence. On the other hand, we have the homotopy equivalence

$$\mu : S_G^{-\infty}(M, \mathrm{rel}\ \partial) \simeq \mathrm{Fib}(\alpha_1) \to F.$$

By chasing the diagram, we see that the composition

$$\lambda^{-1} \circ \mu : S_G^{-\infty}(M, \mathrm{rel}\ \partial) \simeq S_G^{-\infty}(M \times V, \mathrm{rel}\ \partial)$$

is exactly the stablized periodicity map $\Pi^{-\infty}$.

Let $F = \mathrm{Fib}(\beta_2)$ and

$$\phi : S_G^{-\infty}(M \times V, \mathrm{rel}\ \partial) \to F$$

be the map induced by $\alpha_P \circ$Fubini. Then the following diagram, being the diagram of maps on the fibres induced from the diagram (72), is homotopically commutitive.

$$\begin{tikzcd}
S_G^{-\infty}(M \times V, \mathrm{rel}\ \partial) \arrow[d, "\mathrm{incl}"] \arrow[dr, "\lambda"] & \\
S_G^{-\infty}(M \times P, \mathrm{rel}\ \partial) \arrow[r, "\phi"] & F \\
S_G^{-\infty}(M, \mathrm{rel}\ \partial) \arrow[u, "\times P"] \arrow[ur, "\mu"'] &
\end{tikzcd}$$

Consequently,

$$\begin{aligned}
& \mathrm{incl} \circ \Pi^{-\infty} \\
= \ & \mathrm{incl} \circ \lambda^{-1} \circ \mu \\
= \ & \mathrm{incl} \circ \mathrm{incl}^{-1} \circ \phi^{-1} \circ \phi \circ (\times P) \\
= \ & (\times P).
\end{aligned}$$

This completes the proof of the Theorem 5.1.

CHAPTER 6

DESTABLIZATION OF STABLE PERIODICITY

The destablization of the stablized structure space is achieved in the following fibration

$$S_G(M, \mathrm{rel}\ \partial) \to S_G^{-\infty}(M, \mathrm{rel}\ \partial) \to \hat{H}(\mathbf{Z}_2; Wh_G^{Top,\leq 0}(M)).$$

We have the periodicity equivalence on $S_G^{-\infty}$ induced from the periodicities of the stablized normal invariants and surgery obstructions. We want to show that the similar construction would provide the periodicity equivalence for $\hat{H}(\mathbf{Z}_2; Wh_G^{Top,\leq 0}(M))$. Then we will be able to show the periodicity for S_G. Unfortunately, the Tate cohomology functor behaves so badly that the proof for the periodicity on the Tate cohomology of the K-theory breaks down. We therefore could only get a periodicity map for the unstablized spaces and the map may not be an equivalence.

For the basic facts about the K-theory, one is refered to the end of the section 3.4.

6.1. Inclusion Equivalence

Among the operations $M \to M \times P \leftarrow M \times V$, we first consider the maps

$$K((M \times V)/G) \to K((M \times P)/G);$$

$$H((M \times V)/G; K(locM \times V/G)) \to H((M \times P)/G; K(locM \times P/G))$$

induced by the inclusions.

Lemma 6.1.1 *Suppose M has codimension ≥ 3 gap, P is a G-manifold such that P^H is connected and simply connected for any H, and M has the same isotropy everywhere as $M \times P$. Then the inclusion*

$$\pi_i K((M \times V)/G) \to \pi_i K((M \times P)/G)$$

is isomorphic up to dimension 1.

Proof: By the Lemma 4.6, as far as the fundamental groups are concerned, there is no difference between isovariance and equivariance. Now since P^H is connected and simply connected for any H, the inclusion induces isomorphisms on the fundamental groups of the equivariant as well as isovariant components. The Lemma then follows from the fact that the homotopy groups of K up to dimension 1 only depends on the fundamental groups of the components.

The next result is about the total space of the fibration for computing Wh^{Top}. This is the controlled K-theory obstruction space, which turns out to be a homology $H(M/G; K(locM/G))$ over M/G.

Lemma 6.1.2 *Suppose M has codimension ≥ 5 gap, P is a G-manifold such that P^H is* 3*-connected for any $H \in iso(M)$, and M has the same isotropy everywhere as $M \times P$. Then the inclusion*

$$H_i((M \times V)/G; K(locM \times V/G)) \to H_i((M \times P)/G; K(locM \times P/G))$$

is isomorphic up to dimension 1.

Proof: We try to apply the Lemma 2.8 about the high connectivity of maps of homologies to $M \times V \subset M \times P$. Note that the coefficient for the homology of $M \times V$ is the restriction of the coefficient for $M \times P$, because $M \times V$ is a codimension 0 submanifold. Thus we may indeed apply the Lemma.

In our situation, the corresponding homotopy functor is K. Note that with the hypothesis of the Lemma, the G-manifolds M, $M \times P$, $M \times V$ all have codimension ≥ 3 gaps. Hence the local structures $locM$, $locM \times P$, and $locM \times V$ all have simply connected equivariant as well as isovariant components. Since G is finite, we have by the Carter's vanishing theorem [C] that

$$\pi_i K(locM/G) = \pi_i K(locM \times P/G) = \pi_i K(locM \times V/G) = 0 \quad \text{for } i < -1.$$

Therefore we may take the corresponding $j_0 = -1$. In order to have an isomorphism on the homotopy groups of dimension ≤ 1, we need the inclusion to be k-connected for some $k-1>1$. In other words, we demand that $(M\times V)/G\subset(M\times P)/G$ is 3-connected over each strata.

By the Lemma 4.8, the spaces $M\times V$ and $M\times P$ have codimension ≥ 5 gap. Hence by the Lemma 4.6, we do not need to distinguish the isovariant and equivariant components as for as the fundamental groups and the homologies up to dimension $5-2=3$ are concerned. The equivariant components for the spaces are $M_\alpha^K\times V^K$ and $M_\alpha^K\times P^K$ for some equivariant component M_α^K of M with isotropy K. Since P^K is 3-connected, the inclusion $M_\alpha^K\times V^K\to M_\alpha^K\times P^K$ induces isomorphisms on the fundamental groups and the homologies up to dimension 3. Correspondingly, the inclusion induces isomorphisms on the fundamental group and the homologies up to dimension 3 on each isovariant component. By Hurewitz theorem the inclusion is 3-connected on the isovariant components. Consequently, the inclusion is also 3-connected on the strata of orbit spaces.

Remark 6.1.3 If for any $H\subset K$ and $H\neq K$ in $iso(X)$, the codimension of P^K in P^H is ≥ 2, then X only needs to have the standard codimension ≥ 3 gap.

In fact it is shown in the proof of the Lemma 4.8 that $X\times P$ has codimension $\geq 3+2=5$ gap. We may use this in place of the Lemma 4.8 in the proof above.

Lemma 6.1.4 *Suppose M has codimension ≥ 5 gap, P is a G-manifold such that P^H is 3-connected for any $H\in iso(M)$, and M has the same isotropy everywhere as $M\times P$. Then*

$$\mathrm{incl}: Wh_G^{Top,\leq 0}(M\times V)\simeq Wh_G^{Top,\leq 0}(M\times P). \tag{6.1}$$

Proof: Consider the diagram

$$\begin{array}{ccccc} Wh_G^{Top}(M\times V) & \to & H((M\times V)/G;K(locM\times V/G)) & \to & K((M\times V)/G) \\ \downarrow & & \downarrow & & \downarrow \\ Wh_G^{Top}(M\times P) & \to & H((M\times P)/G;K(locM\times P/G)) & \to & K((M\times P)/G) \end{array}$$

where the rows are fibrations. By the Lemma 7.1, the right map induce isomorphisms on the homotopy groups of dimensions up to 1. By the Lemma 7.2, the middle also induces isomorphisms on homotopy groups up to dimension 1. Then by a five lemma arguement on the homotopy exact sequences, we see that the left map induces isomorphsms on the homotopy groups up to dimension 0.

The map on the Wh_G^{Top} induces a commutative diagram

$$\begin{array}{ccc} Wh_G^{Top}(M \times V) & \rightarrow & Wh_G^{Top,\leq 0}(M \times V) \\ \downarrow & & \downarrow \\ Wh_G^{Top}(M \times P) & \rightarrow & Wh_G^{Top,\leq 0}(M \times P) \end{array}$$

Then we look at the homotopy groups. At nonpositive dimensions, the rows are isomorphic on the homotopy groups. Moreover, we just showed that the left map is also isomorphic. Therefore the right map is isomorphic. At positive dimensions, the homotopy groups of both $Wh_G^{Top,\leq 0}$ are 0 and therefore the right map is trivially isomorphic. Consequently, the right map induce isomorphisms on all the homotopy groups.

Now we are able to define the periodicity map for the truncated topological Whitehead obstruction spectrum as a composition.

$$\Pi_K : Wh_G^{Top,\leq 0}(M) \xrightarrow{\times P} Wh_G^{Top,\leq 0}(M \times P) \xrightarrow{\text{incl}^{-1}} Wh_G^{Top,\leq 0}(M \times V). \tag{6.2}$$

We conclude this section by some words about of the other half of the problem, i.e., the equivalence of $\times P$ on the Tate cohomology of the truncated topological Whitehead spectrum. This does not induce equivalence at the level of the truncated spectrum, as in the case of the inclusion. Therefore we must consider its Tate cohomology directly.

The bad thing about the Tate cohomology is that highly connected maps may not induce highly connected maps of Tate cohomology. The reason is that from the spectral sequence

$$\hat{H}^i(\mathbf{Z}_2; \pi_j K) \Rightarrow \pi_{i+j}\hat{H}(\mathbf{Z}_2; K)$$

for the homotopy groups of the Tate cohomology, we see that the computation of $\pi_n \hat{H}(\mathbf{Z}_2; K)$ involves all the homotopy groups of K. Consequently, we can not compute $\hat{H}(\mathbf{Z}_2; Wh^{Top,\leq 0})$ by "approximating" $Wh^{Top,\leq 0}$. On the other hand, since the truncation is not an exact operation, $Wh^{Top,\leq 0}$ does not fit into any fibration. Hence the exactness of the Tate cohomology functor is useless here.

6.2. Periodicity Map

Construct the diagram

$$\begin{array}{ccc}
S_G^{-\infty}(M, \mathrm{rel}\ \partial) & \xrightarrow{\tau} & \hat{H}(\mathbf{Z}_2; Wh^{Top,\leq 0}(M)) \\
\downarrow \times P & & \downarrow \times P \\
S_G^{-\infty}(M \times P, \mathrm{rel}\ \partial) & \xrightarrow{\tau_P} & \hat{H}(\mathbf{Z}_2; Wh^{Top,\leq 0}(M \times P)) \\
\uparrow \mathrm{incl}_S & & \uparrow \mathrm{incl}_K \\
S_G^{-\infty}(M \times V, \mathrm{rel}\ \partial) & \xrightarrow{\tau_V} & \hat{H}(\mathbf{Z}_2; Wh^{Top,\leq 0}(M \times V))
\end{array}$$

By the Lemma 7.4, incl_K induces an equivalence on the truncated Whitehead groups and therefore an equivalence on its Tate cohomologies. By the Theorem 5.2, we have $\mathrm{incl}_S \circ \Pi^{-\infty} = \times P$ and

$$\begin{array}{rl}
& \tau_V \circ \Pi^{-\infty} \\
\simeq & \mathrm{incl}_K^{-1} \circ \tau_P \circ \mathrm{incl}_S \circ \Pi^{-\infty} \\
\simeq & \mathrm{incl}_K^{-1} \circ \tau_P \circ (\times P) \\
\simeq & \mathrm{incl}_K^{-1} \circ (\times P) \circ \tau \\
\simeq & \Pi_K \circ \tau.
\end{array}$$

The homotopy commutativity then induces the unstable periodicity map Π in the following diagram

$$\begin{array}{ccccc}
S_G(M \times V, \mathrm{rel}\ \partial) & \to & S_G^{-\infty}(M \times V, \mathrm{rel}\ \partial) & \xrightarrow{\tau_V} & \hat{H}(\mathbf{Z}_2; Wh_G^{Top,\leq 0}(M \times V)) \\
\uparrow \Pi & & \uparrow \Pi^{-\infty} & & \uparrow \Pi_K \\
S_G(M, \mathrm{rel}\ \partial) & \to & S_G^{-\infty}(M, \mathrm{rel}\ \partial) & \xrightarrow{\tau} & \hat{H}(\mathbf{Z}_2; Wh_G^{Top,\leq 0}(M))
\end{array}$$

where the rows are the destablization fibrations. Since $\Pi^{-\infty}$ is an equivalence, we have by the stable periodicity theorem (Theorem 5.1)

$$Fib(\Pi) \simeq \Omega Fib(\Pi_K) \simeq \Omega Fib(\times P).$$

Theorem 6.2.1 *Let P be a periodicity manifold and V be the tangential G-representation of P at a fixed point. Suppose that M is a homotopically stratified topological G-manifold that has the dimension ≥ 5, has the codimension ≥ 6 gap, P^H is 3-connected for any $H \in iso(X)$, and M, $M \times P$ have the same isotropy everywhere. Then there is a natural periodicity map*

$$\Pi : S_G(M, rel\ \partial) \to S_G(M \times V, rel\ \partial),$$

whose fibre is the loop of the fibre of the map

$$\times P : \hat{H}(\mathbf{Z}_2; Wh_G^{Top,\leq 0}(M)) \to \hat{H}(\mathbf{Z}_2; Wh_G^{Top,\leq 0}(M \times P)).$$

The classical example

$$P = \underbrace{\mathbf{CP}^2 \times \cdots \times \mathbf{CP}^2}_{S}$$

induced from the G-action on S in the section 2.5 does not satisfy the 3-connectedness condition required in the Theorem. However, following the remark after the Lemma 4.15, we may use the quaternionic projective space $\mathbf{HP}^2$ instead of the complex one. This satisfies the 3-connectedness condition with

$$V = \mathbf{R}^8 \times \mathbf{R}S.$$

In particular, if G is an odd order group and $S = G$, then V has all the isotropies and has gap much bigger than 3. It follows that we have the periodicity map

$$\Pi : S_G(M, \text{rel } \partial) \to S_G(M \times D^8 \times D(\mathbf{R}S), \text{rel } \partial) \tag{6.3}$$

for any G-manifold M with codimension$\geq$ 5 gap.

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