SCIE1110 Answers

Exercise 1.

$$\begin{aligned}
2503 &= 1251 \times 2 + 1 = (625 \times 2 + 1) \times 2 + 1\\
&= (312 \times 2 + 1) \times 2^2 + 1 \times 2^1 + 1\\
&= (156 \times 2) \times 2^3 + 1 \times 2^2 + 1 \times 2^1 + 1\\
&= (78 \times 2) \times 2^4 + 1 \times 2^2 + 1 \times 2^1 + 1\\
&= (39 \times 2) \times 2^5 + 1 \times 2^2 + 1 \times 2^1 + 1\\
&= (19 \times 2 + 1) \times 2^6 + 1 \times 2^2 + 1 \times 2^1 + 1\\
&= (9 \times 2 + 1) \times 2^7 + 1 \times 2^6 + 1 \times 2^2 + 1 \times 2^1 + 1\\
&= (4 \times 2 + 1) \times 2^8 + 1 \times 2^7 + 1 \times 2^6 + 1 \times 2^2 + 1 \times 2^1 + 1\\
&= (2 \times 2) \times 2^9 + 1 \times 2^8 + 1 \times 2^7 + 1 \times 2^6 + 1 \times 2^2 + 1 \times 2^1 + 1\\
&= 1 \times 2^{11} + 1 \times 2^8 + 1 \times 2^7 + 1 \times 2^6 + 1 \times 2^2 + 1 \times 2^1 + 1\\
&= 100111000111_{[2]};\\
2503 &= 834 \times 3 + 1 = (278 \times 3) \times 3 + 1 = (92 \times 3 + 2) \times 3^2 + 1\\
&= (30 \times 3 + 2) \times 3^3 + 2 \times 3^2 + 1 = (10 \times 3) \times 3^4 + 2 \times 3^3 + 2 \times 3^2 + 1\\
&= (3 \times 3 + 1) \times 3^5 + 2 \times 3^3 + 2 \times 3^2 + 1\\
&= 1 \times 3^7 + 1 \times 3^5 + 2 \times 3^3 + 2 \times 3^2 + 1 = 1012201_{[3]};\\
2503 &= 500 \times 5 + 3 = 4 \times 5^4 + 3 = 40003_{[5]};\\
2503 &= 1012201_{[3]} = (1_{[3]}01_{[3]}22_{[3]}01_{[3]})_{[9]} = 1181_{[9]};\\
2503 &= 28 \times 12 + 7 = (2 \times 12 + 4) \times 12 + 6\\
&= 2 \times 12^2 + 4 \times 12^1 + 6 = 2, 4, 6_{[12]};\\
2503 &= 40003_{[5]} = (40_{[5]}, 003_{[5]})_{[25]} = 20, 3_{[25]};\\
2503 &= 1012201_{[3]} = (1_{[3]}, 012_{[3]}, 201_{[3]})_{[27]} = 1, 5, 19_{[27]}.
\end{aligned}$$

Exercise 2.

$$
\begin{aligned}
2038040 &= 33967 \times 60 + 20 = (566 \times 60 + 7) \times 60 + 20 \\
&= (9 \times 60 + 26) \times 60^2 + 7 \times 60 + 20 \\
&= 9 \times 60^3 + 26 \times 60^2 + 7 \times 60 + 20 = 9, 26, 7, 20_{[60]}; \\
2038040 &= 2, 03, 80, 40_{[100]} = 2, 3, 80, 40_{[100]}; \\
2038040_{[9]} &= 2 \times 9^6 + 3 \times 9^4 + 8 \times 9^3 + 4 \times 9 = 1088433; \\
2038040_{[9]} &= 203(2 \times 3 + 2)0(2 \times 3 + 2)0_{[3^2]} \\
&= [(0,2)_{[3]}, (0,0)_{[3]}, (0,3)_{[3]}, (2,2)_{[3]}, (0,0)_{[3]}, (2,2)_{[3]}, (0,0)_{[3]}]_{[3^2]} \\
&= 2000322002200_{[3]}; \\
2038040_{[9]} &= 1088433 = 217686 \times 5 + 3 = (43537 \times 5 + 1) \times 5 + 3 \\
&= (8707 \times 5 + 2) \times 5^2 + 1 \times 5 + 3 \\
&= (1741 \times 5 + 2) \times 5^3 + 2 \times 5^2 + 1 \times 5 + 3 \\
&= (348 \times 5 + 1) \times 5^4 + 2 \times 5^3 + 2 \times 5^2 + 1 \times 5 + 3 \\
&= (69 \times 5 + 3) \times 5^5 + 1 \times 5^4 + 2 \times 5^3 + 2 \times 5^2 + 1 \times 5 + 3 \\
&= (13 \times 5 + 4) \times 5^6 + 3 \times 5^5 + 1 \times 5^4 \\
&\quad + 2 \times 5^3 + 2 \times 5^2 + 1 \times 5 + 3 \\
&= (2 \times 5 + 3) \times 5^7 + 4 \times 5^6 + 3 \times 5^5 + 1 \times 5^4 \\
&\quad + 2 \times 5^3 + 2 \times 5^2 + 1 \times 5 + 3 \\
&= 234312213_{[5]}; \\
2038040_{[9]} &= 234312213_{[5]} = [(02)_{[5]}, (34)_{[5]}, (31)_{[5]}, (22)_{[5]}, (13)_{[5]}]_{[5^2]} \\
&= 2, 19, 16, 12, 8_{[25]}; \\
2, 3, 80, 40_{[144]} &= 2 \times 144^3 + 3 \times 144^2 + 80 \times 144^1 + 40 = 6045736; \quad \text{(base 10)} \\
2, 3, 80, 40_{[144]} &= 2, 3, 80, 40_{[12^2]} = [(0,2)_{[12]}, (0,3)_{[12]}, (6, 8_{[12]}), (3, 4_{[12]})]_{[12^2]} \\
&= 2, 0, 3, 6, 8, 3, 4_{[12]}; \\
&\text{(Note: } 80 = 6 \times 12 + 8 = 6, 8_{[12]},\ 40 = 3 \times 12 + 4 = 3, 4_{[12]}) \\
2038040_{[16]} &= 2, 0, 3, 8, 0, 4, 0_{[2^4]} = [10_{[2]}, 0_{[2]}, 11_{[2]}, 1000_{[2]}, 0_{[2]}, 100_{[2]}, 0_{[2]}]_{[2^4]} \\
&= [0010_{[2]}, 0000_{[2]}, 0011_{[2]}, 1000_{[2]}, 0000_{[2]}, 0100_{[2]}, 0000_{[2]}]_{[2^4]} \\
&= 1000000111000000001000000_{[2]}; \\
2038040_{[16]} &= 1000000111000000001000000_{[2]} \\
&= [10_{[2]}, 000_{[2]}, 000_{[2]}, 111_{[2]}, 000_{[2]}, 000_{[2]}, 001_{[2]}, 000_{[2]}, 000_{[2]}]_{[2^3]} \\
&= 2, 0, 0, 7, 0, 0, 1, 0, 0_{[2^3]} = 200700100_{[8]}.
\end{aligned}
$$

Exercise 3.

We omit $\bar{0}+\bar{n}=\bar{n}=\bar{n}+\bar{0}$, $\bar{0}\times\bar{n}=\bar{0}=\bar{n}\times\bar{0}$.

$+$	$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{4}$
$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{4}$	$\bar{0}$
$\bar{2}$	$\bar{3}$	$\bar{4}$	$\bar{0}$	$\bar{1}$
$\bar{3}$	$\bar{4}$	$\bar{0}$	$\bar{1}$	$\bar{2}$
$\bar{4}$	$\bar{0}$	$\bar{1}$	$\bar{2}$	$\bar{3}$

$\times$	$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{4}$
$\bar{1}$	$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{4}$
$\bar{2}$	$\bar{2}$	$\bar{4}$	$\bar{1}$	$\bar{3}$
$\bar{3}$	$\bar{3}$	$\bar{1}$	$\bar{4}$	$\bar{2}$
$\bar{4}$	$\bar{4}$	$\bar{3}$	$\bar{2}$	$\bar{1}$

The table is obtained by keeping the "last digit" of the addition and multiplication tables for base 5 (and deleting all the "non-last digits").

Exercise 4.

Denote by $7|n$ that n is divisible by 7. Then

$$\begin{aligned} 7|10a+b &\iff 7|2(10a+b)=20a+2b=(3\times 7-1)a+2b \\ &\iff 7|-a+2b \iff 7|-(-a+2b)=a-2b. \end{aligned}$$

Exercise 5.

First rule: In $\mathbb{Z}_{21}$, we have

$$\overline{10}=\overline{10},\ \overline{10}^2=-\bar{5},\ \overline{10}^3=-\bar{8},\ \overline{10}^4=\bar{4},\ \overline{10}^5=-\bar{2},\ \overline{10}^6=\bar{1},\ \overline{10}^7=\overline{10},\dots$$

Then we have $\overline{10}^{6k+i}=\overline{10}^i$, and a decimal expression $N_kN_{k-1}\dots N_2N_1N_0$, $N_i\in 0,1,\dots,9$, is divisible by 21 if and only if

$$N_0+10N_1-5N_2-8N_3+4N_4-2N_5+N_6+10N_7-5N_8-8N_9+\cdots$$

is divisible by 21.

Second rule: By $\overline{10}^6=\bar{1}$ in $\mathbb{Z}_{21}$, we may regard a decimal expression as based on 10^6

$$N_kN_{k-1}\dots N_2N_1N_0=(\dots,M_3,M_2,M_1,M_0)_{[10^6]},$$

with

$$M_0=N_5N_4N_3N_2N_1N_0,\ M_1=N_{11}N_{10}N_9N_8N_7N_6,\ \dots$$

Then the number is divisible by 21 if and only if $M_0+M_1+\cdots$ is divisible by 21.

Third rule: Divisible by 21 is equivalent to divisible by 3 and by 7. This means the following hold

1. $N_0+N_1+N_2+N_3+N_4+N_5+N_6+N_7+N_8+\cdots$ is divisible by 3,

2. $N_0 + 3N_1 + 2N_2 - N_3 - 3N_4 - 2N_5 + N_6 + 3N_7 + 2N_8 + \cdots$ is divisible by 7.

Exercise 6.

In $\mathbb{Z}_3$, we have

$$\bar{5} = -\bar{1},\ \bar{5}^2 = (-\bar{1})^2 = \bar{1},\ \bar{5}^3 = (-\bar{1})^3 = -\bar{1},\ \bar{5}^4 = (-\bar{1})^4 = \bar{1},\ \ldots$$

Therefore $(N_k N_{k-1} \ldots N_2 N_1 N_0)_{[5]}$ is divisible by 3 if and only if $N_0 - N_1 + N_2 - N_3 + N_4 - \cdots$ is divisible by 3.

Exercise 7.

(1) In $\mathbb{Z}_T = \mathbb{Z}_{10}$, we have $\overline{11} = \bar{1}$. Therefore for an expression in base 11

$$\cdots N_3 N_2 N_1 N_0 = N_0 + N_1 \times 11 + N_2 \times 11^2 + N_3 \times 11^3 + \cdots,$$

we have

$$\overline{\cdots N_3 N_2 N_1 N_0} = N_0 + N_1 \times \bar{1} + N_2 \times \bar{1}^2 + N_3 \times \bar{1}^3 + \cdots = N_0 + N_1 + N_2 + N_3 + \cdots.$$

Therefore the number is divisible by T if and only if $N_0 + N_1 + N_2 + N_3 + \cdots$ is divisible by T.

(2) $2T2T \cdots 2T$ is divisible by 11 if and only if $T + 2 + T + 2 + \cdots + T + 2 = Tn + 2n$ is divisible by T. This is the same as $2n$ divisible by $T = 10$, which is the same as n being a multiple of 5.

Exercise 8.

Converting to sexagesimal system, we have

$$\begin{aligned}
2.25 &= 2 + \frac{1}{4} = 2 + \frac{15}{60} = 2;15 \\
13.5 &= 13 + \frac{1}{2} = 13 + \frac{30}{60} = 13;30 \\
135 &= 2 \times 60 + 15 = 2,15; \\
225 &= 3 \times 60 + 45 = 3,45;
\end{aligned}$$

Since the Babylonians do not distinguish between ";" and ",", they will denote both 2.25 and 135 as 2,15 (actually 2 15).

Exercise 9.

The Babylonian $36,6$ could mean any of the following in decimal expression

$$\begin{aligned}
&36,6 = 36 \times 60 + 6 = 2166, \\
&36,6 = 36 \times 60^2 + 6 \times 60 = 2166 \times 60 = 129960, \\
&36,6 = 36 \times 60^3 + 6 \times 60^2 = 129960 \times 60 = 7797600.
\end{aligned}$$

The decimal value 7797600 is closest to 7.8 million and is therefore what Babylonians meant.

Exercise 10.

We have

$$1 = \frac{1}{2} + \frac{1}{3} + \frac{1}{6}.$$

The Egyptians can write 1, and can also write $\frac{1}{2} + \frac{1}{3} + \frac{1}{6}$ to express the same number.

We can further get two expressions by unit fractions

$$0.2 = \frac{1}{5} = \frac{1}{2 \times 5} + \frac{1}{3 \times 5} + \frac{1}{6 \times 5} = \frac{1}{10} + \frac{1}{15} + \frac{1}{30}.$$

Exercise 11.

The problem lies in the proper understanding of "no dog". The first axiom says "No (dog has 5 legs)", where "no" and "dog" play separate roles (logical, and terminological). The second axiom says "A dog has 4 more legs than (no dog)", where "no dog" is taken as a combined terminology, and "no" has no logical role.

The fallacy shows that the ambiguity in our common language makes it not adequate for expressing rigorous mathematics. There is a need to develop an "artificial language" in order to express mathematics in the most rigorous way. Such a philosophy underlies "The Principles of Mathematics" (first published in 1903) by Bertrand Russell. In "Principia Mathematica" (3 volumes published in 1910, 1912, 1913) by Alfred North Whitehead and Bertrand Russell, such "artificial language" used to develop the most basic mathematics in the most rigorous way. However, it took them hundreds of pages to prove $1 + 1 = 2$[1]!

Exercise 12.

The 4th number is indeed 16. But the fifth number is 31, not $3^5 = 32$. In fact, there is a general formula for the number (see http://oeis.org/A000127)

$$a(n) = \frac{1}{24}n^4 - \frac{1}{4}n^3 + \frac{23}{24}n^2 - \frac{3}{4}n + 1.$$

The problem show the importance of proof for mathematical statements, no matter how many confirmations you can get.

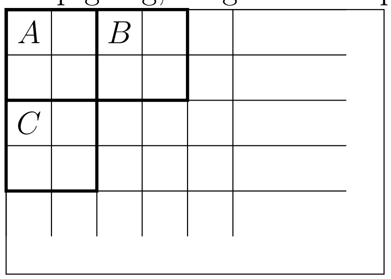
Exercise 13.

The tiling covering A is the unique one as indicated. Then the tile covering B is the unique one as indicated. Keep, going, we find the unique tiles for the first two rows.

[1] The equality is defined in "From this proposition it will follow, when arithmetical addition has been defined, that 1+1=2." on page 379 of volume 1. The actual proof is completed on page 86 of volume 2.

After the first two rows, the tiling covering C is the unique one as indicated. Like the first two rows, we find the the unique tiles for the third and fourth rows.

Keep going, we get the unique tiles covering $2n-1$ and $2n$ rows.



m

n

We conclude that $m \times n$ grid can be covered by 2×2 squares if and only if both m, n are even. Moreover, the tiling is unique.

Exercise 14.

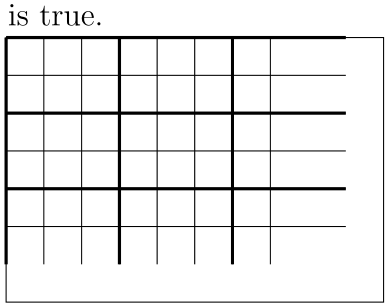
Since 2×3 domino occupies 6 squares, a necessary condition for $m \times n$ to be tiled is that mn is a multiple of 6.

Conversely, suppose mn is a multiple of 6. Then we only need to consider two possibilities: (1) m is divisible by 2, n is divisible by 3; (2) m is divisible by 6, and $n \geq 2$. The other possibilities simply switches m and n and do not affect the answer.

In case (1), the grid can be tiled by putting 2×3 dominos horizontally. See left picture.

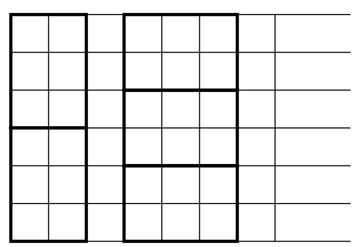
In case (2), we see in the right picture that two 2×3 dominos can tile a 6×2 grid, and three 2×3 dominos can tile a 6×3 grid. By repeating the first tiling a times and the second tiling b times, we can tile a $6 \times (2a + 3b)$ grid. Since any number $n \geq 2$ can be written as $2a + 3b$ for some non-negative integers a, b, we find that $6 \times n$ grid can be tiled. Furthermore, if m is divisible by 6, then stacking $m/6$ copies of such tilings shows that $m \times n$ grid can be tiled.

We conclude that an $m \times n$ grid can be tiled by 2×3 dominos if and only if either (1) or (2) is true.



m

n



Exercise 15.

If $a^3 = 2a^2 + a - 2$, then clearly $F(n) = a^n$ satisfies $F(n) = 2F(n-1) + F(n-2) - F(n-3)$. (We will deal with $F(0) = 0$, $F(1) = 1$, $F(2) = 2$ later.)

We solve $a^3 = 2a^2 + a - 2$ and find three solutions $a = -1, 1, 2$. This means that $F_1(n) = (-1)^n$, $F_2(n) = 1^n = 1$, $F_2(n) = 2^n$ all satisfy $F(n) = 2F(n-1) + F(n-2) - F(n-3)$. Then their combination $F(n) = A1^n + B(-1)^n + C2^n = A + (-1)^n B + C2^n$ satisfies $F(n) = 2F(n-1) + F(n-2) - F(n-3)$. Then we substitute the special values for $n = 0, 1, 2$ to get

$$0 = A + B + C, \quad 1 = A - B + 2C, \quad 2 = A + B + 4C.$$

Solving the system, we get $A = -\frac{1}{2}$, $B = -\frac{1}{6}$, $C = \frac{2}{3}$. Therefore

$$F(n) = -\frac{1}{2} - \frac{1}{6}(-1)^n + \frac{2}{3}2^n = \frac{1}{6}\left(2^{n+2} + (-1)^{n+1} - 3\right).$$

Exercise 16.

To show Axiom 1 is independent, we need an example satisfying Axioms 2 and 3 but fails Axiom 1. This is given by one member M, and two committees $C_1 = \{M\}$ and $C_2 = \{M\}$ (geometrically, two lines on the plane crossing at a single point).

To show Axiom 2 is independent, we need an example satisfying Axioms 1 and 3 but fails Axiom 2. This is given by three members M_1, M_2, M_3, and one committees $C = \{M_1, M_2, M_3\}$ (geometrically, one line and three points on the line).

To show Axiom 3 is independent, we need an example satisfying Axioms 1 and 2 but fails Axiom 3. This is given by three members M_1, M_2, M_3, and two committees $C_1 = \{M_1, M_2, M_3\}$, $C_3 = \{M_1, M_2, M_3\}$ (geometrically, we do not have such model using straight lines, but we can have two S-shaped curves intersecting at three points).

Exercise 17.

The question does not specify the special element 1. We need to assume that the special element to be some $I \in \mathbb{Z}$ (I for "initial", not necessarily equal to our usual 1).

The first axiom fails because $I = (I-1)'$ for $I - 1 \in \mathbb{Z}$.

The second axiom holds because for $a, b \in \mathbb{Z}$, we know $a + 1 = b + 1$ implies $a = b$.

The third axiom fails because $S = (a-1) + \mathbb{N} = \{a, a+1, a+2, \dots\}$ satisfies the property but is not the whole $\mathbb{Z}$.

Exercise 18.

To show that the first axiom is independent of the second and third, we need to find a model such that the first axiom is wrong, but the second and third axioms are correct. Let $X = \{I\}$ and $I' = I$. Then the first axiom fails, the second holds because there is only one element, and the third holds because I is already the whole X.

To show that the second axiom is independent of the first and third, we need to find a model such that the second axiom is wrong, but the first and third axioms are correct. Let

$X = \{I, J\}$ and $I' = J$, $J' = J$. Then the first axiom holds because $x' = J \neq I$ for all x, the second fails because $I' = J'$ and $I \neq J$, and the third holds because if $I \in S$ and $I' = J \in S$, then S must contain $X = \{I, J\}$.

To show that the third axiom is independent of the first and second, we need to find a model such that the third axiom is wrong, but the first and second axioms are correct. Let $X = \mathbb{N} \cup \{a, b\}$. Let the special element $I = 1 \in \mathbb{N}$, and let the successor map be

$$n' = n + 1 \text{ for } n \in \mathbb{N}, \quad a' = b, \quad b' = a.$$

Then the first two axioms hold, and $S = \mathbb{N}$ satisfies the conditions in the third axiom, but $S \neq X$.

Exercise 19.

Let $I = 1$, $1' = 3$, $2' = 2$, and $n' = n + 1$ for $n \geq 3$. Then the first two axioms are satisfied. Moreover, $S = \mathbb{N} - 2 = \{1, 3, 4, 5, \dots\}$ satisfies the conditions in the third axiom, but $S \neq X$.

The example also shows that it is possible to construct counterexample using $I = 1$.

If $n' = n + 1$, then any number ≥ 2 can be expressed as n'. The first axiom then forces us to take $I = 1$. Therefore we get the usual $I = 1$ and $n' = n + 1$.

Exercise 20.

(1)

$$\begin{aligned}
14205 &= 1775 \times 8 + 5 = (221 \times 8 + 7) \times 8 + 5 \\
&= (27 \times 8 + 5) \times 8^2 + 7 \times 8 + 5 \\
&= (3 \times 8 + 3) \times 8^3 + 5 \times 8^2 + 7 \times 8 + 5 \\
&= 3 \times 8^4 + 3 \times 8^3 + 5 \times 8^2 + 7 \times 8 + 5 = 33575_{[8]}.
\end{aligned}$$

(2) In $\mathbb{Z}_7$, we have $\bar{8} = \bar{1}$. Therefore

$$\begin{aligned}
\overline{33575_{[8]}} &= \bar{3} \times \bar{8}^4 + \bar{3} \times \bar{8}^3 + \bar{5} \times \bar{8}^2 + \bar{7} \times \bar{8} + \bar{5} \\
&= \bar{3} \times \bar{1}^4 + \bar{3} \times \bar{1}^3 + \bar{5} \times \bar{1}^2 + \bar{7} \times \bar{1} + \bar{5} \\
&= \bar{3} + \bar{3} + \bar{5} + \bar{7} + \bar{5} = \overline{3 + 3 + 5 + 7 + 5} = \bar{2}.
\end{aligned}$$

Since the result is not $\bar{0}$, the number is not divisible by 7.

(3) In $\mathbb{Z}_3$, we have $\bar{8} = -\bar{1}$. Therefore

$$\begin{aligned}
\overline{33575_{[8]}} &= \bar{3} \times \bar{8}^4 + \bar{3} \times \bar{8}^3 + \bar{5} \times \bar{8}^2 + \bar{7} \times \bar{8} + \bar{5} \\
&= \bar{3} \times (-\bar{1})^4 + \bar{3} \times (-\bar{1})^3 + \bar{5} \times (-\bar{1})^2 + \bar{7} \times (-\bar{1}) + \bar{5} \\
&= \bar{3} - \bar{3} + \bar{5} - \bar{7} + \bar{5} = \overline{3 - 3 + 5 - 7 + 5} = \bar{0}.
\end{aligned}$$

Since the result is $\bar{0}$, the number is divisible by 3.

Exercise 21.

The first and third grids can be tiled.

For the second one, we color the tiles alternatively by white (w), gray (g), and black (b). The squares in any 3×1 domino should have one color each. Therefore a tilable grid should have equal numbers of w, g, b colors. Since the numbers are actually $12w, 10g, 11b$, which are not equal, the grid cannot be tiled.