SCIE1110 Answers

Exercise 1.

We omit $0+n=n=n+0$, $0\times n=0=n\times 0$.

$$\begin{array}{c|cccc} + & 1 & 2 & 3 & 4 \\ \hline 1 & 2 & 3 & 4 & 10 \\ 2 & 3 & 4 & 10 & 11 \\ 3 & 4 & 10 & 11 & 12 \\ 4 & 10 & 11 & 12 & 13 \end{array} \qquad \begin{array}{c|cccc} \times & 1 & 2 & 3 & 4 \\ \hline 1 & 1 & 2 & 3 & 4 \\ 2 & 2 & 4 & 11 & 13 \\ 3 & 3 & 11 & 14 & 22 \\ 4 & 4 & 13 & 22 & 31 \end{array}$$

Exercise 2.

$$\begin{aligned}
2503 &= 1251\times 2+1 = (625\times 2+1)\times 2+1 \\
&= (312\times 2+1)\times 2^2+1\times 2^1+1 \\
&= (156\times 2)\times 2^3+1\times 2^2+1\times 2^1+1 \\
&= (78\times 2)\times 2^4+1\times 2^2+1\times 2^1+1 \\
&= (39\times 2)\times 2^5+1\times 2^2+1\times 2^1+1 \\
&= (19\times 2+1)\times 2^6+1\times 2^2+1\times 2^1+1 \\
&= (9\times 2+1)\times 2^7+1\times 2^6+1\times 2^2+1\times 2^1+1 \\
&= (4\times 2+1)\times 2^8+1\times 2^7+1\times 2^6+1\times 2^2+1\times 2^1+1 \\
&= (2\times 2)\times 2^9+1\times 2^8+1\times 2^7+1\times 2^6+1\times 2^2+1\times 2^1+1 \\
&= 1\times 2^{11}+1\times 2^8+1\times 2^7+1\times 2^6+1\times 2^2+1\times 2^1+1 \\
&= 100111000111_{[2]}; \\
2503 &= 834\times 3+1 = (278\times 3)\times 3+1 = (92\times 3+2)\times 3^2+1 \\
&= (30\times 3+2)\times 3^3+2\times 3^2+1 = (10\times 3)\times 3^4+2\times 3^3+2\times 3^2+1 \\
&= (3\times 3+1)\times 3^5+2\times 3^3+2\times 3^2+1 \\
&= 1\times 3^7+1\times 3^5+2\times 3^3+2\times 3^2+1 = 1012201_{[3]}; \\
2503 &= 500\times 5+3 = 4\times 5^4+3 = 40003_{[5]}; \\
2503 &= 1012201_{[3]} = (1_{[3]}01_{[3]}22_{[3]}01_{[3]})_{[9]} = 1181_{[9]}; \\
2503 &= 28\times 12+7 = (2\times 12+4)\times 12+6 \\
&= 2\times 12^2+4\times 12^1+6 = 2,4,6_{[12]}; \\
2503 &= 40003_{[5]} = (40_{[5]},003_{[5]})_{[25]} = 20,3_{[25]}; \\
2503 &= 1012201_{[3]} = (1_{[3]},012_{[3]},201_{[3]})_{[27]} = 1,5,19_{[27]}.
\end{aligned}$$

Exercise 3.

$$
\begin{aligned}
2038040 &= 33967 \times 60 + 20 = (566 \times 60 + 7) \times 60 + 20\\
&= (9 \times 60 + 26) \times 60^2 + 7 \times 60 + 20\\
&= 9 \times 60^3 + 26 \times 60^2 + 7 \times 60 + 20 = 9, 26, 7, 20_{[60]};\\
2038040 &= 2, 03, 80, 40_{[100]} = 2, 3, 80, 40_{[100]};\\
2038040_{[9]} &= 2 \times 9^6 + 3 \times 9^4 + 8 \times 9^3 + 4 \times 9 = 1088433;\\
2038040_{[9]} &= 203(2 \times 3 + 2)0(2 \times 3 + 2)0_{[3^2]}\\
&= [(0, 2)_{[3]}, (0, 0)_{[3]}, (0, 3)_{[3]}, (2, 2)_{[3]}, (0, 0)_{[3]}, (2, 2)_{[3]}, (0, 0)_{[3]}]_{[3^2]}\\
&= 2000322002200_{[3]};\\
2038040_{[9]} &= 1088433 = 217686 \times 5 + 3 = (43537 \times 5 + 1) \times 5 + 3\\
&= (8707 \times 5 + 2) \times 5^2 + 1 \times 5 + 3\\
&= (1741 \times 5 + 2) \times 5^3 + 2 \times 5^2 + 1 \times 5 + 3\\
&= (348 \times 5 + 1) \times 5^4 + 2 \times 5^3 + 2 \times 5^2 + 1 \times 5 + 3\\
&= (69 \times 5 + 3) \times 5^5 + 1 \times 5^4 + 2 \times 5^3 + 2 \times 5^2 + 1 \times 5 + 3\\
&= (13 \times 5 + 4) \times 5^6 + 3 \times 5^5 + 1 \times 5^4\\
&\quad + 2 \times 5^3 + 2 \times 5^2 + 1 \times 5 + 3\\
&= (2 \times 5 + 3) \times 5^7 + 4 \times 5^6 + 3 \times 5^5 + 1 \times 5^4\\
&\quad + 2 \times 5^3 + 2 \times 5^2 + 1 \times 5 + 3\\
&= 234312213_{[5]};\\
2038040_{[9]} &= 234312213_{[5]} = [(02)_{[5]}, (34)_{[5]}, (31)_{[5]}, (22)_{[5]}, (13)_{[5]}]_{[5^2]}\\
&= 2, 19, 16, 12, 8_{[25]};\\
2, 3, 80, 40_{[144]} &= 2 \times 144^3 + 3 \times 144^2 + 80 \times 144^1 + 40 = 6045736; \quad \text{(base 10)}\\
2, 3, 80, 40_{[144]} &= 2, 3, 80, 40_{[12^2]} = [(0, 2)_{[12]}, (0, 3)_{[12]}, (6, 8_{[12]}), (3, 4_{[12]})]_{[12^2]}\\
&= 2, 0, 3, 6, 8, 3, 4_{[12]};\\
&\text{(Note: } 80 = 6 \times 12 + 8 = 6, 8_{[12]},\ 40 = 3 \times 12 + 4 = 3, 4_{[12]})\\
2038040_{[16]} &= 2, 0, 3, 8, 0, 4, 0_{[2^4]} = [10_{[2]}, 0_{[2]}, 11_{[2]}, 1000_{[2]}, 0_{[2]}, 100_{[2]}, 0_{[2]}]_{[2^4]}\\
&= [0010_{[2]}, 0000_{[2]}, 0011_{[2]}, 1000_{[2]}, 0000_{[2]}, 0100_{[2]}, 0000_{[2]}]_{[2^4]}\\
&= 1000000011100000000001000000_{[2]};\\
2038040_{[16]} &= 1000000011100000000001000000_{[2]}\\
&= [10_{[2]}, 000_{[2]}, 000_{[2]}, 111_{[2]}, 000_{[2]}, 000_{[2]}, 001_{[2]}, 000_{[2]}, 000_{[2]}]_{[2^3]}\\
&= 2, 0, 0, 7, 0, 0, 1, 0, 0_{[2^3]} = 200700100_{[8]}.
\end{aligned}
$$

Exercise 4.

(1)

$$
\begin{array}{cccccccc}
 & 1 & 0 & 1 & 0 & 0 & 1 & 1 \\
+ & & {}_1 & 1 & 0_1 & 1_1 & 0_1 & 1 \\
\hline
 & 1 & 1 & 0 & 1 & 0 & 0 & 0
\end{array}
$$

(2)

$$
\begin{array}{cccccccc}
 & 1 & 0 & 1 & 0 & 0 & 1 & 1 \\
- & {}^1 & {}^1 & 1^1 & 0^1 & 1 & 0 & 1 \\
\hline
 & & 1 & 1 & 1 & 1 & 1 & 0
\end{array}
$$

(3)

$$
\begin{array}{cccccccccccc}
 & & & & & 1 & 0 & 1 & 0 & 0 & 1 & 1 \\
\times & & & & & & & 1 & 0 & 1 & 0 & 1 \\
\hline
 & & & & & 1 & 0 & 1 & 0 & 0 & 1 & 1 \\
 & & & 1 & 0 & 1 & 0 & 0 & 1 & 1 & & \\
 & 1 & 0_1 & 1 & 0_1 & 0_1 & 1_1 & 1 & & & & \\
\hline
 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1
\end{array}
$$

(4)

$$
\begin{array}{cccccccccccc}
 & & & & & & & & & & 1 & 1 \\
\cline{6-12}
1 & 0 & 1 & 0 & 1 & \big) & 1 & 0 & 1 & 0 & 0 & 1 \; 1 \\
 & & & & & & & 1 & 0 & 1 & 0 & 1 \\
\cline{7-12}
 & & & & & & & 1 & 0 & 1 & 0 & 0 \; 1 \\
 & & & & & & & & 1 & 0 & 1 & 0 \; 1 \\
\cline{8-12}
 & & & & & & & & 1 & 0 & 1 & 0 \; 0
\end{array}
$$

The quotient of $1010011 \div 10101$ is 11, with remainder 10100.

(5)

$$
\begin{array}{cccccc}
 & 1 & 0 & 1 & 0 & 1 \\
+ & & & 1 & 0 & 1 \\
\hline
 & 1 & 1 & 0 & 1 & 0 \\
- & & 1 & 0 & 0 & 1 \\
\hline
 & 1 & 0 & 0 & 0 & 1
\end{array}
$$

We get $10101 + 101 - 1001 = 10001$.

(6)

$$
\begin{array}{cccccccc}
 & & & 1 & 0 & 1 & 0 & 1 \\
\times & & & 1 & 0 & 1 & & \\
\hline
 & & & 1 & 0 & 1 & 0 & 1 \\
 & 1 & 0 & 1 & 0 & 1 & & \\
\hline
 & 1 & 1 & 0 & 1 & 0 & 0 & 1
\end{array}
$$

Then

$$
\begin{array}{cccccccccccccccl}
 & & & & & & & & 1 & 0 & 1 & 1. & 1 & 0 & 1 & \\ \cline{5-15}
1 & 0 & 0 & 1 &) & 1 & 1 & 0 & 1 & 0 & 0 & 1 & & & & \\
 & & & & & 1 & 0 & 0 & 1 & & & & & & & \\ \cline{6-11}
 & & & & & & 1 & 0 & 0 & 0 & 0 & & & & & \\
 & & & & & & & 1 & 0 & 0 & 1 & & & & & \\ \cline{7-12}
 & & & & & & & & 1 & 1 & 1 & 1 & & & & \\
 & & & & & & & & 1 & 0 & 0 & 1 & & & & \\ \cline{9-13}
 & & & & & & & & & 1 & 1 & 0 & 0 & & & {\color{red}\Leftarrow} \\
 & & & & & & & & & 1 & 0 & 0 & 1 & & & {\color{blue}\Leftarrow} \\ \cline{10-15}
 & & & & & & & & & & & 1 & 1 & 0 & 0 & {\color{red}\Leftarrow}\ \text{again} \\
 & & & & & & & & & & & 1 & 0 & 0 & 1 & {\color{blue}\Leftarrow}\ \text{again}
\end{array}
$$

We get $10101 \times 101 \div 1001 = 1011.1010101010\cdots = 1011.\overline{10}$.

Exercise 5.

(1)

$5AB + E07 - C5 = 5, 10, 11 + 14, 0, 7 - 12, 5 = 1, 2, 14, 13 = 12ED$.

$$
\begin{array}{cccc}
 & 5 & 10 & 11 \\
+ & 14 & 0 & 7 \\
- & & 12 & 5 \\ \hline
1 & {}^{+1}2 & {}^{-1}14 & 13
\end{array}
$$

Note that we used the following calculations in base 16:

$$
\begin{aligned}
11 + 7 - 5 &= 13, \\
10 + 0 - 12 &= -2 = -1 \times 16 + 14, \\
5 + 14 - 1 &= 1 \times 16 + 2.
\end{aligned}
$$

(2)

$$
\begin{aligned}
5AB \times E07 \div C5 &= (5, 10, 11 \times 14, 0, 7 \div 12, 5)_{[16]} \\
&= (101_{[2]}, 1010_{[2]}, 1011_{[2]})_{[2^4]} \\
&\quad \times (1110_{[2]}, 0_{[2]}, 111_{[2]})_{[2^4]} \\
&\quad \div (1100_{[2]}, 101_{[2]})_{[2^4]} \\
&= (10110101011 \times 111000000111 \div 11000101)_{[2]}
\end{aligned}
$$

$$
\begin{array}{*{24}{c}}
&&&&&&&&&&&&&1&0&1&1&0&1&0&1&0&1&1 \\
\times&&&&&&&&&&&&1&1&1&0&0&0&0&0&0&1&1&1 \\
\hline
&&&&&&&&&&&&&1&0&1&1&0&1&0&1&0&1&1 \\
&&&&&&&&&&&&1&0&1&1&0&1&0&1&0&1&1& \\
&&&&&&&&&&&1&0&1&1&0&1&0&1&0&1&1&& \\
&&&&1&0&1&1&0&1&0&1&0&1&1&&&&&&&&& \\
&&&1&0&1&1&0&1&0&1&0&1&1&&&&&&&&&& \\
+&&1&0&1&1&0&1&0&1&0&1&1&&&&&&&&&&& \\
\hline
&1&0&1&1&1&1&1&1&0&0&0&0&0&0&1&1&0&1&0&1&1&0&1 \\
-&&1&1&0&0&0&1&0&1&&&&&&&&&&&&&& \\
\cline{1-11}
&&1&0&1&1&1&0&0&1&0&&&&&&&&&&&&& \\
&-&&1&1&0&0&0&1&0&1&&&&&&&&&&&&& \\
\cline{2-12}
&&&1&0&1&0&1&1&0&1&0&&&&&&&&&&&& \\
&&-&&1&1&0&0&0&1&0&1&&&&&&&&&&&& \\
\cline{3-13}
&&&&1&0&0&1&0&1&0&1&0&&&&&&&&&&& \\
&&&-&&1&1&0&0&0&1&0&1&&&&&&&&&&& \\
\cline{4-14}
&&&&&&1&1&0&0&1&0&1&0&&&&&&&&&& \\
&&&&-&&1&1&0&0&0&1&0&1&&&&&&&&&& \\
\cline{5-20}
&&&&&&&&&&&1&0&1&0&1&1&0&1&0&&&& \\
&&&&&&&&&&-&&1&1&0&0&0&1&0&1&&&& \\
\cline{11-21}
&&&&&&&&&&&&1&0&0&1&0&1&0&1&1&&& \\
&&&&&&&&&&&-&&1&1&0&0&0&1&0&1&&& \\
\cline{12-23}
&&&&&&&&&&&&&&1&1&0&0&1&1&0&1&0& \\
&&&&&&&&&&&&&-&&1&1&0&0&0&1&0&1& \\
\cline{14-24}
&&&&&&&&&&&&&&&1&1&0&1&0&1&0&1&1 \\
&&&&&&&&&&&&&&-&&1&1&0&0&0&1&0&1 \\
\cline{15-24}
&&&&&&&&&&&&&&&&1&1&1&0&0&1&1&0
\end{array}
$$

Note that the sequence of $-$ signs is doing the division. The $-$ signs mean 1, and gaps between $-$ signs are filled with 0. Specifically, we have 5 consecutive $-$ sign, which means 11111. The 5-th $-$ sign and 6-th $-$ sign has distance 4 gap, which means 0000, etc. Therefore the quotient is

$$11111000011011_{[2]} = [11_{[2]}, 1110_{[2]}, 0001_{[2]}, 1011_{[2]}]_{[2^4]} = 3, 14, 1, 11_{[16]} = 3E1B,$$

with remainder

$$11100110_{[2]} = [1110_{[2]}, 0110_{[2]}]_{[2^4]} = 14, 6_{[16]} = E6.$$

Exercise 6.

We omit $\bar{0} + \bar{n} = \bar{n} = \bar{n} + \bar{0}$, $\bar{0} \times \bar{n} = \bar{0} = \bar{n} \times \bar{0}$.

$$
\begin{array}{c|cccc}
+ & \bar{1} & \bar{2} & \bar{3} & \bar{4} \\
\hline
\bar{1} & \bar{2} & \bar{3} & \bar{4} & \bar{0} \\
\bar{2} & \bar{3} & \bar{4} & \bar{0} & \bar{1} \\
\bar{3} & \bar{4} & \bar{0} & \bar{1} & \bar{2} \\
\bar{4} & \bar{0} & \bar{1} & \bar{2} & \bar{3}
\end{array}
\qquad
\begin{array}{c|cccc}
\times & \bar{1} & \bar{2} & \bar{3} & \bar{4} \\
\hline
\bar{1} & \bar{1} & \bar{2} & \bar{3} & \bar{4} \\
\bar{2} & \bar{2} & \bar{4} & \bar{1} & \bar{3} \\
\bar{3} & \bar{3} & \bar{1} & \bar{4} & \bar{2} \\
\bar{4} & \bar{4} & \bar{3} & \bar{2} & \bar{1}
\end{array}
$$

The table is obtained by keeping the "last digit" of the addition and multiplication tables for base 5 (and deleting all the "non-last digits").

Exercise 7.

Denote by $7|n$ that n is divisible by 7. Then

$$
\begin{aligned}
7|10a+b \iff & 7|2(10a+b) = 20a+2b = (3\times 7-1)a+2b \\
\iff & 7|-a+2b \iff 7|-(-a+2b) = a-2b.
\end{aligned}
$$

Exercise 8.

First rule: In $\mathbb{Z}_{21}$, we have

$$\overline{10} = \overline{10},\ \overline{10}^2 = -\bar{5},\ \overline{10}^3 = -\bar{8},\ \overline{10}^4 = \bar{4},\ \overline{10}^5 = -\bar{2},\ \overline{10}^6 = \bar{1},\ \overline{10}^7 = \overline{10}, \dots$$

Then we have $\overline{10}^{6k+i} = \overline{10}^i$, and a decimal expression $N_k N_{k-1} \dots N_2 N_1 N_0,\ N_i \in 0, 1, \dots, 9$, is divisible by 21 if and only if

$$N_0 + 10N_1 - 5N_2 - 8N_3 + 4N_4 - 2N_5 + N_6 + 10N_7 - 5N_8 - 8N_9 + \cdots$$

is divisible by 21.

Second rule: By $\overline{10}^6 = \bar{1}$ in $\mathbb{Z}_{21}$, we may regard a decimal expression as based on 10^6

$$N_k N_{k-1} \dots N_2 N_1 N_0 = (\dots, M_3, M_2, M_1, M_0)_{[10^6]},$$

with

$$M_0 = N_5 N_4 N_3 N_2 N_1 N_0,\ M_1 = N_{11} N_{10} N_9 N_8 N_7 N_6,\ \dots$$

Then the number is divisible by 21 if and only if $M_0 + M_1 + \cdots$ is divisible by 21.

Third rule: Divisible by 21 is equivalent to divisible by 3 and by 7. This means the following hold

1. $N_0 + N_1 + N_2 + N_3 + N_4 + N_5 + N_6 + N_7 + N_8 + \cdots$ is divisible by 3,
2. $N_0 + 3N_1 + 2N_2 - N_3 - 3N_4 - 2N_5 + N_6 + 3N_7 + 2N_8 + \cdots$ is divisible by 7.

Exercise 9.

In $\mathbb{Z}_3$, we have

$$\bar{5} = -\bar{1},\ \bar{5}^2 = (-\bar{1})^2 = \bar{1},\ \bar{5}^3 = (-\bar{1})^3 = -\bar{1},\ \bar{5}^4 = (-\bar{1})^4 = \bar{1},\ \ldots$$

Therefore $(N_k N_{k-1} \ldots N_2 N_1 N_0)_{[5]}$ is divisible by 3 if and only if $N_0 - N_1 + N_2 - N_3 + N_4 - \cdots$ is divisible by 3.

Exercise 10.

In the Chinese multiplication table, one only needs to multiply 1 through 9. This means 9^2 multiplications. However, $m \times n$ and $n \times m$ are the same multiplication. For $m \neq n$, therefore, the two multiplications are counted the same. Alternatively, we double count the squares $n \times n$ (there are totally 9 of these), and get the final number of multiplications

$$\frac{1}{2}(9^2 + 9) = 45.$$

Given the sexagesimal system, the Babylonians need to multiply 1 through 59. By the similar argument, the number of multiplications they need is

$$\frac{1}{2}(59^2 + 59) = 1770.$$

Exercise 11.

(1) $235 + T3T - 1T9 = 2,3,5 + 10,3,10 - 1,10,9 = 10,7,6 = T76$.

$$\begin{array}{rrrr} & 2 & 3 & 5 \\ + & 10 & 3 & 10 \\ - & 1 & 10 & 9 \\ \hline & 10 & {}^{-1}7 & 6 \end{array}$$

$3T7 \times 203 = 3,10,7 \times 2,0,3 = 8,0,3,9,10 = 8039T$.

$$\begin{array}{rrrrr} & & 3 & 10 & 7 \\ \times & & 2 & 0 & 3 \\ \hline & 1 & 0 & 9 & 10 \\ 7 & 10 & 3 & & \\ \hline 8 & 0 & 3 & 9 & 10 \end{array}$$

(2) In $\mathbb{Z}_T = \mathbb{Z}_{10}$, we have $\overline{11} = \bar{1}$. Therefore for an expression in base 11

$$\cdots N_3 N_2 N_1 N_0 = N_0 + N_1 \times 11 + N_2 \times 11^2 + N_3 \times 11^3 + \cdots,$$

we have

$$\overline{\cdots N_3 N_2 N_1 N_0} = N_0 + N_1 \times \bar{1} + N_2 \times \bar{1}^2 + N_3 \times \bar{1}^3 + \cdots = N_0 + N_1 + N_2 + N_3 + \cdots .$$

Therefore the number is divisible by T if and only if $N_0 + N_1 + N_2 + N_3 + \cdots$ is divisible by T.

(3) $2T2T \cdots 2T$ is divisible by 11 if and only if $T + 2 + T + 2 + \cdots + T + 2 = Tn + 2n$ is divisible by T. This is the same as $2n$ divisible by $T = 10$, which is the same as n being a multiple of 5.

Exercise 12.

Converting to sexagesimal system, we have

$$\begin{aligned} 2.25 &= 2 + \frac{1}{4} = 2 + \frac{15}{60} = 2;15 \\ 13.5 &= 13 + \frac{1}{2} = 13 + \frac{30}{60} = 13;30 \\ 135 &= 2 \times 60 + 15 = 2,15; \\ 225 &= 3 \times 60 + 45 = 3,45; \end{aligned}$$

Since the Babylonians do not distinguish between ";" and ",", they will denote both 2.25 and 135 as 2,15 (actually 2 15).

Exercise 13.

The Babylonian $36,6$ could mean any of the following in decimal expression

$$\begin{aligned} &36,6 = 36 \times 60 + 6 = 2166, \\ &36,6 = 36 \times 60^2 + 6 \times 60 = 2166 \times 60 = 129960, \\ &36,6 = 36 \times 60^3 + 6 \times 60^2 = 129960 \times 60 = 7797600. \end{aligned}$$

The decimal value 7797600 is closest to 7.8 million and is therefore what Babylonians meant.

Exercise 14.

We have

$$1 = \frac{1}{2} + \frac{1}{3} + \frac{1}{6}.$$

The Egyptians can write 1, and can also write $\frac{1}{2} + \frac{1}{3} + \frac{1}{6}$ to express the same number.

We can further get two expressions by unit fractions

$$0.2 = \frac{1}{5} = \frac{1}{2 \times 5} + \frac{1}{3 \times 5} + \frac{1}{6 \times 5} = \frac{1}{10} + \frac{1}{15} + \frac{1}{30}.$$

Exercise 15.

Let x be the age of Diophantus. The epigram means

$$\frac{1}{6}x+\frac{1}{12}x+\frac{1}{7}x+5+\frac{1}{2}x+4=x.$$

The solution is $x=84$.

Exercise 16.

The problem lies in the proper understanding of "no dog". The first axiom says "No (dog has 5 legs)", where "no" and "dog" play separate roles (logical, and terminological). The second axiom says "A dog has 4 more legs than (no dog)", where "no dog" is taken as a combined terminology, and "no" has no logical role.

The fallacy shows that the ambiguity in our common language makes it not adequate for expressing rigorous mathematics. There is a need to develop an "artificial language" in order to express mathematics in the most rigorous way. Such a philosophy underlies "The Principles of Mathematics" (first published in 1903) by Bertrand Russell. In "Principia Mathematica" (3 volumes published in 1910, 1912, 1913) by Alfred North Whitehead and Bertrand Russell, such "artificial language" used to develop the most basic mathematics in the most rigorous way. However, it took them hundreds of pages to prove $1+1=2$[1]!

Exercise 17.

The 4th number is indeed 16. But the fifth number is 31, not $3^5=32$. In fact, there is a general formula for the number (see http://oeis.org/A000127)

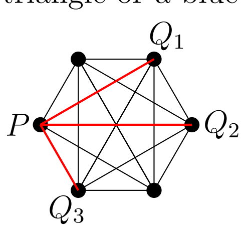
$$a(n)=\frac{1}{24}n^4-\frac{1}{4}n^3+\frac{23}{24}n^2-\frac{3}{4}n+1.$$

The problem show the importance of proof for mathematical statements, no matter how many confirmations you can get.

Exercise 18.

It is sufficient to pick any 6 from the party and study the relations between the six.

Denote each of the six person by a dot. Connect each pair by a red line if they are mutual friends, and by blue line if they are mutual strangers. The question becomes the existence of a red triangle or a blue triangle.



[1] The equality is defined in "From this proposition it will follow, when arithmetical addition has been defined, that 1+1=2." on page 379 of volume 1. The actual proof is completed on page 86 of volume 2.

Pick any person, say P. Between P and the other 5, at least 3 lines have the same color. For example, suppose P is connected to Q_1, Q_2, Q_3 in red color. If any one of 3 lines between Q_1, Q_2, Q_3 is red, then we get a red triangle. Otherwise all of 3 lines between Q_1, Q_2, Q_3 are blue, and we get a blue triangle.

Exercise 19.

Let $S_n = \frac{1}{1} + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{2n-1}$ be the partial sum. Then

$$\begin{aligned}
S_{10} &= \frac{1}{1} + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{19} > \frac{10}{19} > \frac{10}{30} = \frac{1}{3}, \\
S_{100} - S_{10} &= \frac{1}{21} + \frac{1}{23} + \frac{1}{25} + \cdots + \frac{1}{199} > \frac{90}{199} > \frac{90}{270} = \frac{1}{3}, \\
S_{1000} - S_{100} &= \frac{1}{201} + \frac{1}{203} + \frac{1}{205} + \cdots + \frac{1}{1999} > \frac{900}{1999} > \frac{90}{2700} = \frac{1}{3}, \\
&\vdots
\end{aligned}$$

Adding up, we get

$$S_{10} > \frac{1}{3}, \quad S_{100} > \frac{2}{3}, \quad S_{1000} > \frac{3}{3}, \cdots, S_{10^n} > \frac{n}{3}, \cdots$$

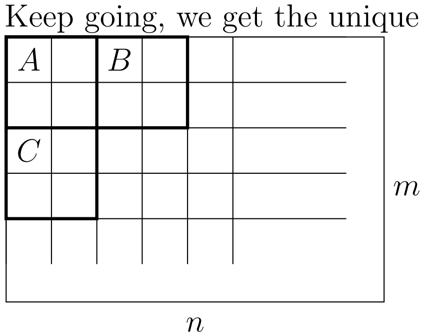
Therefore S_n diverges to infinity.

Exercise 20.

The tiling covering A is the unique one as indicated. Then the tile covering B is the unique one as indicated. Keep, going, we find the unique tiles for the first two rows.

After the first two rows, the tiling covering C is the unique one as indicated. Like the first two rows, we find the the unique tiles for the third and fourth rows.

Keep going, we get the unique tiles covering $2n-1$ and $2n$ rows.



We conclude that $m \times n$ grid can be covered by 2×2 squares if and only if both m, n are even. Moreover, the tiling is unique.

Exercise 21.

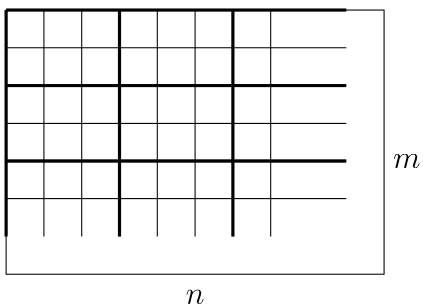
Since 2×3 domino occupies 6 squares, a necessary condition for $m \times n$ to be tiled is that mn is a multiple of 6.

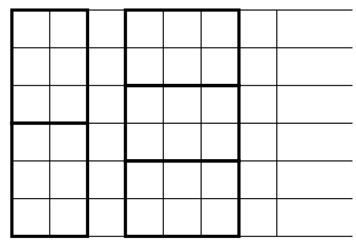
Conversely, suppose mn is a multiple of 6. Then we only need to consider two possibilities: (1) m is divisible by 2, n is divisible by 3; (2) m is divisible by 6, and $n \geq 2$. The other possibilities simply switches m and n and do not affect the answer.

In case (1), the grid can be tiled by putting 2×3 dominos horizontally. See left picture.

In case (2), we see in the right picture that two 2×3 dominos can tile a 6×2 grid, and three 2×3 dominos can tile a 6×3 grid. By repeating the first tiling a times and the second tiling b times, we can tile a $6 \times (2a + 3b)$ grid. Since any number $n \geq 2$ can be written as $2a + 3b$ for some non-negative integers a, b, we find that $6 \times n$ grid can be tiled. Furthermore, if m is divisible by 6, then stacking $m/6$ copies of such tilings shows that $m \times n$ grid can be tiled.

We conclude that an $m \times n$ grid can be tiled by 2×3 dominos if and only if either (1) or (2) is true.





Exercise 22.

If $a^3 = 2x^2 + a - 2$, then clearly $F(n) = a^n$ satisfies $F(n) = 2F(n-1) + F(n-2) - F(n-3)$. (We will deal with $F(0) = 0$, $F(1) = 1$, $F(2) = 2$ later.)

We solve $a^3 = 2x^2 + a - 2$ and find three solutions $a = -1, 1, 2$. This means that $F_1(n) = (-1)^n$, $F_2(n) = 1^n = 1$, $F_2(n) = 2^n$ all satisfy $F(n) = 2F(n-1) + F(n-2) - F(n-3)$. Then their combination $F(n) = A1^n + B(-1)^n + C2^n = A + (-1)^n B + C2^n$ satisfies $F(n) = 2F(n-1) + F(n-2) - F(n-3)$. Then we substitute the special values for $n = 0, 1, 2$ to get

$$0 = A + B + C, \quad 1 = A - B + 2C, \quad 2 = A + B + 4C.$$

Solving the system, we get $A = -\frac{1}{2}$, $B = -\frac{1}{6}$, $C = \frac{2}{3}$. Therefore

$$F(n) = -\frac{1}{2} - \frac{1}{6}(-1)^n + \frac{2}{3}2^n = \frac{1}{6}\left(2^{n+2} + (-1)^{n+1} - 3\right).$$