Geometric phase and the generalized invariant formulation

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An alternative concept of the basic invariants is introduced. The Lewis-Riesenfeld invariant theory is extended to obtain a generalized invariant formulation. The formulation is then used to establish four facts: (i) Any invariant for a quantum system can be constructed in terms of the basic invariants. (ii) It is possible to introduce a solution-generating technique by making use of the basic invariants. (iii) The path integral in the generalized invariant formulation reduces to an ordinary integral. (iv) The study of noncyclic evolution of a quantum system reduces explicitly to the study of the cyclic evolution. Finally, phase factors and general solution for the driven generalized time-dependent harmonic oscillator are studied as an illustrative example.

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I. INTRODUCTION

The geometric phase in quantum adiabatic evolution was first discussed by Berry [1]. It was recognized immediately by Simon [2] that this phase can be interpreted as a holonomy of the Hermitian fiber bundle over the parameter space. This quantum holonomy phenomenon, referred to as the Berry phase, has attracted great interest for its close relation to the study of gauge theory [3,4], anomaly [5–7], fractional statistics [8,9], and quantum Hall effect [10], and for its having been verified repeatedly experimentally [11–15]. In a fundamental generalization of Berry's idea, Aharonov and Anandan [16] removed the adiabatic condition and studied the geometric phase for any cyclic evolution. This Aharonov-Anandan (AA) phase is related to a holonomy associated with the parallel transport around a circuit in the projective Hilbert space and has been verified in optical and NMR interferometry experiments [17,18]. In a recent paper [19], Anandan pointed out that, in principle, the study of any noncyclic evolution may reduce to the study of cyclic evolution. However, no explicit example in this direction has been seen in the literature, perhaps because of the difficulty associated with the diagonalization of the time-evolution operator, which is usually a time-ordered product of an infinite number of unitary matrices of infinite rank.

Recently, making use of the Lewis-Resenfeld invariant theory (LRIT) [20], we and other authors [21–23] investigated the generalized time-dependent harmonic oscillator, spin-j and two-level systems. From these works, we can see the applicability of the LRIT to the study of the geometric phase problem. In this paper, it is further shown that the invariant theory will become much more suitable for the study of the geometric phase and other problems if the LRIT is improved by introducing the concept of the basic invariants. In Sec. II, the basic invariants are defined, and the LRIT can therefore be extended to obtain a generalized invariant formulation. The formulation is then used to establish four facts: (i) Any invariant for a quantum system can be constructed in terms of basic invariants. In this sense, the basic invariants can be referred to as invariant generators. (ii) It is possible to introduce a solution-generating technique by making use of the basic invariants. This is to say that with the help of a chosen basic invariant, a complete set of the solutions of a Schrödinger equation can be generated from one solution of it. In this sense, the basic invariant may be called a solution generator. (iii) The path integral in the GI formulation reduces to an ordinary integral. (iv) The study of the noncyclic evolution of a quantum system reduces explicitly to the study of the cyclic evolution. In Sec. III, the driven generalized time-dependent harmonic oscillator (DGTHO) is discussed as an explicit example to illustrate the results obtained in Sec. II. We then point out the following: (1) There is some similarity between the Berry phase and the AA phase in the formulation. (2) Taking the adiabatic limit, the formulation can be employed not only to obtain the Berry phase, but also to get the corrections to arbitrary order in the adiabatic approximation to the Berry phase. (3) There is a new kind of adiabatic geometric phase different from the Berry phase. In Sec. IV, there is a brief discussion of some related topics.

II. THE BASIC INVARIANTS AND THE GENERALIZED INVARIANT FORMULATION

We consider a system whose Hamiltonian $H(t)$ is time dependent. A Hermitian operator $I(t)$ is called an invariant for the system if it satisfies

$$\frac{\partial I(t)}{\partial t}-i\hbar^{-1}[I(t),H(t)]=0 . \tag{1}$$

The eigenvalue equation of $I(t)$ can be written as

$$I(t)|n,t\rangle=\lambda_n|n,t\rangle, \quad n=1,2,\ldots . \tag{2}$$

With the help of Eq. (1), it is easy to show

$$\frac{\partial\lambda_n}{\partial t}=0 . \tag{3}$$

The Schrödinger equation for the system is

$$i\hbar\frac{\partial|\psi(t)\rangle_s}{\partial t}=H(t)|\psi(t)\rangle_s , \tag{4}$$

which has particular solutions $|n,t\rangle_s$ different from $|n,t\rangle$ in Eq. (2) only by a phase factor $e^{i\varphi_n(t)}$. The general solution of the Schröinger equation can be shown to be

$$|\psi(t)\rangle_S=\sum_n C_n|n,t\rangle_S=\sum_n C_n e^{i\varphi_n(t)}|n,t\rangle ,$$
$$\varphi_n(t)=\int_0^t\left\langle n,t'\left|\frac{i\partial}{\partial t'}-\hbar^{-1}H(t')\right|n,t'\right\rangle dt' , \tag{5}$$
$$C_n=\langle n,0|\psi(0)\rangle_S .$$

The statement outlined above is the basic content of the LRIT [20].

Now we proceed to introduce the basic invariants. It is easy to see that the formal solution of Eq. (1) is

$$I(t)=U(t)I(0)U^\dagger(t) , \tag{6}$$

where the time-evolution operator $U(t)$ for the system is of the form

$$U(t)=P\left[\exp\left[-i\hbar^{-1}\int_0^t H(t')dt'\right]\right] . \tag{7}$$

Although the formal solution is not very useful in actual calculations, it helps us to introduce a new concept of the basic invariants. For a one-dimensional system, two linearly independent basic invariants can be defined either to be $q(t)=U(t)qU^\dagger(t)$, $p(t)=U(t)pU^\dagger(t)$, or to be any two linearly independent combinations (not necessarily Hermitian) of $q(t)$ and $p(t)$.

If I_1 and I_2 are both invariants, it is easy to show that I_1I_2 is also an invariant. We are then led to the conclusion that any invariant $I(t)=U(t)I(0)U^\dagger(t)$ can be expressed by a power series in $q(t)$ and $p(t)$ as long as $I(0)$ can be expressed by a power series in q and p. In this sense, the basic invariants can be called invariant generators. It is worth pointing out that the concept of the basic invariants is not present in the literature to our knowledge. For example, the "general form" of the invariants for the displaced harmonic oscillator was discussed by Xin Ma [24]. However, his form is not really general, since it fails to contain our basic invariants.

If $|\psi(t)\rangle_s$ is a solution of the Schrödinger equation and $I(t)$ is an invariant, it is readily seen that $I(t)|\psi(t)\rangle_s$ is also a solution of the equation. With this in mind, we will show later in Sec. III that a complete set of the solutions of the Schrödinger equation can be obtained from one solution of the equation by means of a chosen basic invariant. Here we want to emphasize that the complete set of the solutions cannot be obtained in this way without the basic invariant (in general, not Hermitian). In this sense, the basic invariant may be called a solution generator.

It is easy to see that the concept of the basic invariants can readily be generalized to more than one-dimensional cases. This concept combined with the original picture and representation theory in quantum mechanics leads to a different formulation. It is apparent that, for a basic invariant $I_e(t)$, the spectrum of the eigenvalues corresponding to a complete set of the eigenfunctions (if it exists) of $I_e(t)$ is, in general, continuous. The eigenvalue equation of $I_e(t)$ can be written as

$$I_e(t)|\lambda,t\rangle=\lambda|\lambda,t\rangle , \tag{8}$$

with λ varied in a continuous range, where λ is time independent, since $I_e(t)$ is an invariant. In this case, it is useful to calculate $|\psi(t)\rangle_s$ with the path-integral technique

$$|\psi(t)\rangle_s=U(t,0)|\psi(0)\rangle_s$$
$$=\int[d\lambda]C_\lambda U(t,0)|\lambda,0\rangle , \tag{9}$$

where $[d\lambda]$ is the corresponding measure that, in general, is not ordinary $d\lambda$. It is easy to get

$$U(t,0)|\lambda,0\rangle=\int[d\lambda_0]|\lambda_0,t\rangle\langle\lambda_0,t|U(t,0)|\lambda,0\rangle , \tag{10a}$$

$$\begin{aligned}\langle\lambda_0,t|U(t,0)|\lambda,0\rangle=\int_{N\to\infty}&[d\lambda_1]\cdots[d\lambda_{N-1}]\langle\lambda_0,t|U(t,t-\Delta t)|\lambda_1,t-\Delta t\rangle\\ &\times\langle\lambda_1,t-\Delta t|U(t-\Delta t,t-2\Delta t)|\lambda_2,t-2\Delta t\rangle\\ &\times\cdots\langle\lambda_m,t-m\Delta t|U[t-m\Delta t,t-(m+1)\Delta t]|\lambda_{m+1},t-(m+1)\Delta t\rangle\\ &\times\cdots\langle\lambda_{N-1},t-(N-1)\Delta t|U[t-(N-1)\Delta t,t-N\Delta t]|\lambda_N,t-N\Delta t\rangle ,\end{aligned} \tag{10b}$$

where $N\Delta t=t$ and $\lambda_N=\lambda$. With the help of the fact that λ_m is time independent, we get

$$\langle \lambda_m, t-m\Delta t | U[t-m\Delta t, t-(m+1)\Delta t] | \lambda_{m+1}, t-(m+1)\Delta t \rangle = \delta(\lambda_m - \lambda_{m+1}) \exp\left[i \int_{t-(m+1)\Delta t}^{t-m\Delta t} \left\langle \lambda_m, t' \left| i\frac{\partial}{\partial t'} - \hbar^{-1} H(t') \right| \lambda_m, t' \right\rangle dt' \right] . \quad (11)$$

Substitution of Eq. (11) into Eqs. (9) and (10) gives

$$|\psi(t)\rangle_s = \int [d\lambda] C_\lambda \exp\left[i \int_0^t \left\langle \lambda, t' \left| i\frac{\partial}{\partial t'} - \hbar^{-1} H(t') \right| \lambda, t' \right\rangle dt' \right] \Big| \lambda, t \rangle , \quad (12)$$

where

$$\exp[-i\hbar^{-1}\int_0^t \langle \lambda, t' | H(t') | \lambda, t' \rangle dt'] = \exp[i\varphi_\lambda^{(d)}(t)]$$

is the dynamical phase factor and $\exp(i\int_0^t \langle \lambda, t' | i\partial/\partial t' | \lambda, t' \rangle dt') = \exp[i\varphi_\lambda^{(g)}(t)]$ is the geometric phase factor. It is worthwhile to note the following: (a) It is the time-independent property of λ_m that makes the path integral reduce to an ordinary integral. (b) In this formulation, the study of the time evolution is closely related to the study of the phase factors. (c) In the discrete spectrum case, it is easy to see that Eq. (12) becomes Eq. (5). However, the method that Lewis and Riesenfeld used to obtain Eq. (5) is entirely different from that adopted here.

We now turn to the problem of noncyclic evolution. For any given time interval $[0,T]$, if an invariant $I(t)$ can be found to possess the following properties: (1) the set of the eigenstates $|n,t\rangle$ of $I(t)$ is complete; (2) λ_n is not degenerate; (3) $I(0)=I(T)$, which leads to $|n,T\rangle = |n,0\rangle$, then it is not difficult to find that the study of the noncyclic evolution of $|\psi(t)\rangle_s$ reduces to the study of the cyclic evolution of $|n,t\rangle$

$$|\psi(t)\rangle_s = \sum_n C_n e^{i\varphi_n(t)} |n,t\rangle , \qquad |n,T\rangle = |n,0\rangle, \quad C_n = \langle n,0|\psi(0)\rangle_s . \quad (13)$$

We will later show in Sec. III how the invariant $I(t)$ with the properties mentioned above can always be found for some systems.

III. APPLICATION OF THE GENERALIZED INVARIANT FORMULATION TO THE STUDY OF THE DRIVEN GENERALIZED TIME-DEPENDENT HARMONIC OSCILLATOR

Recently, the problem of the solution for the DGTHO was studied by Engineer [25]. In his paper, he used a tentative ansatz to discuss only the evolution of the ground state and did not find the general solution for the system. In this section, the GI formulation is employed to find the general solution for the DGTHO without introducing any ansatz.

The Hamiltonian for the DGTHO is

$$H(t) = \tfrac{1}{2}[X(t)q^2 + Y(t)(qp+pq) + Z(t)p^2] + Fq , \quad (14)$$

with $\mathbf{R}(t) = (X(t), Y(t), Z(t))$ being time-dependent parameters that satisfy $XZ - Y^2 > 0$ and F being constant. Since H, q,p, and 1 constitute a quasialgebra, it is not difficult to see that a general basic invariant is of the form

$$I_e(t) = \left[\frac{1}{x(t)} \cos[\Theta(t)+\theta_0] + \frac{\dot{x}(t) - Y(t)x(t)}{Z(t)} \sin[\Theta(t)+\theta_0] \right] [q - y(t)] - x(t)\sin[\Theta(t)+\theta_0] \times \left[p - \frac{\dot{y}(t) - Y(t)y(t)}{Z(t)} \right] ,$$

$$\Theta(t) = \int_0^t dt' Z(t')/x^2(t') , \quad (15)$$

where $x(t)$ and $y(t)$ are c-number solutions of the auxiliary equations

$$\frac{d}{dt}\left[\frac{\dot{x}}{Z} \right] + \left[\frac{XZ - Y^2}{Z} - \frac{d}{dt}\left[\frac{Y}{Z} \right] \right] x = \frac{Z}{x^3} , \quad (16a)$$

$$\frac{d}{dt}\left[\frac{\dot{y}}{Z} \right] + \left[\frac{XZ - Y^2}{Z} - \frac{d}{dt}\left[\frac{Y}{Z} \right] \right] y = -F , \quad (16b)$$

and θ_0 is an initial phase angle.

It is apparent that the initial conditions imposed on $x, \dot{x}, y, \dot{y}$ can be arbitrarily chosen. With appropriate choices, two linearly independent basic invariants can be obtained as follows:

$$I_b(t) = \frac{e^{i\Theta}}{\sqrt{2\hbar}} \left\{ \frac{q-y}{x} + i \left[x \left[p + \frac{Y}{Z}q - \frac{\dot{y}}{Z} \right] - \frac{\dot{x}}{Z}(q-y) \right] \right\} \equiv e^{i\Theta(t)} b(t) , \quad (17a)$$

$$I_b^\dagger(t) = \frac{e^{-i\Theta}}{\sqrt{2\hbar}} \left\{ \frac{q-y}{x} - i \left[x \left[p + \frac{Y}{Z}q - \frac{\dot{y}}{Z} \right] - \frac{\dot{x}}{Z}(q-y) \right] \right\} \equiv e^{-i\Theta(t)} b^\dagger(t) , \quad (17b)$$

$$[I_b(t), I_b^\dagger(t)] = 1 . \quad (18)$$

From the basic invariants, we can get another invariant $I_c(t)$:

$$I_c(t)=\hbar[I_b^\dagger(t)I_b(t)+\tfrac{1}{2}]$$
$$=\tfrac{1}{2}\left\{\frac{(q-y)^2}{x^2}+\left[x\left[p+\frac{Y}{Z}q-\frac{\dot y}{Z}\right]-\frac{\dot x}{Z}(q-y)\right]^2\right\}. \quad (19)$$

These invariants can be used to study the DGHTO. It is easy to establish

$$H=\hbar w(a^\dagger a+\tfrac{1}{2})-F^2Z/(2w^2),\quad w=(XZ-Y^2)^{1/2}, \quad (20a)$$

$$a=\left[\frac{w}{2\hbar Z}\right]^{1/2}\left[\left[q+\frac{FZ}{w^2}\right]+i\frac{Z}{w}\left[p+\frac{Y}{Z}q\right]\right],$$
$$a^\dagger=\left[\frac{w}{2\hbar Z}\right]^{1/2}\left[\left[q+\frac{FZ}{w^2}\right]-i\frac{Z}{w}\left[p+\frac{Y}{Z}q\right]\right], \quad (20b)$$

$$[a,a^\dagger]=1,\quad a^\dagger a|n,t\rangle_a=n|n,t\rangle_a\quad (n=0,1,\ldots),$$
$$a|n,t\rangle_a=\sqrt{n}\,|n-1,t\rangle_a, \quad (20c)$$
$$a^\dagger|n,t\rangle_a=\sqrt{n+1}|n+1,t\rangle_a,$$

$$I_c=\hbar(b^\dagger b+\tfrac{1}{2}), \quad (21a)$$

$$b=\frac{1}{\sqrt{2\hbar}}\left\{\frac{q-y}{x}+i\left[x\left[p+\frac{Y}{Z}q-\frac{\dot y}{Z}\right]-\frac{\dot x}{Z}(q-y)\right]\right\}, \quad (21b)$$

$$b^\dagger=\frac{1}{\sqrt{2\hbar}}\left\{\frac{q-y}{x}-i\left[x\left[p+\frac{Y}{Z}q-\frac{\dot y}{Z}\right]-\frac{\dot x}{Z}(q-y)\right]\right\},$$

$$[b,b^\dagger]=1,\quad b^\dagger b|n,t\rangle_b=n|n,t\rangle_b\quad (n=0,1,\ldots),$$
$$b|n,t\rangle_b=\sqrt{n}\,|n-1,t\rangle_b, \quad (21c)$$
$$b^\dagger|n,t\rangle_b=\sqrt{n+1}|n+1,t\rangle_b,$$

and

$$|n,t\rangle_{bS}=\frac{[I_b^\dagger(t)]^n}{(n!)^{1/2}}|0,t\rangle_{bS}$$
$$=e^{in\Theta(t)}\frac{[b^\dagger(t)]^n}{(n!)^{1/2}}|0,t\rangle_{bS}, \quad (22)$$

where $|n,t\rangle_{bS}$ differs from $|n,t\rangle_b$ by a phase factor $\exp(i\varphi_n(t))$ and is the solution of the Schrödinger equation.

We now show that b and $b^\dagger$ can always be found to satisfy the cyclic condition

$$b(T)=b(0),\quad b^\dagger(T)=b^\dagger(0). \quad (23)$$

It is only necessary to study the condition $b(T)=b(0)$, since it implies $b^\dagger(T)=b^\dagger(0)$. The equation $b(T)=b(0)$ leads to four equations for $x(0)$, $\dot x(0)$, $x(T)$, $\dot x(T)$, $y(0)$, $\dot y(0)$, $y(T)$, and $\dot y(T)$ by equating the coefficients of q, p, and 1 on both sides of Eq. (23). With the help of the auxiliary equations, $x(T)$, $\dot x(T)$, $y(T)$, and $\dot y(T)$ are determined by $x(0)$, $\dot x(0)$, $y(0)$, and $\dot y(0)$. It is therefore apparent that $x(0)$, $\dot x(0)$ $y(0)$, and $\dot y(0)$ are completely determined by the condition $b(T)=b(0)$. This means that an appropriate choice of $x(0)$, $\dot x(0)$, $y(0)$, and $\dot y(0)$ leads to $b(T)=b(0)$ and hence $b^\dagger(T)=b^\dagger(0)$, $I_c(T)=I_c(0)$. From this and the relevant result obtained in Sec. II, it follows that

$$|\psi(t)\rangle_s=\sum_n C_n\exp(i\varphi_n(t))|n,t\rangle_b,$$
$$\varphi_n(t)=\varphi_n^{(g)}(t)+\varphi_n^{(d)}(t), \quad (24a)$$

$$\varphi_n^{(g)}(t)=\int_0^t {}_b\left\langle n,t'\left|\frac{i\partial}{\partial t'}\right|n,t'\right\rangle_b dt',$$
$$\varphi_n^{(d)}(t)=-\hbar^{-1}\int_0^t {}_b\langle n,t'|H(t')|n,t'\rangle_b dt', \quad (24b)$$

$$|n,0\rangle_b=|n,T\rangle_b. \quad (25)$$

This means that, for the DGTHO, the study of the noncyclic evolution of a general state $|\psi(t)\rangle_s$ reduces to the study of the cyclic evolution of the eigenstates $|n,t\rangle_b$ of $I_c(t)$ in Eq. (21) and the corresponding phase $\varphi_n^{(g)}(t)+\varphi_n^{(d)}(t)$. It is worthwhile to point out two facts: (i) The geometric phase $\varphi_n^{(g)}(T)$ is nothing but the AA phase for which the expression is $\int_{0b}^T\langle n,t'|i\partial/\partial t'|n,t'\rangle_b dt'$. (ii) If the system is initially in the eigenstate $|n,0\rangle_b$ of $I_c(0)$, it will remain in the eigenstate $|n,t\rangle_b$ of $I_c(t)$ all the time. From these two facts, we clearly see that there is some similarity between the evolution of the $|n,t\rangle_b$ and the adiabatic evolution of $|n,t\rangle_a$.

In the following, we study the evolution of the ground state $|0,t\rangle_b$ of $I_c(t)$ to get $|0,t\rangle_{bS}$ and then to get $|n,t\rangle_{bS}=[(I_b^\dagger)^n/(n!)^{1/2}]|0,t\rangle_{bS}$ by means of the solution-generating technique.

The solution of the Schrödinger equation $|0,t\rangle_{bS}$ can be written as

$$|0,t\rangle_{bS}=U(t,0)|0,0\rangle_b=\exp[i\varphi_0(t)]|0,t\rangle_b, \quad (26a)$$
$$\varphi_0(t)=\int_0^t {}_b\langle 0,t'|i\partial/\partial t'-\hbar^{-1}H(t')|0,t'\rangle_b dt', \quad (26b)$$

where $-\hbar^{-1}\int_{0b}^t\langle 0,t'|H(t')|0,t'\rangle_b dt'=\varphi_0^{(d)}(t)$ is the dynamical phase and $\int_{0b}^t\langle 0,t'|i\partial/\partial t'|0,t'\rangle_b dt'=\varphi_0^{(g)}(t)$ is the geometric phase [$\varphi_0^{(g)}(T)$ is the AA phase]. From Eqs. (20) and (21), we get

$$\varphi_0^{(d)}(t)=-\int_0^t\left[\frac{1}{4}\left[\frac{w^2x^2}{Z}+\frac{\dot x^2}{Z}+\frac{Z}{x^2}\right]+\left[\frac{w^2y^2}{2\hbar Z}+\frac{Fy}{\hbar}+\frac{\dot y^2}{2\hbar Z}\right]\right]dt'. \quad (27)$$

In order to calculate the ground state $|0,t\rangle_b$ of $I_c(t)$, we study the connection between the $I_c(t)$ and the operator $I_0=\frac{1}{2}[(X_0/Z_0)^{1/2}q^2+(Z_0/X_0)^{1/2}p^2]$ with $X_0=1$, $Z_0=1$ in the same units as used for X,Z in Eq. (3.1). It is easy to find

$$I_c(t)=D(t)S(t)I_0S^\dagger(t)D^\dagger(t), \quad (28a)$$
$$D(t)=\exp\{-i[py-q(\dot y-Yy)/Z]/\hbar\}, \quad (28b)$$

$$S(t)=\exp\{i\hbar^{-1}r[(\sin\delta)(q^2-p^2)+(\cos\delta)(qp+pq)]/4\} , \quad (28c)$$

in which r and δ are determined by

$$\cosh r=\frac{1}{2}\left[\frac{1}{x^2}+x^2+\left[\frac{\dot{x}-Yx}{Z}\right]^2\right] \quad (29a)$$

$$e^{i\delta}\sinh r=\frac{1}{2}\left[\frac{1}{x^2}-x^2+\left[\frac{\dot{x}-Yx}{Z}\right]^2\right]+ix\frac{\dot{x}-Yx}{Z} . \quad (29b)$$

It is easy to get

$$|0,t\rangle_b=D(t)S(t)|0\rangle , \quad (30)$$

where $I_0|0\rangle=\frac{1}{2}\hbar|0\rangle$. The geometric phase $\varphi_0^{(g)}(t)$ is then obtained to be

$$\varphi_0^{(g)}(t)=\int_0^t {}_b\left\langle 0,t'\left|S^\dagger(t')D^\dagger(t')\frac{i\partial}{\partial t'}D(t')S(t')\right|0,t'\right\rangle_b dt'$$

$$=\frac{1}{4}\int_0^t\left\{\left[\frac{\dot{x}^2}{Z}+x^2\frac{d}{dt}\left[\frac{Y}{Z}\right]-x\frac{d}{dt}\left[\frac{\dot{x}}{Z}\right]\right]+\frac{\dot{r}\sin\delta+\dot{\delta}}{\cosh r-\sinh r\cos\delta}-\dot{\delta}\right\}dt'$$

$$+\frac{1}{2\hbar}\int_0^t\left\{\left[\frac{\dot{y}^2}{Z}-y\frac{d}{dt'}\left[\frac{\dot{y}}{Z}\right]\right]+y^2\frac{d}{dt'}\left[\frac{Y}{Z}\right]\right\}dt' . \quad (31)$$

The total phase $\varphi_0(t)$ is found to be

$$\varphi_0(t)=-\frac{1}{2}\int_0^t\frac{Z}{x^2}dt'+\int_0^t\left\{-\left[\frac{w^2y^2}{2\hbar Z}+\frac{Fy}{\hbar}+\frac{\dot{y}^2}{2\hbar Z}\right]+\frac{1}{2\hbar}y^2\frac{d}{dt'}\left[\frac{Y}{Z}\right]+\frac{1}{2\hbar}\left[\frac{\dot{y}^2}{Z}-y\frac{d}{dt'}\left[\frac{\dot{y}}{Z}\right]\right]\right\}$$

$$+\frac{1}{4}\int_0^t\left[\frac{\dot{r}\sin\delta+\dot{\delta}}{\cosh r-\sinh r\cos\delta}-\dot{\delta}\right]dt' , \quad (32)$$

where $(\dot{r}\sin\delta+\dot{\delta})/(\cosh r-\sinh r\cos\delta)-\dot{\delta}$ is a total time derivative. From the above derivation, we can see clearly the origin of the nontrivial topological property of the ground state $|0,t\rangle_{bS}$.

From Eq. (21), we get

$$q=\left[\frac{\hbar}{2}\right]^{1/2}x(b+b^\dagger)+y , \quad (33a)$$

$$p=\left[\frac{\hbar}{2}\right]^{1/2}\left[\frac{1}{x}\frac{b-b^\dagger}{i}+\frac{\dot{x}-Yx}{Z}(b+b^\dagger)\right]+\frac{\dot{y}-Yy}{Z} . \quad (33b)$$

This leads to

$$\langle q\rangle={}_{bS}\langle 0,t|q|0,t\rangle_{bS}=y ,$$

$$\langle p\rangle={}_{bS}\langle 0,t|p|0,t\rangle_{bS}=(\dot{y}-Yy)/Z , \quad (34a)$$

$${}_{bS}\langle 0,t|q^2-\langle q\rangle^2|0,t\rangle_{bS}=\hbar x^2/2 ,$$

$${}_{bS}\langle 0,t|p^2-\langle p\rangle^2|0,t\rangle_{bS}=\frac{\hbar}{2}\left[\frac{1}{x^2}+\left[\frac{\dot{x}-Yx}{Z}\right]^2\right] . \quad (34b)$$

From these expressions, the physical meaning of x, $\dot{x}$, y, and $\dot{y}$ is clearly seen.

With the help of $I_b^\dagger(t)$ and Eqs. (20) and (21), we obtain

$$|n,t\rangle_{bS}=\frac{[I_b^\dagger(t)]^n}{(n!)^{1/2}}|0,t\rangle_{bS}$$

$$=\exp\{i[\varphi_0(t)+n\Theta(t)]\}\frac{[b^\dagger(t)]^n}{(n!)^{1/2}}|0,t\rangle_b$$

$$=\exp[i\varphi_n(t)]|n,t\rangle_b , \quad (35)$$

and

$$\varphi_n(t)=\varphi_0(t)+n\Theta(t) ,$$

$$\varphi_n^{(g)}(t)=\varphi_0^{(g)}(t)+\frac{n}{2}\int_0^t\left[\frac{\dot{x}^2}{Z}+x^2\frac{d}{dt'}\left[\frac{Y}{Z}\right]-x\frac{d}{dt'}\left[\frac{\dot{x}}{Z}\right]\right]dt' , \quad (36)$$

$$\varphi_n^{(d)}(t)=\varphi_0^{(d)}(t)-\frac{n}{2}\int_0^t\left[\frac{w^2x^2}{Z}+\frac{\dot{x}^2}{Z}+\frac{Z}{x^2}\right]dt' ,$$

where $|n,T\rangle_b=|n,0\rangle_b$ is implied by $b(T)=b(0)$. Apparently, the set of the states $|n,t\rangle_b$ $(n=0,1,\ldots)$ is complete in the sense that the general solution of the Schrödinger equation can be expressed in terms of them:

$$|\psi(t)\rangle_S=\sum_n C_n|n,t\rangle_{bS}=\sum_n C_n\exp[i\varphi_n(t)]|n,t\rangle_b\ , \tag{37a}$$

$$C_n=\langle n,0|\psi(0)\rangle_S\ . \tag{37b}$$

The invariant $I_b^\dagger(t)$ is what we call the solution generator. Equation (35) provides an illustrative example of the solution-generating technique.

We want to indicate that the phase factor $\exp(i\varphi_0)$ associated with the ground state $|0,t\rangle_b$ appears also in the phase factor $\exp(i\varphi_n)$ for any n. The phase factor $\exp(i\varphi_0)$ is therefore an overall phase factor of the time-evolution operator $U(t)$. Thus, it will not affect any invariant $I(t)=U(t)I(0)U^\dagger(t)$. Or, we may say that $I(t)$ does not contain the information of the phase factor $\exp(i\varphi_0)$ for the ground state.

From the foregoing discussion, we see that, in some sense, the evolution of the ground state $|0,t\rangle_{bS}$ and the basic invariant $I_b^\dagger(t)$ determine the evolution of the excited state $|n,t\rangle_{bS}$ and hence the general evolution.

Finally, we discuss the adiabatic limit. In this limit, the choices $\dot{x}(0)=0$, $\dot{y}(0)=0$, $x(0)=[Z(0)/W(0)]^{1/2}$, and $y(0)=-FZ(0)/w^2(0)$ lead to $w(0)I(0)=H(0)+F^2Z/(2w^2)$, and we find the Berry phase associated with a closed circuit in the parameter space as follows:

$$\varphi_n^{(g)}(T)=(n+\tfrac{1}{2})\int_0^T\frac{Z}{2w}\frac{d}{dt}\left[\frac{Y}{Z}\right]dt+\frac{1}{2\hbar}\int_0^T\frac{F^2Z^2}{w^4}\frac{d}{dt}\left[\frac{Y}{Z}\right]dt\ . \tag{38}$$

Since the system is precisely in the eigenstate $|n,t\rangle_b$, it is easy to calculate the correction to arbitrary order in the adiabatic approximation to the Berry phase.

If the initial condition $\dot{x}(0)\neq 0$ is chosen, we will have $|n,0\rangle_b\neq|n,0\rangle_a$. This makes $\varphi_n^{(g)}(T)=\int_0^T {}_b\langle n,t|i\partial/\partial t|n,t\rangle dt$ corresponding to a cyclic evolution appreciably different from the Berry phase (38).

IV. DISCUSSION

(1) The classical basic invariants may be obtained by taking the classical limit of the quantum basic invariants: that is, any classical invariant can be generated by the basic invariants.

(2) The basic invariants can be applied to the study of the coherent states and squeezed states. Work in this direction will be presented elsewhere.

(3) It is interesting to investigate the possibility of extending the concept of the basic invariants to field theory.

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