The Hong Kong University of Science & Technology
Department of Mathematics
Introduction to Scientific Computation I
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CLASSICAL FRACTALS

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Abstract. *The word 'fractal' was coined by Benoit Mandelbrot in the 1970s. This paper reviews some classical fractals, namely the Koch curve and snowflake, the Sierpinski gasket and carpet, and the Menger sponge, and describes their properties, including the concept of 'fractal dimension'.*

1. Introduction

In 1970s, Benoit Mandelbrot coined the word 'fractal', whose structure is made up of parts exactly the same as its whole. In this paper we shall examine some interesting classical fractals, and show these objects possess exact 'self-similarity' from its diagram constructions. Examples used to illustrated in this paper are fractals constructed by Koch, Sierpinski and Menger.

2. Classical Fractals

A fractal is a pattern that appears self-similar at various scale of magnification. Those objects possessing exact self-similarity are called regular fractals, which can be constructed mathematically by repetitions of a given operation. Classical fractals are regular fractals generated on a fairly simple way.

Whenever we magnify a fractal at any scale, it shows similarity with its original pattern, thus preserving the details upon magnification. The recurrence of the same pattern over a range of scale is called 'self-similarity', which is a remarkable characteristic of fractals.

There is a non-integer, representing fractal dimension, to characterize fractal objects over a range of scales.

Fractals are non-smooth because iterations produce a line or surface at each generation at all scale of magnification. The non-smoothness cause the non-differentibility of fractal objects.

3. The Koch Curve and snowflake

3.1. The Koch Curve

The Koch curve is generated by Helge von Koch (1870-1924) in 1904. Its construction is illustrated in figure 1. The curve is constructed from a line segment of unit length. This starting form, called the initiator, is the 0^{th} generation in the construction, as shown in step $k = 0$ in the figure. Its middle third portion is then extracted and replaced by 2 equal lines of length $1/3$. This is known as the generator of the curve. We thus obtained the 1^{st} generation, which is a curve of 4 equal line segments of length $1/3$, as shown in step $k = 1$ in the figure. At the next generation (step $k = 2$), each line segment is replaced by

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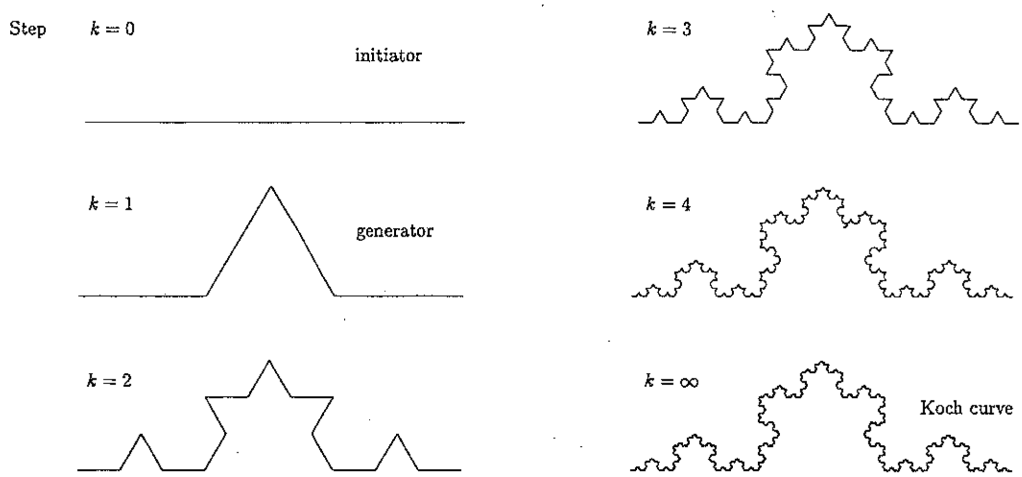

FIGURE 1. *The construction of the Koch curve.*

a scaled-down version of the generator, with the orientation of the replacement always on the same side of the curve. This process is repeated infintely to produce the Koch curve.

The total length of the curve is multiplied by $4/3$ after each generation, and therefore the total length of the Koch curve is infinite.

Now consider the area between the Koch curve and the original line. At step $k = 1$, one triangle is added, its area is

$$A_1 = \frac{1}{2} \times \frac{1}{3} \times \frac{\sqrt{3}}{6} = \frac{\sqrt{3}}{36} . \tag{1}$$

At the next step $k = 2$, 4 new triangles are added, their sizes scaled down by a factor of $1/3$. Thus their areas are $(1/3)^2 A_1 = (1/9)A_1$. The total area now is

$$A_2 = \frac{4}{9} A_1 . \tag{2}$$

At each step, 4 times as many triangles are added to the total area, the area of each triangle added being $1/9$ those added at the previous step. Thus the increase in area at any step is $4/9$ the area added at the previous step. This gives the formula for A_n, the total area at step n,

$$A_n = \left[1 + \frac{4}{9} + \left(\frac{4}{9}\right)^2 + \left(\frac{4}{9}\right)^3 + \cdots + \left(\frac{4}{9}\right)^n \right] A_1 , \tag{3}$$

which, when summing to infinity, has a finite sum,

$$A_\infty = \frac{9A_1}{5} = \frac{\sqrt{3}}{20} . \tag{4}$$

Therefore, the Koch curve is a infinite length curve enclosing a finite area.

This curve is nowhere differentiable, this is, it does not have a well defined slope at any point.

Also, whenever you magnify the curve to examine its detail, it shows exact similarity of its original pattern. As the curve is infinite in length, any scaled down subimage is also of infinite length. Therefore, for any 2 points on the curve, no matter how close they are, the curve between them is of infinite length.

3.2. The Koch snowflake

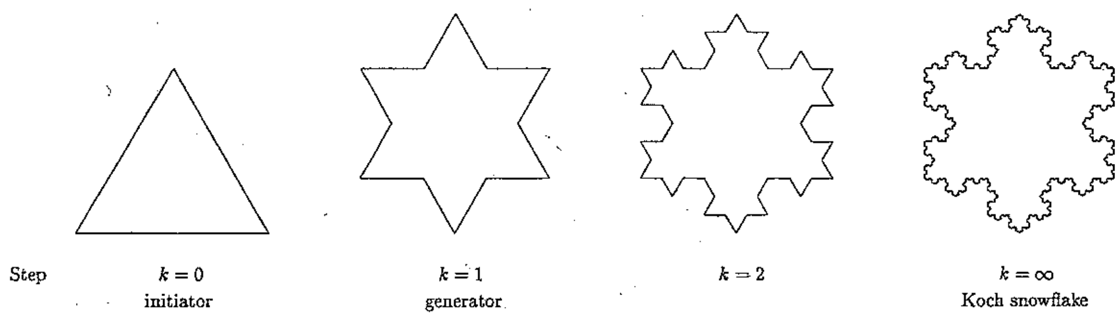

FIGURE 2. *The construction of the Koch snowflake.*

The Koch snowflake is made up of three Koch curves with a triangle as the initiator. For simplicity, we use a equilateral triangle with each side of unit length as the initiator. Its construction is shown the figure 2.

It follows from the Koch curve that the length of the coastline of the Koch snowflake is infinite.

The bounded area in the initiator is given by half of the base multiplied by the height of the equilateral triangle,

$$A' = \frac{1}{2} \times 1 \times \frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{4} . \tag{5}$$

The total area bounded is thus given by

$$A'_{\infty} = A' + 3 \times A_{\infty} = \frac{2}{5}\sqrt{3} , \tag{6}$$

by using eq. (4) and eq. (5).

Therefore the Koch snowflake has a finite area but a immeasurable perimeter.

3.3. The Randomized Koch curve and snowflake

Figure 3 illustrates the construction of a random version of the Koch curve. The orientation of the replacement of the generator is randomly placed either side of the removed segment. Similarly, a randomized Koch snowflake can be constructed and is illustrated in figure 4.

The resulting random fractals show irregularity compared with its regular counterparts, respectively the exactly self-similar Koch curve and snowflake. Like its regular

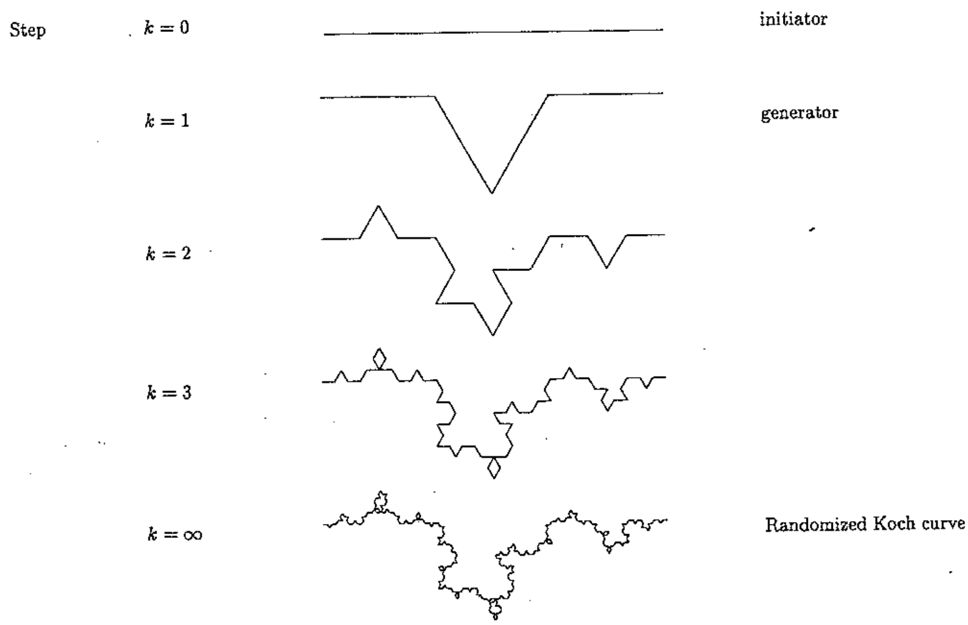

FIGURE 3. *The randomized Koch curve.*

counterparts, these random fractals have respective infinite length and perimeter, and non-differentibility of its respective curve or boundary. However, these random fractals still retain a certain degree of regularity, since at each generation, the regular features are placed randomly either side of the extracted line segment.

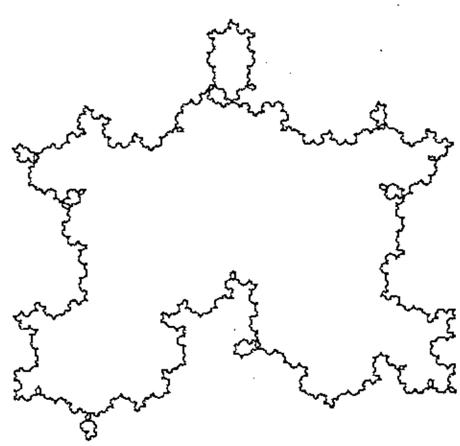

FIGURE 4. *The randomized Koch snowflake.*

4. The Sierpinski gasket and carpet

The Sierpinski gasket (or triangle) is generated by Waclaw Sierpinski (b 1882). Figure 5 illustrated the construction of the sierpinski gasket. The gasket is constructed from

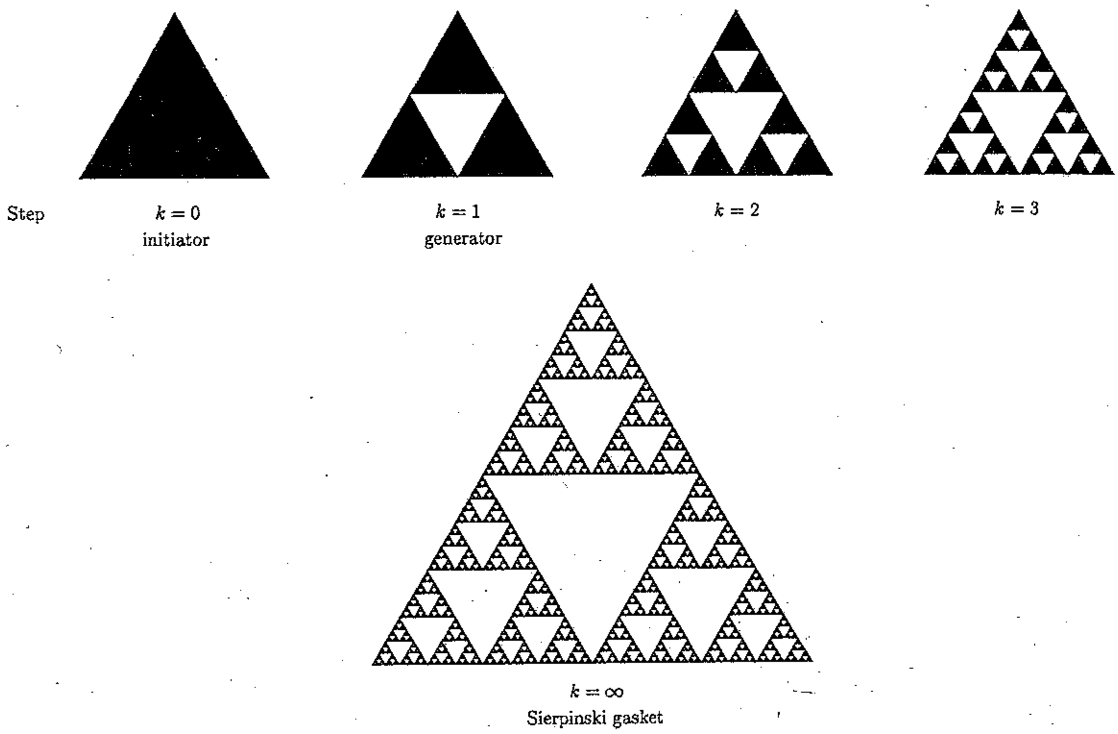

FIGURE 5. *The construction of the Sierpinski gasket.*

a filled triangle as the initiator in 2–dimensional space. The generator is the triangle with a inverted triangle formed by joining the mid-points of the 3 sides of the initiator being removed. For simplicity, again we use a equilateral triangle each side of unit length as the initiator. 3 half-scale triangles now remain, so $1/4$ of the area of the original triangle has been removed. The process is repeated for each triangle remaining. At the 2^{nd} generation (step $k = 2$), $1/4$ of the area of 3 triangles is removed, each of which is $1/4$ of the area of the original triangle's area. Therefore the total area R_∞ removed by the repetition process is given by, with the original area R,

$$R_\infty = R\left[\frac{1}{4} + 3\left(\frac{1}{4}\right)^2 + 3^2\left(\frac{1}{4}\right)^3 + 3^3\left(\frac{1}{4}\right)^4 + \cdots\right] \tag{7}$$

$$= \frac{R}{4}\left[1 + \frac{3}{4} + \left(\frac{3}{4}\right)^2 + \left(\frac{3}{4}\right)^3 + \cdots\right] \tag{8}$$

$$= R\,. \tag{9}$$

Therefore the same size as the original space has been extracted. However, there are points still left in the Sierpinski gasket. It means that those points existing in the Sierpinski gasket with area zero are separated.

A similar procedure can be performed by using a square as the initiator. Then subtract a square at $1/3$ scale from centre, leaving 8 sub-squares. This gives a Sierpinski carpet, as shown in figure 6.

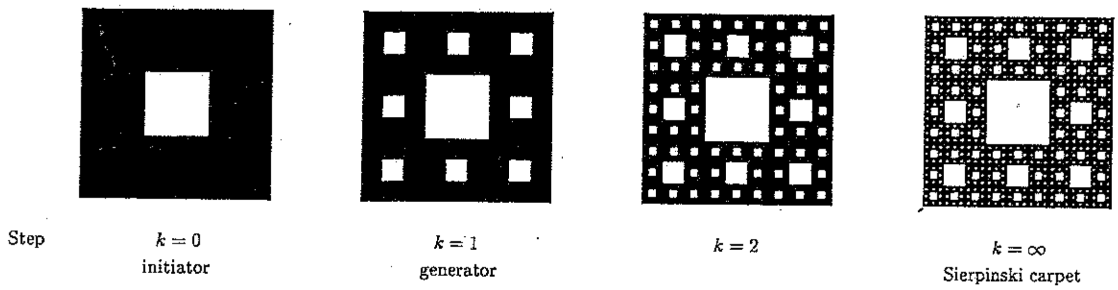

FIGURE 6. *The construction of the Sierpinski carpet.*

5. The Menger sponge

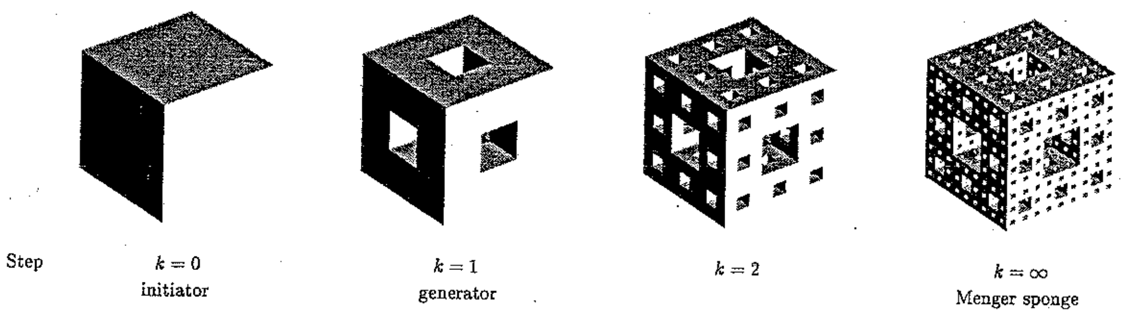

FIGURE 7. *The construction of the Menger sponge.*

The Menger sponge is generated by Karl Menger (1926). Figure 7 illustrated the construction of the Menger sponge. The sponge is constructed from a cube as the initiator in 3–dimensional space. The generator is formed by subdividing faces of cube into 9 congruent squares and drilling through each central square to the oppsite central square. At the 1^{st} generation, seven $1/3$ scale cubes has been removed, remaining twenty $1/3$ scale cubes, so $7/27$ of the volume of the original cube have been removed. The process is repeated for each cube remaining. At the 2^{nd} generation, $7/27$ of the volume of 20 cubes is extracted, each of which is $1/27$ of the volume of the original cube's volume. For V to be the original volume, the total volume V_∞ extracted by this process is

$$V_\infty = V\left[7\left(\frac{1}{3}\right)^3 + 7\left[\left(\frac{1}{3}\right)^3\right]^2 \times 20 + 7\left[\left(\frac{1}{3}\right)^3\right]^3 \times 20^2 + \cdots\right] \tag{10}$$

$$= \frac{7V}{27}\left[1 + \frac{20}{27} + \left(\frac{20}{27}\right)^2 + \cdots\right] \tag{11}$$

$$= V\,. \tag{12}$$

So again we have removed a region of the same size as the original space, but we still have points left in the Menger sponge. Again, those points existing in the Menger sponge with volume zero are separated.

6. Fractal Dimension

Fractal objects have self-similarity at various scale of magnification. Small parts of the object contain scaled down versions of the whole. Consider an object to be subdivided into N copies of itself at scale ϵ, its dimension D satisfies

$$N = \left(\frac{1}{\epsilon}\right)^D , \tag{13}$$

from which D can be obtained directly by

$$D = \frac{\log(N)}{\log(1/\epsilon)} . \tag{14}$$

For example, squares can be subdivided into 4 copies at $1/2$ scale, 9 copies at $1/3$ scale, 16 copies at $1/4$ scale and so on. In this case,

$$D = \frac{\log 4}{\log 2} = \frac{\log 9}{\log 3} = \frac{\log 16}{\log 4} = 2 , \tag{15}$$

the dimension equals 2. A cube can be subdivided into 8 copies at $1/2$ scale, 27 copies at $1/3$ scale. Its dimension is given by

$$D = \frac{\log 8}{\log 2} = \frac{\log 27}{\log 3} , \tag{16}$$

which is in consistent with our usual sense of integer dimension of a square and a cube. For the Koch curve we have 4 copies at $1/3$ scale. Thus

$$D = \frac{\log 4}{\log 3} = 1.2619 , \tag{17}$$

to 4 decimal places, which is a non-integer dimension between 1 and 2. The Sierpinski gasket has 3 copies at $1/2$ scale, so has dimension

$$D = \frac{\log 3}{\log 2} = 1.5850 , \tag{18}$$

to 4 decimal places. The Menger sponge contains 20 copies at $1/3$ scale, giving dimension of

$$D = \frac{\log 20}{\log 3} = 2.7268 , \tag{19}$$

to 4 decimal places, this is a non-integer dimension between 2 and 3.

This definition of dimension applys only for objects containing exact subcopies of themselves. These exact fractals have the same dimension for all their parts. However, many realistic objects are not so well behaved. Their dimension just can be defined only locally and therefore should be computed computationally.

Consider a small regular region centre on a particular point. In 1–dimensional space a neighbourhood is a short line segment, in 2–dimensionals it is a small circle, in 3–dimensions it is a small sphere, each having the reference point at its centre. Now define the correlation sum, C_r for the particular radius r:

$$C_r = \lim_{N\to\infty} \frac{1}{N(N-1)} \sum_{i=1}^{N} \sum_{j=1;j\neq i}^{N} \mathcal{G}(r - \|x_i - x_j\|)\,, \tag{20}$$

where

$$\mathcal{G}(\alpha) = \begin{cases} 1 & \text{if } \alpha > 0 \\ 0 & \text{if } \alpha < 0\,. \end{cases} \tag{21}$$

The power-law relation for the scaling region says that

$$C_r \propto r^{D_C}\,, \tag{22}$$

where D_C is called the correlation dimension.

Figure 8 shows a correlation dimension plot of $\log(C_r)$ against $\log(r)$ for the Koch curve. The slope represents the correlation dimension D_C of the koch curve. By Linear Regression analysis, the slope D_C computed is 1.2667 with error 0.0317, which agrees with the theoratical result, 1.2619.

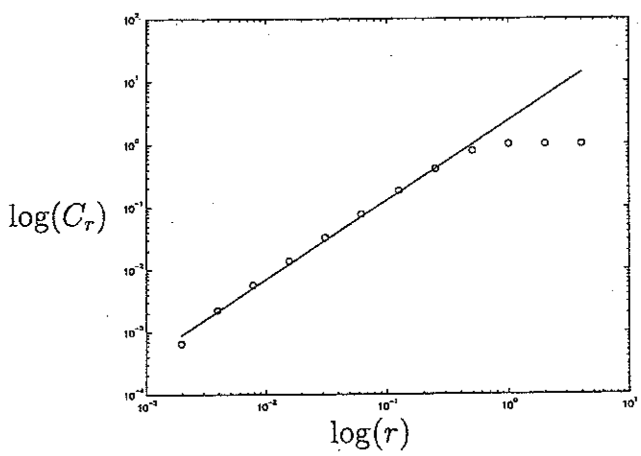

FIGURE 8. *Determining the correlation dimension for Koch curve, the '$\log(r) - \log(C_r)$' plot, with slope $D_C = 1.2667 \pm 0.0317$.*

7. Conclusion

We have already illustrated some constructions of classical fractals, namely the Koch curve and snowflake, the Sierpinski gasket and carpet, and the Menger sponge, which have self-similarity at all scales. Each small portion of the object contains identical copies of the whole.

Fractals are not smooth, they generally look rough. Iterations would not produce a smooth object. It produce a line or surface at various scale, causing non-smoothness and hence non-differentibility of fractal objects.

We also introduced the concept of fractal dimension, which is non-integer in nature, to quantitatively characterize the geometrical structure of complicated fractal objects over a range of scales.

REFERENCES

[1] ADDISON PAUL S. (edt), *Fractals and chaos: an illustrated course*, Institute of Physics Publishing, 1997

[2] CRILLY A. J., EARNSHA R. A., JONES H., *Fractals and Chaos*, Springer-Verlag, 1991

[3] FEDER J., *Fractals*, Plenum Press, 1988

[4] PEITGEN H. O., JÜRGENS H., SAUPE D., *Chaos and Fractals: New Frontiers of Science*, Springer-Verlag, 1992

[5] WILLIAMS G. P., *Chaos Theory Tamed*, Joseph Henry Press, 1997