The Hong Kong University of Science & Technology
Department of Mathematics
Introduction to Scientific Computation I
Spring 2000

LORENZ EQUATION

[redacted]

Abstract. *Lorenz equation was generated by E.N.Lorenz. In 1963,Lorenz studied a two-dimensional fluid cell warmed from below and cooled from above.The resulting convective motion is modelled by a partial differential equation where all the variables which expanded into an infinite modes are set identically to zero except three of them, this gives the lorenz equation.*

1. Introduction

Lornez equation is a coulped set of three quadratic ordinary ordinary differential equations, one in fluid velocity and two in temperature, for fluid convection in a two dimensional layer heated from below. Clearly the original purpose is to provide a model of convection process in our atmosphere which involved air warms ,rises cools and falls again. But amazingly ,these equations completely equivalent to the set of Lorenz equations occur in laser physics explaining the phenomenon of irregularly spiking of lasers. So this equation is worth to study. In this paper we will first analyse the Lorenz equation and find out the stationary points of this equation. Then we will study the chaotic behaviour of it. We will also discuss the period doubling and bifurcation and finally show how to calculate the Hausdorff dimension.

2. Governing equation

The Lorenz equations is given by

$$\begin{aligned} \dot{X} &= \sigma(Y - X), \\ \dot{Y} &= X(R - Z) - Y, \\ \dot{Z} &= XY - bZ. \end{aligned} \tag{1}$$

where
X measures the rate of convective overturning.
Y measures the horizontal temperature variation.
Z measures the vertical temperature variation.
$\sigma \propto$ the Prandtl number (the ratio of the fluid viscosity to the thermal conductivity).
$R \propto$ the Rayleigh number (the difference on temperature between the top and bottom of the gaseous system).
b measures the size of the area.
$X, Y, Z \in \Re$, σ, R, b are positive.

3. Analysis of the Lorenz Equations

By consdering the Liapunov function $V = X^2 + \sigma Y^2 + \sigma Z^2$ where V is a volume element shows that the origin is stable and globally attracting for $R < 1$.For $R > 1$ there are two other stationary points called C_1 and C_2. The stationary points are $(\pm\sqrt{b(R-1)}, \pm\sqrt{b(R-1)}, R-1)$ and C_2 is the one lying in $X > 0$.

---

*Email: [redacted]@ust.hk

Use the following linearized transformaton to deduce the stability properties of the critical points:

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix}' = \begin{pmatrix} -\sigma & \sigma & 0 \\ (R-Z) & -1 & -X \\ Y & X & -b \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} \tag{2}$$

For $R > 1$, there are three eigenvalues,

$$\lambda_1, \lambda_2 = \frac{1}{2} - \sigma - 1 \pm \sqrt{(\sigma-1)^2 + 4\sigma)R}, \lambda_3 = -b.$$

When $R < \frac{\sigma(\sigma+b+3)}{(\sigma-b-1)}$, all three roots have negative real part. This implies that when $\sigma = 10$, $b = \frac{8}{3}$, C_1 and C_2 are stable in the parameter range $1 < R < \frac{470}{19} \approx 24.74$. Let us call this critical R-value be R_H. When $R > R_H$, the complex roots of the equation above have positive real part C_1 and C_2 are non-stable and the real root is negative for all R. When $R = R_H$, the complex eigenvalues cross the imaginary axis and Hopf bifurcation is formed where C_1 and C_2 become non-stable.

4. Strange attractor

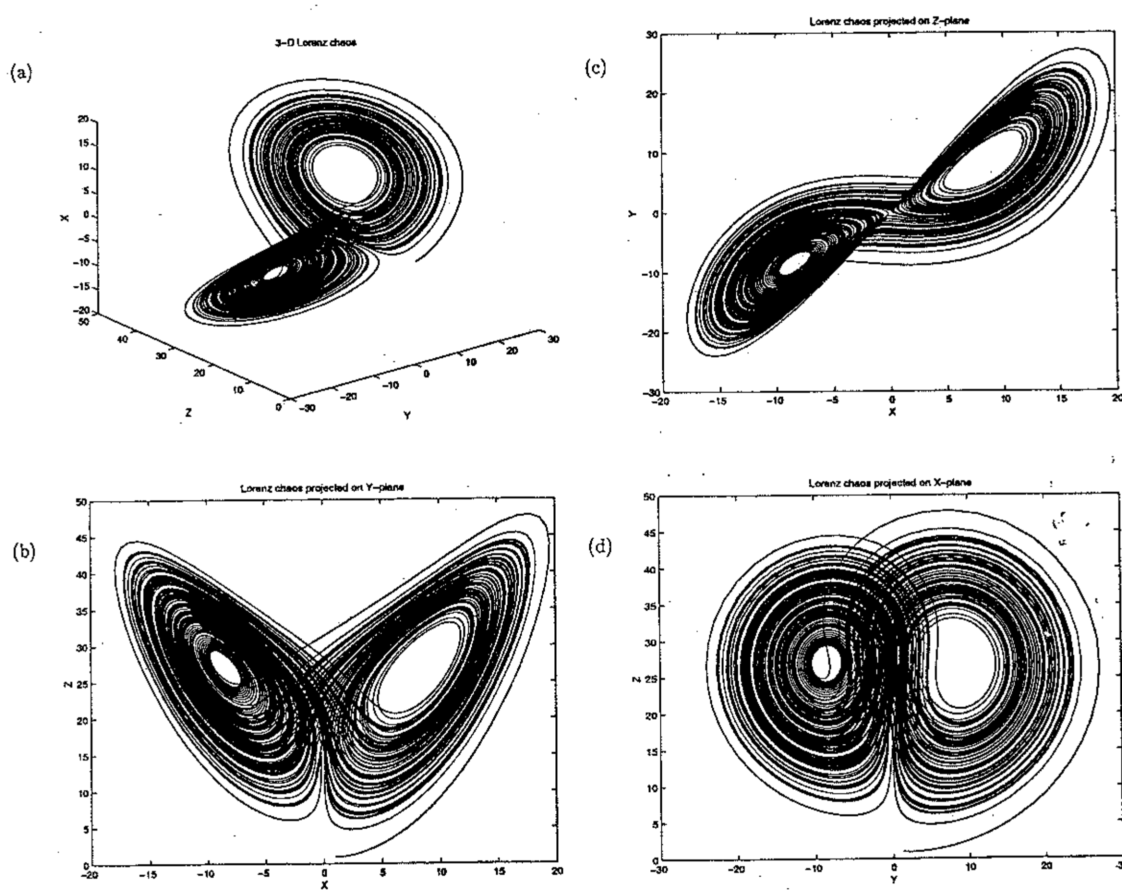


FIGURE 1. *The strange attractor with r=28, σ =10,b = $\frac{8}{3}$ and initial condition (1,1,1)*

By setting $\sigma = 10$, $b = \frac{8}{3}$ and $R = 28$, integrated the Lorenz equations numerically and projected onto the XYZ-space, XY-plane, XZ-plane, YZ-plane respectively(fig1), we can see that the trajectory shown does not interect itself if we consider the XYZ-space(fig1a). Also the trajectories shown are not periodic. Furthermore, we can observed that the trajectories do not appear to show a transient phenomenon. However as we continue the numerical integration the trajectories continue to wind around and around, first on one side and then on the other and never closing up. It is not possible to predict how the trajectory will develop over a long time interval as in **section 6** , you will see that the exact squence of loops which the trajectory makes is extremely sensitive to the changes in initial conditions.

5. Attractor with different R

5.1. Range of R between 0 to 99.524

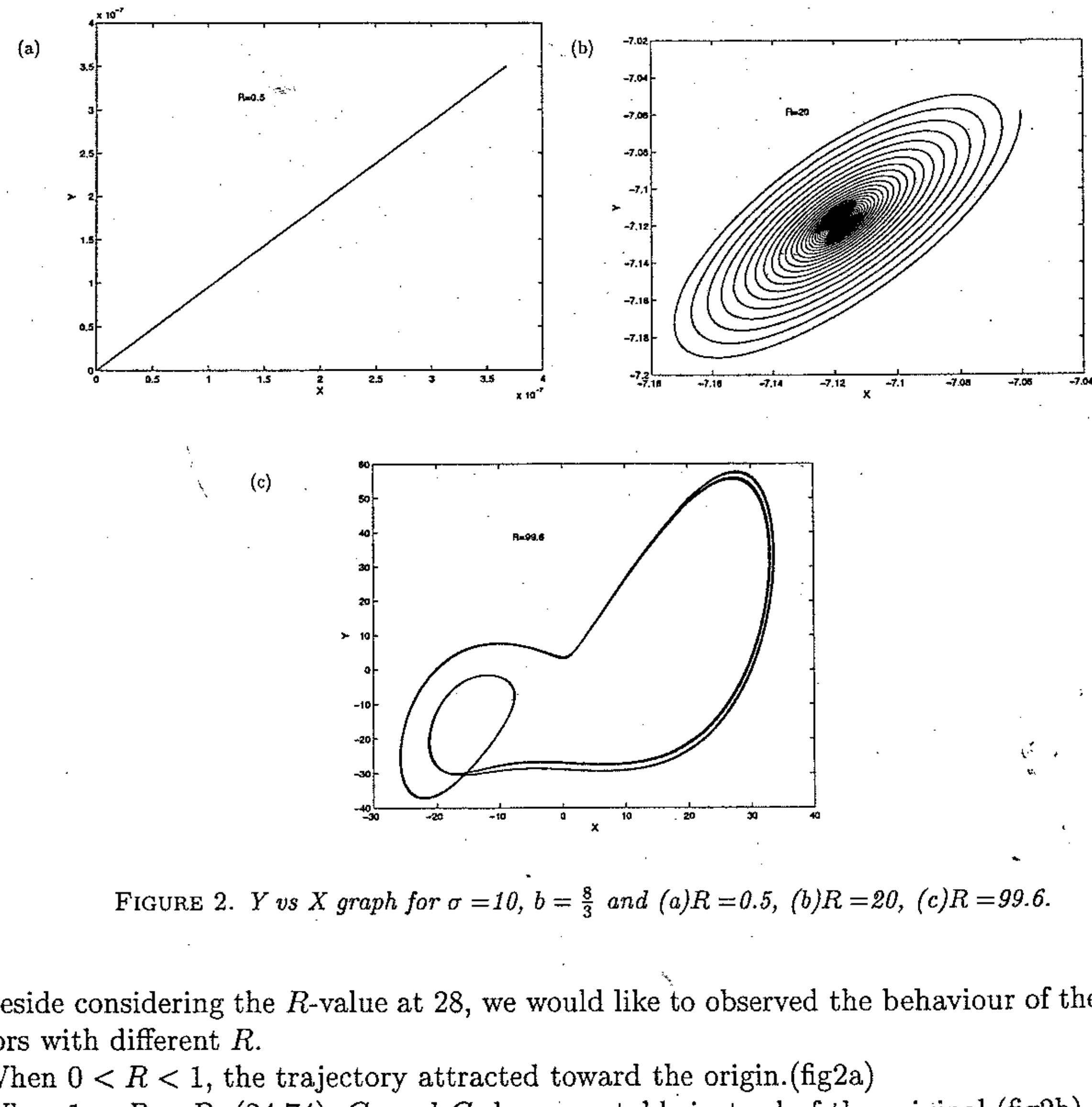


FIGURE 2. *Y vs X graph for σ =10, $b = \frac{8}{3}$ and (a)R =0.5, (b)R =20, (c)R =99.6.*

Beside considering the R-value at 28, we would like to observed the behaviour of the attractors with different R.
When $0 < R < 1$, the trajectory attracted toward the origin.(fig2a)
When $1 < R < R_H(24.74)$, C_1 and C_2 become stable instead of the original.(fig2b)
When $R_H < R < 99.524$, chaotic occurs which has analysis in section 3.(fig1a)

5.2. Period Doubling for R between 99.524 to 100.75

When $R > 99.524$, periodic doubling occur with no attractor.(fig2c)

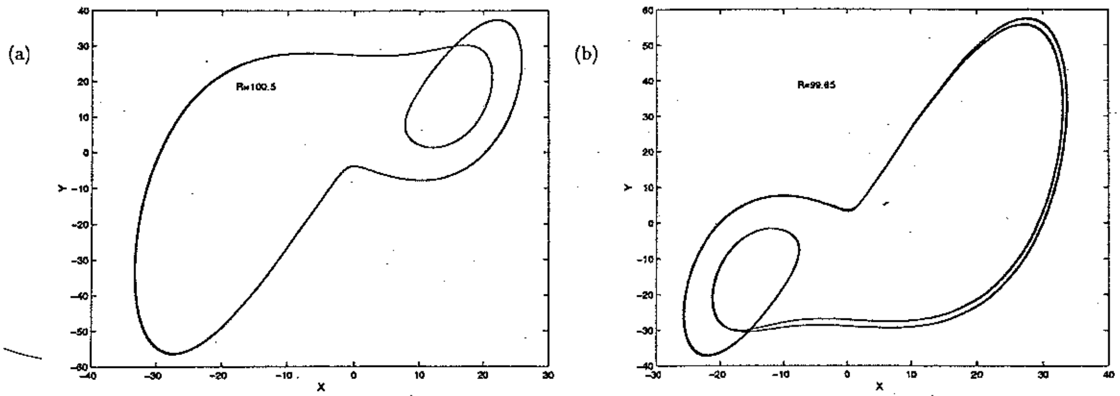


FIGURE 3. *Y vs X graph for σ =10, $b = \frac{8}{3}$ and (a)R =100.5, (b)R =99.65.*

i)Between $R \approx 99.98$ and $R \approx 100.795$ there is a stable periodic X^2Y orbit.
ii)Between $R \approx 99.629$ and $R \approx 99.98$ there is a stableX^2YX^2Y periodic obrit. As R decrease to 99.8, the two loops get closer and eventually merge. This is an example of the period doubling bifurcation.
iii)Between $R \approx 99.547$ and $R \approx 99.629$, the period has doubled again.
Period doubling occur again for the range $145 < R < 166$ and the range $215.364 < R < \infty$.

5.3. Bifurcation

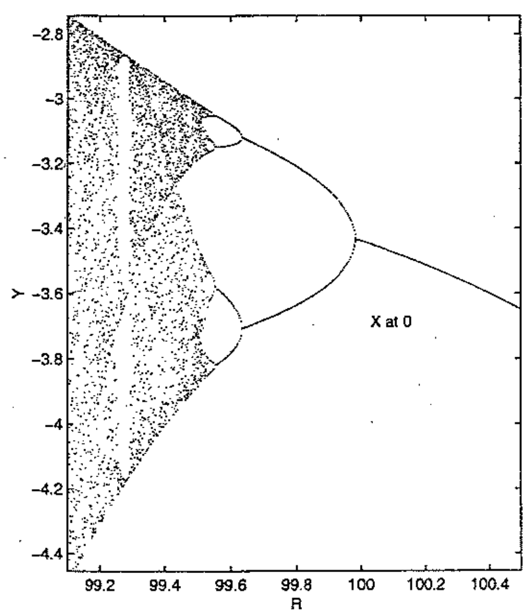


FIGURE 4. *Bifurcation diagram with 100.5> R >99.1.*

As R decreases further, more period doubling bifurcations occur. Franceschini stated that the next bifurcations occur at $R \approx 99.529$ and $R \approx 99.5255$ and the intervial between bifurcations is decreasing rather rapidly as R decreases.

5.4. Semi- periodicity

From the above topic, we know that where is the range of R for period doubling occured, then what is the behaviour just below an period doubling windows?Let us examine this below.

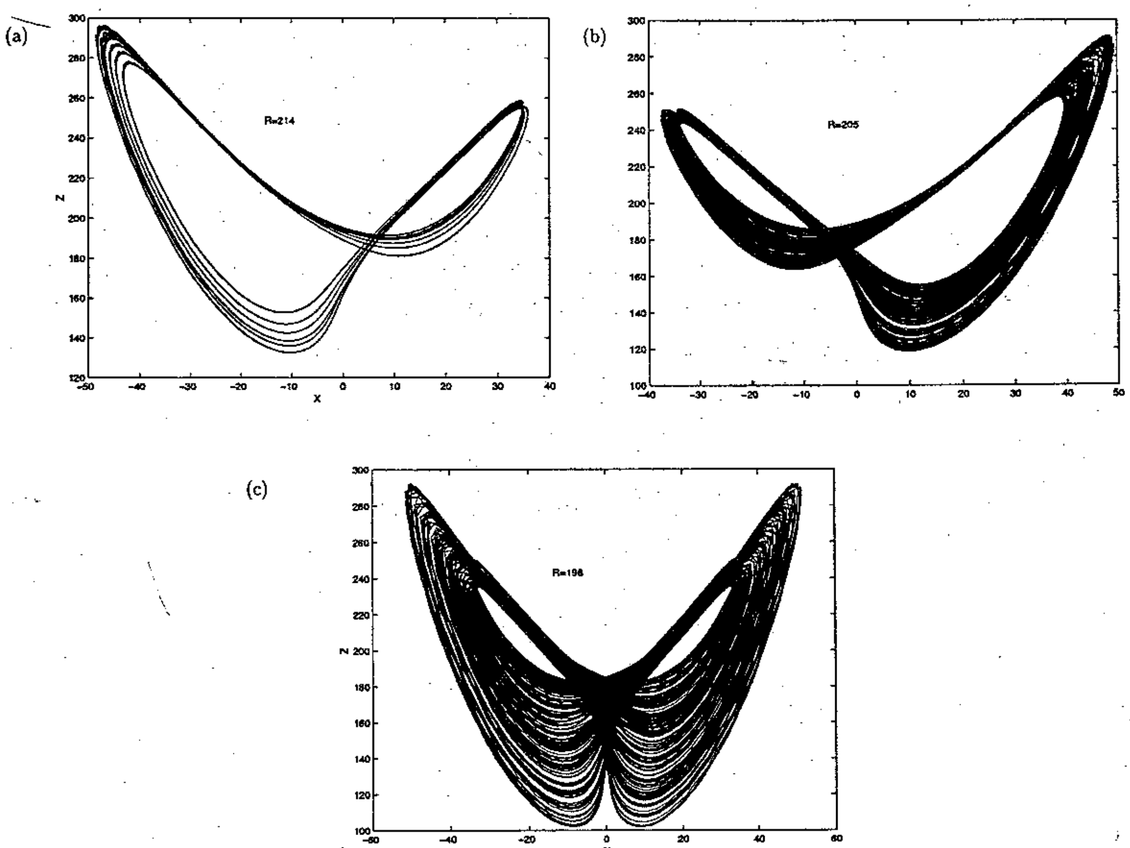
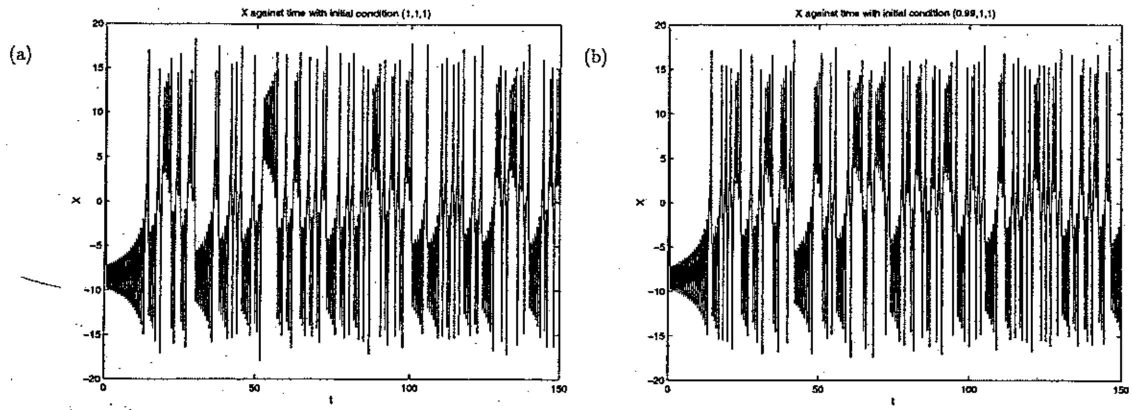


FIGURE 5. *Z vs X graph for* σ *=10,* $b = \frac{8}{3}$ *and (a)R =214, (b)R =205, (c)R =198.*

Fig.5(a) show the behaviour when $R = 214$(The periodic orbit lost its stability in aperiod doubling bifurcation at $R \approx 215.97$); looks like stable periodic behaviour. The tubes here are very thin. Fig.5(b) show the behaviour at $R = 205$, the behaviour is XY semi-periodic and have two non-symmeatric regions. In each of them we can see non-symmetric chaotic behaviour. Fig.5(c) shows the behaviour when $R = 198$. The two non-symmetric tubes expanded enough to overlap and become symmetric chaos.
Semi- periodicity can also be observed in intervals below the other period doubling windows.

6. Sensitivity to Initial Conditions



The two X against time graphs above show that with two slightly different initial conditions(X=1 and X=0.99 respectively and Y,Z remain unchanged), The trajectories are similar from t=0 to t≈18,but after t≥18,the paths of the trajectories have a large difference.This is one of the characteristics of chaos and this explain why long term prediction of the weather is not possible.

7. Hausdorff dimension

By giving an upper bound for the Hausdorff dimension of this attractor, we can write the Lorenz system in the form

$$\begin{aligned} \dot{X} + \sigma(X - Y) &= 0, \\ \dot{Y} + \sigma X + Y + XZ &= 0, \\ \dot{Z} + bZ - XY &= -b(R + \sigma). \end{aligned} \tag{3}$$

making the change of variable $X \to X$, $Y \to Y$, $Z \to Z - R - \sigma$.It is then easy to see that the inequality

$$\begin{aligned} \frac{1}{2}|u|^2 + X^2 + Y^2 + bZ^2 &= b(R + \sigma)Z \\ &\leq (b-1)Z^2 + \frac{b^2}{4(b-1)}(R - \sigma)^2. \end{aligned} \tag{4}$$

deuce that

$$\lim_{t \to \infty} sup|u(t)| \leq \frac{b(R + \sigma)}{2\sqrt{l(b-1)}}$$

where $u = (X, Y, Z)$, l =min(1,σ)(with a bounded initial condition, the trajectory remains bounded).

If S(t) is the mapping $u_0 \to$ u(t), its derivative L(t,u_0) is the linear mapping of $\Re^3$ into $\Re$, $\xi \to L(t,u_0)\xi$ where $L(t,u_0)\xi$ =U(t) is the t-volume of the linearization of (3):

$$U' + \Lambda(u)U = 0, U(0) = \xi$$

where $\lambda(u)U = A_1U + A_2U + B(u)U$ and.

$$A_1 = \begin{pmatrix} \sigma & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}, A_2 = \begin{pmatrix} 0 & -\sigma & 0 \\ \sigma & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, B(u) = \begin{pmatrix} 0 & 0 & 0 \\ Z & 0 & X \\ -Y & -X & 0 \end{pmatrix}. \tag{5}$$

Now determine the numbers $\omega_j(L(t,u_0))$, j= 1,2,3:

$$\begin{aligned} \omega_3(L(t,u_0)) &\le exp[-\sigma + b + 1)t] \\ \omega_2(L(t,u_0)) &\le exp(k_2 - \delta)t \end{aligned} \tag{6}$$

where $k_2 = \frac{-(\sigma+b+1)+m+b(R+\sigma)}{[4\sqrt{l(b-1)}]}$, $m = max(1,b,\sigma)$ and $\delta > 0$ is arbitrarily small. Finally, $\omega_d(L(t,u_0)) < 1$, u_0, if for $d = 1+s$, we have

$$d \le 2 + \frac{k_2}{1+b+\sigma+k_2}$$

This calculation shows that the Hausdorff dimension d of Lorenz attractor with $b = \frac{8}{3}, \sigma = 10, R = 28$ satisfies d<2.538....

8. Conclusion

We have studied some characteristics behaviour of Lorenz equations. We know that for different value of Reyleigh number R, the behaviour of the equation is very difference but for some range of R, the behavoiur repeated. For a certain range of R, period doubling occur and we can observed clearly from the bifurcation diagram. Also we have studied the chaotic behaviour of the lorenz equation and find that it is very sensitive to both changes in initial conditions and changes in the integrating routine.

Lorenz equation has been studied by mathematicians and others for many years and which has generated several significant problems, and there is still some uncertainties.Hope in the future we can anaylsis this equation more deeply and can apply it to other fields like laser physics.

REFERENCES

[1] B.Branner & P.Hjorth,*Real and Complex Dynamical Systems*,Hillerod, Demark, 1993.

[2] C.Sparrow, *The Lorenz Equations: Bifurcation, Chaos, and Strange Attractors*, Springer- Verlag, New York, Heidelberg, 1982.

[3] G.Cherbit, *Fractals Non-integral Dimensions and Applications*, John Wiley & Sons Ltd, 1991.

[4] Kotik K.Lee. *Lectures on Dynamical Systems, Structural Stability and their Applications*, World Scientific Publishing Co. Pte Ltd,1992.