A Transmission Tomography Problem Based on Multiple Arrivals from Paraxial Liouville Equations

Shingyu Leung and Jianliang Qian, UCLA*

Summary

A level set based Eulerian method for paraxial geometrical optics was developed recently by Qian, Leung and Osher (Qian and Leung, 2004; Qian and Leung, 2003; Leung et al., 2004). This provides an efficient method for computing the multivalued arrivaltimes from a point source. We combine these techniques with adjoint state methods to solve a transmission tomography problem where **multivalued** traveltimes are measured on the boundary while the velocity in the media will be determined.

Introduction

Travel-time tomography (Bishop et al., 1985; Delprat-Jannaud and Lailly, 1995; Clarke et al., 2001; Bube and Langan, 1997; Minkoff, 1996; Washbourne et al., 2002; Dahlen et al., 2000; Hung et al., 2000; Hung et al., 2001; Montelli et al., 2004b; Montelli et al., 2004a) is an important class of inverse problems. Theoretically it amounts to determining uniquely a Riemannian metric by knowing the length of geodesics joining points of the boundary of a two-dimensional compact domain. Those geodesics solve a family of Hamilton-Jacobi equations. Computationally, it amounts to designing fast, high resolution methods to invert given data for the unknown velocities or other material parameters (Sei and Symes, 1994; Sei and Symes, 1995; Uhlmann, 2001; Uhlmann and Hansen, 2002; Berryman, 1990; Berryman, 2000a; Berryman, 2000b; Harris et al., 1995; Hyndman and Harris, 1996).

Geophysical travel-time tomography is closely related to X-ray computerized tomography used in medical diagnosis (Dines and Lytle, 1979). The so-called transmission traveltime tomography uses travel-time data between boreholes to invert velocity as a discretized field (Bube and Langan, 1997; Washbourne et al., 2002; Dahlen et al., 2000; Hung et al., 2000; Montelli et al., 2004b; Montelli et al., 2004a). Seismic tomography is usually formulated as a minimization problem that produces a velocity model which minimizes the difference between traveltimes generated by tracing rays through the model and traveltimes measured from the data (Bishop et al., 1985; Bube and Langan, 1997; Washbourne et al., 2002; Berryman, 1990; Berryman, 2000a; Berryman, 2000b; Harris et al., 1995; Hyndman and Harris, 1996). The ray tracing in the above methods is based on Fermat's Principle and the resulting ray path is computed from an explicit discretization of a nonlinear ray path integral; as such, the methods inherit some shortcomings from typical ray tracing methods, such as shadow zones due to non-uniform coverage of the computational domain.

On the other hand, the work in (Sei and Symes, 1994; Sei and Symes, 1995) has shown that it is possible to formulate the transmission tomography problem in a purely Eulerian framework based on finite difference eikonal solvers; (Sei and Symes, 1994; Sei and Symes, 1995) have also derived the linearized eikonal equation and the traveltime gradient based on the adjoint state method borrowed from optimal control.

However, all the above cited methods are based on first arrival traveltimes only. As far as we know, no attempt has been made to formulate transmission tomography problem by using all arrivals, including multivalued traveltimes. The works presented in (Delprat-Jannaud and Lailly, 1995; Clarke et al., 2001) have used multivalued traveltimes from ray tracing methods, but those works are on reflection traveltime tomography which is different from transmission tomography in that rays start at the surface, reflect off interfaces whose depths are to be determined, and return to the surface.

Because multivalued traveltimes and resulting multipathings are common in complex velocity structures, it is necessary to take into account all the arrivals systematically. To achieve this purpose, we propose to formulate transmission tomography by using the Liouville equation PDE framework in phase space.

Based on the paraxial Liouville equations, we implement Liouville equation based adjoint state methods for 2-D transmission traveltime tomography.

Level Set Method for Paraxial Geometrical Optics

In a series of studies on applying Eulerian methods to capture the multivalued solutions in geometrical optics, Qian and Leung developed a level set based paraxial formulation for the eikonal equation (Qian and Leung, 2004; Qian and Leung, 2003). A three-dimensional generalization has been made recently by Leung, Qian and Osher (Leung et al., 2004). This technique provides a computationally efficient way for determining the multi-arrivals for waves traveling in an inhomogeneous media.

Consider the eikonal equation with a point source condition in an isotropic medium which occupies an open, bounded domain $\Omega \subset \mathbf{R}^2$. By isotropy here we mean the wave velocity has no directional dependence. The equation is as follows,

$$|\nabla_{\mathbf{x}} \tau(\mathbf{x}, \mathbf{x}_s)| = \frac{1}{c(\mathbf{x})}\,, \tag{1}$$

A Transmission Tomography Problem Based on Level Set Formulations

where $\mathbf{x}_s$ is the given source point, $c \in C^1(\Omega)$ is the positive velocity. Here $\tau(\mathbf{x}, \mathbf{x}_s)$ denotes the time ("traveltime") taken by a particle moving at velocity $c(\mathbf{x})$ to travel from the source point $\mathbf{x}_s$ to a target point $\mathbf{x} \in \Omega$.

Applying the method of characteristics and the so-called sub-horizontal condition, we obtain a ray tracing system which governs the trajectories of rays in the phase space (x, z, θ). Now we define $\phi = \phi(z, x, \theta)$ such that the zero level set, $\{(x(z), \theta(z)) : \phi(z, x(z), \theta(z)) = 0\}$, gives the location of the reduced bicharacteristic strip $(x(z), \theta(z))$ at z. In essence, we embed the reduced system from the eikonal equation (1) as the velocity field into the level set equation which governs the motion of the bicharacteristic strips in the phase space. For more details, see (Qian and Leung, 2004; Qian and Leung, 2003; Leung et al., 2004).

Governing Equations for Transmission Tomography

Using the level set formulation, we have

$$\begin{aligned} \phi_z + u\phi_x + v\phi_\theta &= 0 \\ T_z + uT_x + vT_\theta &= \frac{1}{c\cos\theta}\,, \end{aligned} \tag{2}$$

where $\mathbf{u} = (u, v) = (\tan\theta, m_z \tan\theta - m_x)$, $m = m(c) = \log c$ with the initial conditions and boundary conditions

$$\begin{aligned} \phi(z_0, \cdot, \cdot) &= x \\ T(z_0, \cdot, \cdot) &= 0 \\ \phi(z, \cdot, \cdot)|_{\partial\Omega} &= \begin{cases} \phi^* & \text{if}(\mathbf{u}\cdot\mathbf{n}) < 0 \\ \text{no b.c. needed} & \text{if}(\mathbf{u}\cdot\mathbf{n}) \geq 0 \end{cases} \\ T(z, \cdot, \cdot)|_{\partial\Omega} &= \begin{cases} T^* & \text{if}(\mathbf{u}\cdot\mathbf{n}) < 0 \\ \text{no b.c. needed} & \text{if}(\mathbf{u}\cdot\mathbf{n}) \geq 0 \end{cases} \end{aligned} \tag{3}$$

where $\mathbf{n}$ is the outward normal vector of $\partial\Omega$, $(\cdot)^*$ denotes the measured values on both the boundaries $\partial\Omega$ and the level $z = z_f$, and $\Omega = (x_{\min}, x_{\max}) \times (\theta_{\min}, \theta_{\max})$. We further define $\tilde{\Omega} = \Omega \times (z_0, z_f)$ and $\Omega_p = (x_{\min}, x_{\max}) \times (z_0, z_f)$.

Assuming that we can measure the data $\phi(z, \cdot, \cdot)|_{\partial\Omega}$, $\phi(z_f, \cdot, \cdot)$, $T(z, \cdot, \cdot)|_{\partial\Omega}$, $T(z_f, \cdot, \cdot)$ and $m|_{\partial\Omega_p}$, the transmission tomography problem is to determine m, and therefore c, in Ω_p such that the solutions from the system (2) together with the boundary conditions (3) would be as close to these measurements as possible.

In this paper, we propose to minimize the following energy

$$\begin{aligned} E(m) &= \frac{1}{2}\int_\Omega (\phi - \phi^*)^2|_{z=z_f} + \frac{\beta}{2}\int_\Omega (T - T^*)^2|_{z=z_f} \\ &\quad + \frac{1}{2}\int_z\int_{\partial\Omega} (\mathbf{u}\cdot\mathbf{n})(\phi - \phi^*)^2 \\ &\quad + \frac{\beta}{2}\int_z\int_{\partial\Omega} (\mathbf{u}\cdot\mathbf{n})(T - T^*)^2\,. \end{aligned} \tag{4}$$

Now, we perturb m by $\epsilon\tilde{m}$. Using the adjoint state method, the corresponding change in E, denoted by δE, is given by

$$\delta E = \epsilon\int_{\Omega_p} \tilde{m}\left\{(f_1)_x - (f_2)_z + \frac{\beta}{c}f_3 + f_4\right\}\,, \tag{5}$$

where

$$\begin{aligned} f_1(x,z) &= \int_\theta \lambda_1\phi_\theta + \beta\lambda_2 T_\theta \\ f_2(x,z) &= \int_\theta \tan\theta\,(\lambda_1\phi_\theta + \beta\lambda_2 T_\theta) \\ f_3(x,z) &= \int_\theta \frac{\lambda_2}{\cos\theta} \\ f_4(x,z) &= \frac{1}{2}\left\{\frac{\partial}{\partial x}\left[(\phi-\phi^*)^2 + \beta(T-T^*)^2\right] - \tan\theta\cdot\right. \\ &\quad \left.\frac{\partial}{\partial z}\left[(\phi-\phi^*)^2 + \beta(T-T^*)^2\right]\right\}\Bigg|_{\theta=\theta_{\min}}^{\theta=\theta_{\max}} \end{aligned} \tag{6}$$

with λ_1 and λ_2 satisfying

$$\begin{aligned} (\lambda_1)_z + (u\lambda_1)_x + (v\lambda_1)_\theta &= 0 \\ (\lambda_2)_z + (u\lambda_2)_x + (v\lambda_2)_\theta &= 0\,, \end{aligned} \tag{7}$$

with the *initial* conditions on $z = z_f$

$$\lambda_1(z = z_f) = \phi^* - \phi \ \text{ and } \ \lambda_2(z = z_f) = T^* - T \tag{8}$$

and the boundary conditions

$$\begin{aligned} \lambda_1|_{\partial\Omega} &= \begin{cases} \phi^* - \phi & \text{if} \quad (\mathbf{u}\cdot\mathbf{n}) > 0 \\ \text{no b.c. needed} & \text{if} \quad (\mathbf{u}\cdot\mathbf{n}) \leq 0 \end{cases} \\ \lambda_2|_{\partial\Omega} &= \begin{cases} T^* - T & \text{if} \quad (\mathbf{u}\cdot\mathbf{n}) > 0 \\ \text{no b.c. needed} & \text{if} \quad (\mathbf{u}\cdot\mathbf{n}) \leq 0\,. \end{cases} \end{aligned} \tag{9}$$

To have fast convergence for the method of gradient descent, we use

$$\tilde{m} = -(I - \nu\Delta)^{-1}\left\{(f_1)_x - (f_2)_z + \frac{\beta}{c}f_3 + f_4\right\}\,, \tag{10}$$

where I is the identity operator, Δ is the Laplace operator and $\nu \geq 0$ controls the weight of regularity one wants. Zero boundary condition is imposed in inverting the operator $(I - \nu\Delta)$. With this $\tilde{m}$, we have

$$\delta E = -\epsilon\int_{\Omega_p} (|\tilde{m}|^2 + \nu|\nabla\tilde{m}|^2) \leq 0\,. \tag{11}$$

The equality is achieved when $\tilde{m} \equiv 0$.

Examples

We applied the proposed formulations to a waveguide model (Qian and Leung, 2004; Qian and Leung, 2003) and a Gaussian model where the exact velocities are given by

$$
\begin{aligned}
c_1(x,z) &= 1.1 - \exp(-0.5x^2) \\
c_2(x,z) &= 1.5 - 0.6\exp\left(-\frac{x^2+(z-0.75)^2}{0.5^2}\right) \quad (12)
\end{aligned}
$$

Figures 1 and 2 show the initial conditions, converged solutions, the errors in the approximated velocities and also the changes in the energy of the corresponding tests.

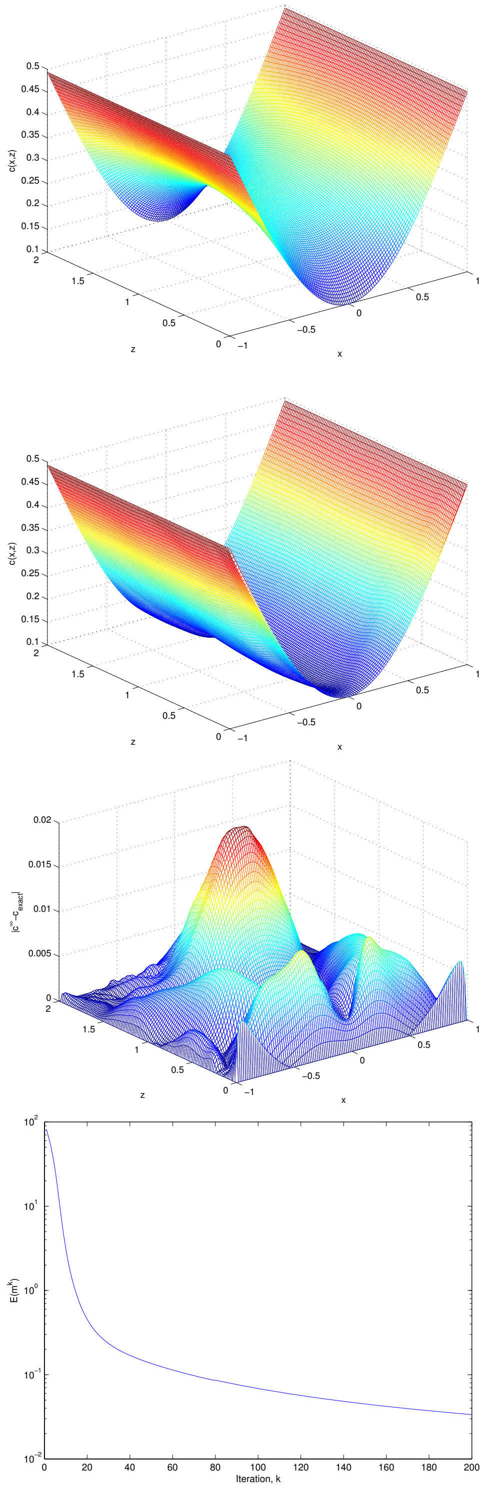

Fig. 1: Waveguide model.

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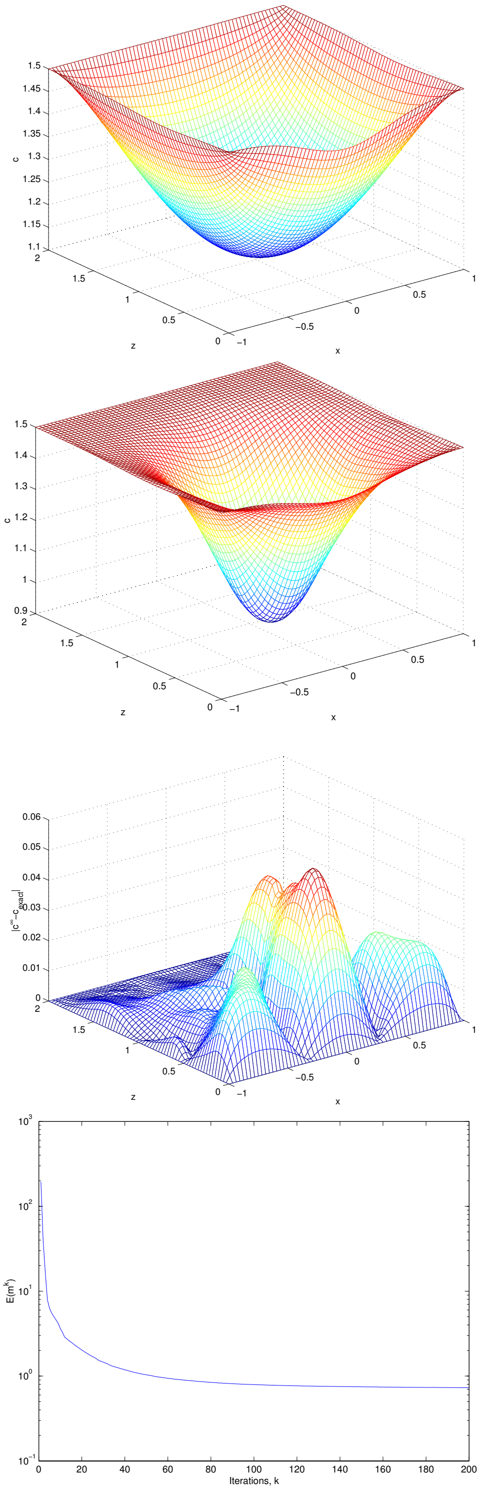

Fig. 2: Gaussian model.

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