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DYNAMICS OF DOMAIN WALL IN THIN FILM DRIVEN BY SPIN CURRENT

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Abstract. The dynamics of magnetization under the applied spin current is modeled by the generalized Landau-Lifshitz-Gilbert equation with a spin transfer torque term. Using matched asymptotic expansion with the domain wall thickness ϵ as the small parameter, we derive analytically the dynamic law for the domain wall motion induced by the spin current. We show that the domain wall driven by adiabatic current spin-transfer torque moves with a decreasing velocity and eventually stops. With a pinning potential, the domain wall motion is a damped oscillation around the pinning site with an intrinsic frequency which is independent of the strength of the current. When the AC current is applied, the dynamic law shows that the frequency of the applied current can be turned to maximize the amplitude of the oscillation. The results obtained are consistent with the recent experimental and numerical results.

1. Introduction.

The subject of current induced magnetic reversal has received considerable interest recently due to its promising applications for magnetic nanodevices [1], [2], [3], [4], [5], [6] and [10]. The physics of the current induced magnetization reversal involves interplay between nonequilibrium conduction electrons and local magnetization. When a spin-polarized current goes through a domain wall, it produces a spin torque on the magnetization. This current spin-transfer torque can be written in the following form [7]

$$\Gamma_{st} = -\frac{b_J}{M_s^2}\mathbf{M} \times (\mathbf{M} \times \frac{\partial \mathbf{M}}{\partial x}) - \frac{c_J}{M_s}\mathbf{M} \times \frac{\partial \mathbf{M}}{\partial x}, \tag{1}$$

where $\mathbf{M}$ is the magnetization vector, M_s is the saturation magnetization, $b_J = Pj_e\mu_B/eM_s$ and $c_J = \xi b_J$, P is the spin polarization of the current, j_e is the current density in the x direction, μ_B is Bohr magneton, and ξ is a dimensionless constant which describes the degree of the nonadiabaticity between the spin of the nonequilibrium conduction electrons and local magnetization. The "b_j" and "c_j" terms are called "adiabatic" and "nonadiabatic" spin torque respectively.

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With the spin-transfer torque (1), the generalized Landau-Lifshitz-Gilbert equation (LLG) describing dynamics of magnetization can be written as:

$$\frac{\partial \mathbf{M}}{\partial t} = -\gamma \mathbf{M} \times \mathbf{H}_{eff} + \frac{\alpha}{M_s} \mathbf{M} \times \frac{\partial \mathbf{M}}{\partial t} + \Gamma_{st}, \tag{2}$$

where γ is the gyromagnetic ratio, α is the damping parameter, and $\mathbf{H}_{eff}$ is the effective field including anisotropy field, exchange field, demagnetization field, the external magnetic field.

We consider a ferromagnetic thin film strip in the xy plane, with the easy axis x along the long length of the strip as shown in Fig. 1. The in plane transverse domain wall separates two domains in the strip. Therefore the demagnetization field can be simplified as the shape anisotropy $4\pi M_3$ along the z direction. The effective field $\mathbf{H}_{eff}$ is written explicitly as:

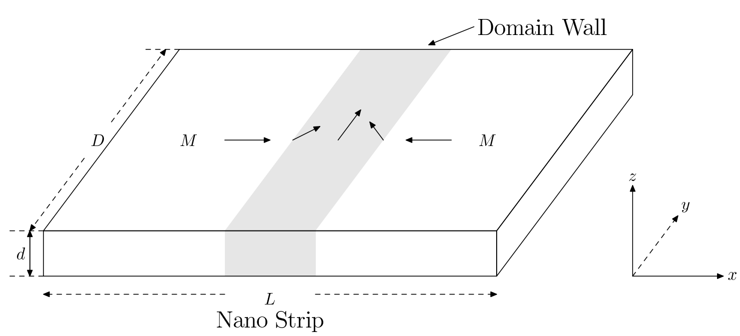
$$\mathbf{H}_{eff} \quad = \quad \frac{H_K M_1}{M_s} e_x + \frac{2A}{M_s^2} \triangle \mathbf{M} - 4\pi M_3 e_z + H_e e_x, \tag{3}$$

where H_K is the anisotropy constant, A is the exchange coefficient, and $4\pi M_3$ is the simplified demagnetization field as the shape anisotropy. H_e is the applied magnetic field in the x-direction. M_1 and M_3 are x and z components of $\mathbf{M}$ respectively. The dynamics of the domain wall under the applied magnetic field has been studied in various situations (see e.g., [14] and [13]). However, the dynamics of the domain wall under the applied current is expected to be very different.

We assume that the current flows along the x direction. In this case, the magnetization $\mathbf{M}$ is assumed to be a function of x and t only. The system is then reduced to a 1D model. Li and Zhang [8] solved the 1D Landau-Lifshitz-Gilbert equation numerically and analytically and their analysis follow the Walker's analysis of domain wall motion by introducing a trial funtion [16]. Their numerical simulations of dynamics of the in-plane Neel wall driven by adiabatic spin-transfer torque have shown that the domain wall moves at the initial application of the current. The velocity then decreases in time and eventually stops [8]. During the motion, the domain wall also develops a small out-of-plane angle and the wall width decreases.

In realistic nano-wires or nano-strips, there are various pinning sources such as defects and roughness and the domain walls are not free to move. To model the effect of defect, a pinning potential is usually introduced. One form of the pinning potential is the following [10] and [15]

$$H_p = \begin{cases} 0, & |d - d_0| >= q_0, \\ \pm 2K_p \dfrac{d - d_0}{q_0}, & |d - d_0| < q_0, \end{cases} \tag{4}$$



where d is the position of the domain wall center, d_0 is the defect position. K_p and q_0 measure the strength and the width of the defect. The sign before K_p is positive for the tail-to-tail wall and negative for the head-to-head wall. Therefore the effective field $\mathbf{H}_{eff}$ including the pinning potential can be written as:

$$\mathbf{H}_{eff} = \frac{H_K M_1}{M_s} e_x + \frac{2A}{M_s^2} \triangle \mathbf{M} - 4\pi M_3 e_z + H_e e_x + H_p e_x, \tag{5}$$

Recent experimental results [15] have shown that the injection of spin-polarized current below a threshold value through a domain wall confined to a pinning potential results in its precessional motion within the potential well. By using a short train of current pulses, whose length and spacing are tuned to this precession frequency, the domain wall's oscillations can be resonantly amplified.

The numerical results in [17] have shown that when confined in a pinning potential, the domain wall driven by a dc current oscillates around the equilibrium position. There is a critical current below which the oscillation amplitude decreases with time due to the damping. The critical current is defined as the minimum current required for the depinning. The numerical results also show that the oscillation amplitude can be resonantly amplified when the frequency of the current pulse is appropriately chosen.

To gain analytical insight into the domain wall dynamics, we study the Landau-Lifshitz-Gilbert system (2) by the matched asymptotic expansions. In Sec. 2, we derive the dynamic law of the domain wall motion driven by the current. The asymptotic results are also compared with the numerical solutions. In Sec. 3, we derive the current driven dynamic law of the domain wall confined to a pinning potential. The oscillatory motion law of the domain wall driven by both dc or ac currents are obtained. The resonant amplification of the domain wall oscillation is also shown from the dynamic law obtained.

2. **Asymptotic analysis of domain wall motion driven by current.** The generalized LLG equation (2) with the spin current torque can be rewritten in terms of the polar angle θ and azimuth angle ϕ defined relative to the z axis. Let's assume

$$\begin{cases} M_1 = M_s \sin\theta \cos\varphi, \\ M_2 = M_s \cos\theta, \\ M_3 = M_s \sin\theta \sin\varphi. \end{cases} \tag{6}$$

We then have

$$\alpha \frac{\partial \theta}{\partial t} - \sin\theta \frac{\partial \varphi}{\partial t} = \gamma \frac{2A}{M_s}\left(\frac{\partial^2 \theta}{\partial x^2} - \sin\theta\cos\theta\left(\frac{\partial \varphi}{\partial x}\right)^2\right) - \gamma H_k \sin\theta\cos\theta$$
$$- \gamma 4\pi M_s \sin\theta \cos\theta \sin^2\varphi - \gamma H_e \sin\theta - b_J \sin\theta \frac{\partial \varphi}{\partial x} + c_J \frac{\partial \theta}{\partial x}, \tag{7}$$

$$\frac{\partial \theta}{\partial t} + \alpha \sin\theta \frac{\partial \varphi}{\partial t} = \gamma \frac{2A}{M_s}\left(2\cos\theta \frac{\partial \varphi}{\partial x}\frac{\partial \theta}{\partial x} + \sin\theta \frac{\partial^2 \varphi}{\partial x^2}\right)$$
$$- \gamma 4\pi M_s \sin\theta \sin\varphi\cos\varphi + b_J \frac{\partial \theta}{\partial x} + c_J \sin\theta \frac{\partial \varphi}{\partial x}. \tag{8}$$

To write the equations into the dimensionless form, we scale the spatial variable x by l_0 where l_0 is the characteristic size of the strip. We denote the dimensionless domain wall thickness by

$$\varepsilon = \frac{1}{l_0}\sqrt{\frac{2A}{M_s H_K}},$$

and study the behavior of the solution for small ε. Since H_k, H_e have the same physical dimension as M_s, we write $H_k/M_s = \bar{h}_k$, $H_e/M_s = \varepsilon \bar{h}_e$ so that the applied field is weak and of ε order. The numerical results in [17] suggested that when current is applied, the wall motion occurs on a short time scale. Also note that $M_s l_0^2/(2A\gamma)$ has the dimension of time, we introduce the fast time scale

$$\tau = \frac{t}{\varepsilon^2 M_s l_0^2/(2A\gamma)}.$$

We assume that the applied current is weak and denote

$$\frac{b_J}{(\gamma l_0 H_k)} = b_j \varepsilon^2, \qquad \frac{c_J}{(\gamma l_0 H_k)} = c_j \varepsilon^2.$$

With the above scalings, Eqs. (7) and (8) can now be written in the dimensionless form as:

$$\begin{aligned}\alpha\frac{\partial\theta}{\partial\tau} - \sin\theta\frac{\partial\varphi}{\partial\tau} =& \varepsilon^2\left(\frac{\partial^2\theta}{\partial x^2} - \sin\theta\cos\theta\left(\frac{\partial\varphi}{\partial x}\right)^2\right) - \sin\theta\cos\theta - p\sin\theta\cos\theta\sin^2\varphi \\ & - \varepsilon h_e\sin\theta - \varepsilon^2 b_j\sin\theta\frac{\partial\varphi}{\partial x} + \varepsilon^2 c_j\frac{\partial\theta}{\partial x}, \end{aligned} \tag{9}$$

$$\begin{aligned}\frac{\partial\theta}{\partial\tau} + \alpha\sin\theta\frac{\partial\varphi}{\partial\tau} =& \varepsilon^2\left(2\cos\theta\frac{\partial\varphi}{\partial x}\frac{\partial\theta}{\partial x} + \sin\theta\frac{\partial^2\varphi}{\partial x^2}\right) \\ & - p\sin\theta\sin\varphi\cos\varphi + \varepsilon^2 b_j\frac{\partial\theta}{\partial x} + \varepsilon^2 c_j\sin\theta\frac{\partial\varphi}{\partial x}. \end{aligned} \tag{10}$$

where $p = 4\pi/\bar{h}_k$ and $h_e = \bar{h}_e/\bar{h}_k$.

The initial condition is an in-plane Neel wall given by

$$\begin{cases} \theta = 2\tan^{-1}(\exp\frac{x-d_0}{\varepsilon}), \\ \varphi = 0, \end{cases} \tag{11}$$

where d_0 is the initial position of the domain wall.

2.1. **Outer expansion.** We now assume that, away from the domain wall, the solutions θ and φ can be expanded in powers of ε as:

$$\theta(x,\tau) = \theta_0(x,\tau) + \varepsilon\theta_1(x,\tau) + \varepsilon^2\theta_2(x,\tau) + \ldots, \tag{12}$$

$$\varphi(x,\tau) = \varphi_0(x,\tau) + \varepsilon\varphi_1(x,\tau) + \varepsilon^2\varphi_2(x,\tau) + \ldots. \tag{13}$$

It is easy to see that the outer expansions of the initial conditions give

$$\theta_j(x,0) = \begin{cases} 0, & x < d_0, \\ \pi, & x > d_0, \end{cases} \tag{14}$$

$$\varphi_j(x,0) = 0, \tag{15}$$

for all $j = 0, 1, 2....$

Substituting the expansions (12) and (13) into Eqs. (9) and (10) gives equations in leading order $O(1)$:

$$\alpha\frac{\partial\theta_0}{\partial\tau} - \sin\theta_0\frac{\partial\varphi_0}{\partial\tau} = \sin\theta_0\cos\theta_0(1 + p\sin^2\varphi_0), \tag{16}$$

$$\frac{\partial\theta_0}{\partial\tau} + \alpha\sin\theta_0\frac{\partial\varphi_0}{\partial\tau} = p\sin\theta_0\sin\varphi_0\cos\varphi_0. \tag{17}$$

We then have:

$$(1+\alpha^2)\frac{\partial\theta_0}{\partial\tau} = \alpha\sin\theta_0\cos\theta_0(1+p\sin^2\varphi_0) + p\sin\theta_0\sin\varphi_0\cos\varphi_0, \tag{18}$$

$$(1+\alpha^2)\sin\theta_0\frac{\partial\varphi_0}{\partial\tau} = \alpha p\sin\theta_0\sin\varphi_0\cos\varphi_0 - \sin\theta_0\cos\theta_0(1+p\sin^2\varphi_0). \tag{19}$$

Obviously, $\theta_0(x,\tau)\equiv\theta_0(x,0)=0$ or π is a solution. Then $\phi_0(0,\tau)$ can be arbitrary and we choose it to be the same as the initial condition, i.e., $\phi_0(0,\tau)=0$.

The equations in order $O(\varepsilon)$ are written as:

$$\alpha\frac{\partial\theta_1}{\partial\tau} - \theta_1\frac{\partial\varphi_0}{\partial\tau} = \theta_1(1-p\sin^2\varphi_0), \tag{20}$$

$$\frac{\partial\theta_1}{\partial\tau} - \alpha\theta_1\frac{\partial\varphi_0}{\partial\tau} = -p\theta_1\sin\varphi_0\cos\varphi_0. \tag{21}$$

Obviously, we have $\theta_1\equiv 0$ and $\varphi_1\equiv 0$.

The equations in order $O(\varepsilon^2)$ are:

$$\alpha\frac{\partial\theta_2}{\partial\tau} - \theta_2\frac{\partial\varphi_0}{\partial\tau} = \theta_1(-1+p\sin^2\varphi_0), \tag{22}$$

$$\frac{\partial\theta_2}{\partial\tau} - \alpha\theta_2\frac{\partial\varphi_0}{\partial\tau} = -p\theta_2\sin\varphi_0\cos\varphi_0. \tag{23}$$

We again have $\theta_2=0$.

Similarly we can show that $\theta_i=\theta_0$ and $\varphi_i=0$ for all $i\geq 1$. Therefore, the outer solution should be $\theta=0$ or $\theta=\pi$ and $\varphi=0$. If the domain wall is head-to-head mode, θ is from 0 to π, otherwise, it is from π to 0 for the tail-to-tail wall.

2.2. **Inner expansion.** We now study the behavior of the solution within the domain wall. We introduce an inner variable

$$\eta = \frac{x-d(\tau)}{\varepsilon},$$

where $d(\tau)$ is the position of the domain wall. Numerical results also suggested that the angle φ is almost a constant inside the domain wall. Therefore we look for solutions of the following form:

$$\theta(\eta,\tau) = \theta_0(\eta) + \varepsilon\theta_1(\eta) + \varepsilon^2\theta_2(\eta,\tau) + \dots, \tag{24}$$

$$\varphi(\eta,\tau) = \varphi_0(\tau) + \varepsilon\varphi_1(\tau) + \varepsilon^2\varphi_2(\eta,\tau) + \dots, \tag{25}$$

$$d(\tau) = d_0(\tau) + \varepsilon d_1(\tau) + \varepsilon^2 d_2(\tau) + \dots. \tag{26}$$

We now have

$$\frac{d\theta}{d\tau} = -\frac{d_\tau(\tau)}{\varepsilon}\theta_\eta.$$

Substituting the above expansions into Eq. (9) and we have, at the leading order $O(1/\varepsilon)$,

$$-\alpha d_{0\tau}\theta_{0\eta} = 0, \tag{27a}$$

$$d_{0\tau}\theta_{0\eta} = 0. \tag{27b}$$

which gives $d_0=0$, i.e., the domain wall is not moving in leading order.

At the next order $O(1)$, we have:

$$\alpha(-d_{1\tau})\theta_{0\eta} - \sin\theta_0\varphi_{0\tau} = \theta_{0\eta\eta} - \sin\theta_0\cos\theta_0(1+p\sin^2\varphi_0), \tag{28a}$$

$$(-d_{1\tau})\theta_{0\eta} + \alpha\sin\theta_0\varphi_{0\tau} = -p\sin\theta_0\sin\varphi_0\cos\varphi_0. \tag{28b}$$

The solutions are:

$$d_{1\tau} = 0, \quad \varphi_0 = 0, \quad \theta_0 = 2\tan^{-1}(\exp(\eta)), \tag{29}$$

which gives the steady state Neel wall solution.

The $O(\varepsilon)$ equations give:

$$\alpha(-d_{2\tau})\theta_{0\eta} - \sin\theta_0\varphi_{1\tau} = \theta_{1\eta\eta} - \theta_1\cos 2\theta_0 - h_e\sin\theta_0 + c_j\theta_{0\eta}, \tag{30a}$$

$$(-d_{2\tau})\theta_{0\eta} + \alpha\sin\theta_0\varphi_{1\tau} = -p\sin\theta_0\varphi_1 + b_j\theta_{0\eta}. \tag{30b}$$

From (29), we have $\theta_{0\eta} = \sin\theta_0$. Substituting into Eq. (30b), we have

$$-d_{2\tau} + \alpha\varphi_{1\tau} = -p\varphi_1 + b_j. \tag{31}$$

From Eq. (30a), we have:

$$\alpha(-d_{2\tau})\sin\theta_0 - \sin\theta_0\varphi_{1\tau} = \theta_{1\eta\eta} - \theta_1\cos 2\theta_0 + (c_j - h_e)\sin\theta_0. \tag{32}$$

Multipling the equation by $\theta_{0\eta}$ and integrating with respect to η, we obtain the following:

$$\int_{-\infty}^{\infty}(\theta_{1\eta\eta} - \theta_1\cos 2\theta_0 + (\alpha d_{2\tau} + \varphi_{1\tau} + c_j - h_e)\sin\theta_0)\theta_{0\eta}d_\eta = 0. \tag{33}$$

The outer solution provides the boundary conditions of θ at $\pm\infty$

$$\theta = \begin{cases} 0, & \eta \to -\infty, \\ \pi, & \eta \to \infty, \end{cases} \tag{34}$$

and its all the derivatives with respect η approach to 0 on both sides.

Integrating by parts of the first two terms in (33) gives:

$$\int_{-\infty}^{\infty}\theta_{1\eta}(\theta_{0\eta\eta} - \sin\theta_0\cos\theta_0)d_\eta = 0, \tag{35}$$

since $\theta_{0\eta\eta} = \sin\theta_0\cos\theta_0$.

Then we are left with:

$$\int_0^{\pi}(\alpha d_{2\tau} + \varphi_{1\tau} + c_j - h_e)\sin\theta_0 d_{\theta_0} = 0, \tag{36}$$

which is written as

$$(\alpha d_{2\tau} + \varphi_{1\tau} + c_j - h_e)\int_0^{\pi}\sin\theta_0 d_{\theta_0} = 0. \tag{37}$$

Since

$$\int_0^{\pi}\sin\theta_0 d_{\theta_0} = 1,$$

we obtain:

$$\alpha d_{2\tau} + \varphi_{1\tau} + c_j - h_e = 0. \tag{38}$$

From Eq. (32), we also have:

$$\theta_{1\eta\eta} - \theta_1\cos 2\theta_0 = 0, \tag{39}$$

which can be solved to give $\theta_1 = \sin\theta_0$.

Therefore, the results obtained from the $O(\varepsilon)$ equations are:

$$\alpha d_{2\tau} + \varphi_{1\tau} = h_e - c_j, \tag{40}$$

$$-d_{2\tau} + \alpha\varphi_{1\tau} = -p\varphi_1 + b_j, \tag{41}$$

$$\theta_1 = \sin\theta_0. \tag{42}$$

Eqs. (40)-(41) can be solved to give

$$\varphi_1 = \frac{h_e - c_j + \alpha b_j}{\alpha p}\Big(1 - \exp(\frac{-\alpha p \tau}{1+\alpha^2})\Big), \tag{43}$$

$$d_{2\tau} = \frac{h_e - c_j}{\alpha} - \frac{h_e - c_j + \alpha b_j}{\alpha(1+\alpha^2)} \exp\Big(\frac{-\alpha p \tau}{1+\alpha^2}\Big). \tag{44}$$

These are the dynamic laws for the out of plane angle φ and the domain wall position in the higher orders. It is easy to see that

$$d_{2\tau}|_{\tau=0} = \frac{\alpha(h_e - c_j) - b_j}{1+\alpha^2}, \qquad \varphi_1|_{\tau=0} = 0,$$
$$d_{2\tau}|_{\tau=\infty} = \frac{h_e - c_j}{\alpha}, \qquad \varphi_1|_{\tau=\infty} = \frac{h_e - c_j + \alpha b_j}{\alpha p}.$$

Consider the case when there is no applied magnetic field ($h_e = 0$) and the non-adiabatic effect of the current is negligible ($c_j \approx 0$). At the initial application of the current ($\tau = 0$), we have $\varphi_1 = 0$ and the domain wall velocity

$$v = \varepsilon^2 d_{2\tau} = -\varepsilon^2 \frac{b_j}{1+\alpha^2}.$$

On the other hand, as τ increases, the domain wall velocity $v \to 0$ exponentially fast as $\tau \to \infty$. This shows that the domain wall stops quickly and current alone cannot drive the domain wall for long distance. As τ increases, φ_1 is non-zero indicating domain wall distortion. The distortion reaches maximum at

$$\varphi = \varepsilon\varphi_1 = \varepsilon\frac{b_j}{p}, \quad \text{as} \quad \tau \to \infty.$$

When the applied magnetic field h_e is nonzero, the velocity of the domain wall will be alerted initially when a current is applied. But the terminal velocity is independent of the applied current.

We now compare the asymptotic results (43) and (44) with the solution from the full equations (9) and (10) for two cases. In our comparison, we consider a Co nanowire, $4\pi M_s = 1.8 \times 10^4$ Oe, $H_k = 500$ Oe, $\gamma = 1.9 \times 10^7$ Oe^{-1}s^{-1}, $M_s = 14.46 \times 10^5$ A/m, and $A = 2.0 \times 10^{-11}$ J/m. In Fig. 1, we show the results with $H_e = 100$ Oe, $b_j = 0$ m/s and in Fig. 2, we show the results with $H_e = 0$ Oe, $b_J = -300$ m/s. In both cases, the out-of-plane magnetization component m_3 at the domain wall center and the domain wall displacement are plotted as functions of time. The asymptotic dynamic laws agree well with the numerical results of the full equations.

3. **Effect of pinning potential on domain wall motion driven by current.**

We now study the dynamics of the domain wall in the presence of a pinning potential H_p given by (4). We assume that the pinning is strong and is of order $1/\varepsilon$, i.e.,

$$H_p = \frac{k_p(d - d_0)}{\varepsilon},$$

where d is the position of the domain wall and d_0 is the position of the defect and will be assumed to be $d_0 = 0$. The dimensionless equations are then written as:

$$\begin{aligned}\alpha\frac{\partial\theta}{\partial\tau} - \sin\theta\frac{\partial\varphi}{\partial\tau} =& \varepsilon^2\Big(\frac{\partial^2\theta}{\partial x^2} - \sin\theta\cos\theta(\frac{\partial\varphi}{\partial x})^2\Big)\\ &- \sin\theta\cos\theta - p\sin\theta\cos\theta\sin^2\varphi - \varepsilon h_e\sin\theta\\ &- \varepsilon^2 b_j\sin\theta\frac{\partial\varphi}{\partial x} + \varepsilon^2 c_j\frac{\partial\theta}{\partial x} + k_p\frac{d}{\varepsilon}\sin\theta, \qquad (45a)\end{aligned}$$

$$\begin{aligned}\frac{\partial\theta}{\partial\tau} + \alpha\sin\theta\frac{\partial\varphi}{\partial\tau} =& \varepsilon^2(2\cos\theta\frac{\partial\varphi}{\partial x}\frac{\partial\theta}{\partial x} + \sin\theta\frac{\partial^2\varphi}{\partial x^2})\\ &- p\sin\theta\sin\varphi\cos\varphi + \varepsilon^2 b_j\frac{\partial\theta}{\partial x}. \qquad (45b)\end{aligned}$$

The initial condition is the same in-plane Neel wall given by (11).

3.1. **Outer expansion.** It is easy to see that the leading order outer expansion gives $sin\theta \equiv 0$. We again have that the outer solutions of θ and φ are

$$\theta(x,\tau) = \begin{cases} 0, & x < d_0, \\ \pi, & x > d_0. \end{cases} \qquad (46)$$

$$\varphi(x,\tau) = 0. \qquad (47)$$

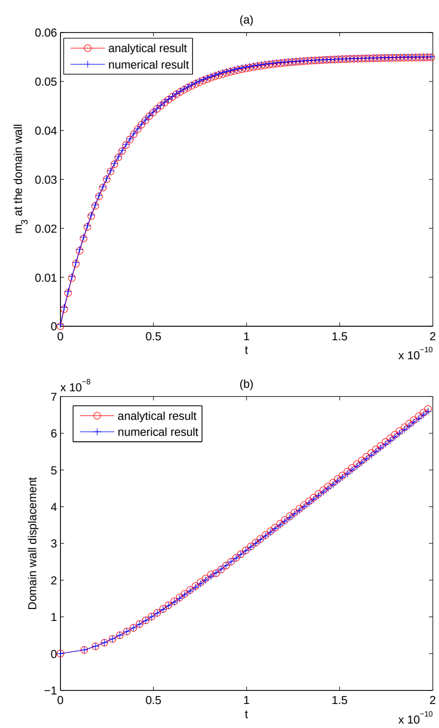


FIGURE 1. (a) The magnetization component m_3 as a function of time (s) by numerical calculation and asymptotic expansion. (b) The displacement (m) of domain wall as a function of time (s) of numerical and asymptotic results. $H_e = 100$ Oe, $\alpha = 0.1$.

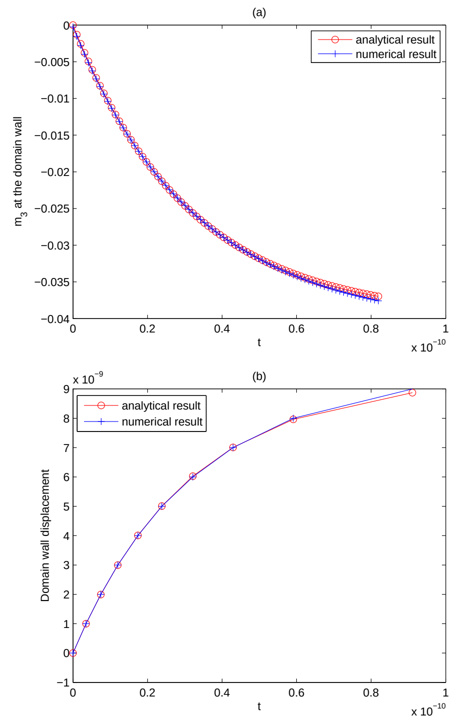


FIGURE 2. (a) The magnetization component m_3 as a function of time (s) of numerical and asymptotic results. (b) The displacement (m) of domain wall as a function of time (s) of numerical and asymptotic results. $b_J = -300$ m/s, $\alpha = 0.1$.

3.2. **Inner expansion.** We now study the behavior of the solution within the domain wall. We again introduce a rescaled variable

$$\eta = \frac{x - d(\tau)}{\varepsilon}.$$

We look for solutions θ, φ and $d(\tau)$ that can be expanded in the following form

$$\theta = \theta_0(\eta) + \varepsilon\theta_1(\eta) + \varepsilon^2\theta_2(\eta, \tau) + \dots, \tag{48}$$

$$\varphi = \varphi_0(\tau) + \varepsilon\varphi_1(\tau) + \varepsilon^2\varphi_2(\eta, \tau) + \dots, \tag{49}$$

$$d = d_0(\tau) + \varepsilon d_1(\tau) + \varepsilon^2 d_2(\tau) + \dots. \tag{50}$$

At the leading order $O(1/\varepsilon)$, we have:

$$-\alpha d_{0\tau}\theta_{0\eta} = k_p d_0 \sin\theta_0, \tag{51a}$$

$$d_{0\tau}\theta_{0\eta} = 0. \tag{51b}$$

The solution is $d_0 \equiv 0$.

At the next order $O(1)$, we have:

$$\alpha(-d_{1\tau})\theta_{0\eta} - \sin\theta_0\varphi_{0\tau} = \theta_{0\eta\eta} - \sin\theta_0\cos\theta_0(1 + p\sin^2\varphi_0) + k_p d_1 \sin\theta_0, \tag{52a}$$

$$-d_{1\tau}\theta_{0\eta} + \alpha\sin\theta_0\varphi_{0\tau} = -p\sin\theta_0\sin\varphi_0\cos\varphi_0. \tag{52b}$$

The solutions are:

$$d_{1\tau} = 0, \quad \varphi_0 = 0, \quad \theta_0 = 2\tan^{-1}(\exp(\eta)), \tag{53}$$

which also gives $d_1 = 0$.

The $O(\varepsilon)$ equations give:

$$\begin{aligned} -\alpha d_{2\tau}\theta_{0\eta} - \sin\theta_0\varphi_{1\tau} &= \theta_{1\eta\eta} - \theta_1\cos 2\theta_0 - h_e\sin\theta_0 \\ &\quad + c_j\theta_{0\eta} + k_p d_2\sin\theta_0, \end{aligned} \tag{54a}$$

$$-d_{2\tau}\theta_{0\eta} + \alpha\sin\theta_0\varphi_{1\tau} = -p\sin\theta_0\varphi_1 + b_j\theta_{0\eta}. \tag{54b}$$

Since $\theta_{0\eta} = \sin\theta_0$, we have from Eq. (54b) that

$$-d_{2\tau} + \alpha\varphi_{1\tau} = -p\varphi_1 + b_j. \tag{55}$$

From the first equation in Eq. (54a), we have:

$$\alpha(-d_{2\tau})\sin\theta_0 - \sin\theta_0\varphi_{1\tau} = \theta_{1\eta\eta} - \theta_1\cos 2\theta_0 + (c_j - h_e + k_p d_2)\sin\theta_0. \tag{56}$$

Multiplying the equation by $\theta_{0\eta}$ and integrating with respect to η, we obtain the following:

$$\int_{-\infty}^{\infty}\Big(\theta_{1\eta\eta} - \theta_1\cos 2\theta_0 + (\alpha d_{2\tau} + \varphi_{1\tau} + c_j - h_e + k_p d_2)\sin\theta_0\Big)\theta_{0\eta}d_\eta = 0. \tag{57}$$

Integrating by parts of the first two terms in (57) gives:

$$\int_{-\infty}^{\infty}\theta_{1\eta}(\theta_{0\eta\eta} - \sin\theta_0\cos\theta_0)d_\eta = 0. \tag{58}$$

Then we are left with:

$$\int_0^{\pi}(\alpha d_{2\tau} + \varphi_{1\tau} + c_j - h_e + k_p d_2)\sin\theta_0 d_{\theta_0} = 0, \tag{59}$$

which is written as

$$(\alpha d_{2\tau} + \varphi_{1\tau} + c_j - h_e + k_p d_2)\int_0^{\pi}\sin\theta_0 d_{\theta_0} = 0. \tag{60}$$

Since

$$\int_0^{\pi}\sin\theta_0 d_{\theta_0} = 1,$$

we obtain:

$$\alpha d_{2\tau} + \varphi_{1\tau} + c_j - h_e + k_p d_2 = 0. \tag{61}$$

From Eq. (56), we have $\theta_1 = \sin\theta_0$.

Therefore, the results obtained from the $O(\varepsilon)$ equations are:

$$\alpha d_{2\tau} + \varphi_{1\tau} = h_e - k_p d_2 - c_j, \tag{62}$$

$$-d_{2\tau} + \alpha\varphi_{1\tau} = -p\varphi_1 + b_j, \tag{63}$$

$$\theta_1 = \sin\theta_0. \tag{64}$$

Eqs. (62) and (63) are the dynamic equations for the position of the domain wall and the out of plane angle (in higher ε^2 order).

3.2.1. *Oscillation driven by dc current.* We first study the effect of the pinning potential on the dynamics of domain wall under a constant dc current. Use the relation $c_j = \xi b_j$ and from (62) and (63), we have:

$$(1+\alpha^2)d_{2\tau} = p\varphi_1 + \alpha h_e - \alpha k_p d_2 - (\alpha\xi + 1)b_j, \tag{65}$$

$$(1+\alpha^2)\varphi_{1\tau} = -\alpha p\varphi_1 + h_e - k_p d_2 + (\alpha - \xi)b_j. \tag{66}$$

The matrix form of the system is:

$$\begin{pmatrix} d_2 \\ \varphi_1 \end{pmatrix}_\tau = \frac{1}{1+\alpha^2}\begin{pmatrix} -\alpha k_p & p \\ -k_p & -\alpha p \end{pmatrix}\begin{pmatrix} d_2 \\ \varphi_1 \end{pmatrix} + \frac{1}{1+\alpha^2}\begin{pmatrix} \alpha h_e - (\alpha\xi+1)b_j \\ h_e + (\alpha-\xi)b_j \end{pmatrix}. \tag{67}$$

The eigenvalues of the coefficient matrix are:

$$\lambda_\pm = \frac{-\alpha(k_p+p) \pm \sqrt{\alpha^2(k_p+p)^2 - 4(1+\alpha^2)pk_p}}{2(1+\alpha^2)}. \tag{68}$$

It is easy to see that the eigenvalues $\lambda_\pm$ are complex when

$$\Delta = \alpha^2(k_p+p)^2 - 4(1+\alpha^2)pk_p < 0.$$

This is the case when the damping coefficient α is small. Denote

$$r = \frac{-\alpha(k_p+p)}{2(1+\alpha^2)}, \quad \text{and} \quad \omega = \frac{\sqrt{4pk_p - \alpha^2(k_p-p)^2}}{2(1+\alpha^2)}.$$

We then have the general solutions

$$\begin{pmatrix} d_2 \\ \varphi_1 \end{pmatrix} = c_1\begin{pmatrix} \cos\omega\tau \\ \frac{(\alpha k_p + r)}{p}\cos\omega\tau - \frac{\omega}{p}\sin\omega\tau \end{pmatrix}e^{r\tau} + c_2\begin{pmatrix} \sin\omega\tau \\ \frac{(\alpha k_p + r)}{p}\sin\omega\tau + \frac{\omega}{p}\cos\omega\tau \end{pmatrix}e^{r\tau} + \begin{pmatrix} (h_e - \xi b_j)/k_p \\ b_j/p \end{pmatrix}. \tag{69}$$

Here c_1 and c_2 are determined from the initial conditions of $d(0) = 0$, $\varphi(0) = 0$.

When $h_e = 0$ and $\xi = 0$, the solutions in (69) represent exponentially damped oscillations of d and φ around their equilibrium positions 0 and b_j/p respectively. ω is the intrinsic frequency of the oscillation which is independent of the applied current. If $\alpha = 0$, we have $r = 0$. Therefore the solutions describe undamped oscillation with the intrinsic frequency $\omega = \sqrt{pk_p}$.

3.2.2. *Resonant amplification of domain wall oscillation by ac current.* The behavior of the damped oscillation of the domain wall around the pinning site with an intrinsic frequency described in the previous section motivates the use of ac current or current pulses to drive the domain wall. Experimental results in [10] and [15] have shown that oscillations in the domain wall position and momentum can be resonantly amplified by using a short sequence of current pulses, whose lengths and separations are turned to its oscillation frequency. In this section, we show this resonant amplification effect from the dynamic laws derived in the last section. We consider an ac current $b_j = b_a \cos\omega_1\tau$ with frequency ω_1 and amplitude b_a. For simplicity, we assume $h_e = 0$, $\xi = 0$. From the dynamic laws (62) and (63), we can write down the general solution d_2 as

$$d_2 = (c_1 \cos \omega\tau + c_2 \sin \omega\tau)e^{r\tau} + d_{2p}, \tag{70}$$

where r, ω are defined in the previous section and d_{2p} is a particular solution of the form:

$$d_{2p} = A \cos \omega_1\tau + B \sin \omega_1\tau, \tag{71}$$

where A, and B are determined to be

$$A = \frac{b_a \omega_1^2 \alpha(k_p + p)}{(pk_p - (1+\alpha^2)\omega_1^2)^2 + \alpha^2(k_p+p)^2\omega_1^2},$$
$$B = \frac{b_a \omega_1 (k_p p - (1+\alpha^2)\omega_1^2)}{(pk_p - (1+\alpha^2)\omega_1^2)^2 + \alpha^2(k_p+p)^2\omega_1^2}.$$

The first part of the solution (70) will again be exponentially damped. However, the second part d_{2p} is undamped and has an fixed amplitude

$$C = \frac{b_a \omega_1}{\sqrt{(pk_p - (1+\alpha^2)\omega_1^2)^2 + \alpha^2(k_p+p)^2\omega_1^2}}.$$

It is easy to see that C reaches a maximum value

$$C_m = \frac{b_a}{\alpha(k_p + p)},$$

when

$$\omega_1 = \sqrt{\frac{pk_p}{1+\alpha^2}},$$

which is the resonance frequency. Note that if there is no damping in the system, i.e., if $\alpha = 0$, $C_m \to \infty$ when $\omega_1 \to \sqrt{pk_p}$, which is the intrinsic frequency $\omega = \sqrt{pk_p}$ for the undamped system.

4. **Conclusions.** Using matched asymptotic expansions, we have derived the dynamic laws for the transverse domain wall motion induced by spin current. The analytic results can be used to explain many of the experimentally observed results. Without the pinning effect, domain wall driven by adiabatic current spin-transfer torque moves with a decreasing velocity and eventually stops. With a pinning potential, the domain wall oscillates around the pinning site with an intrinsic frequency that is independent of the strength of the current. When the AC current is applied, the frequency of the applied current can be turned to maximized the amplitude of the oscillation amplitude. The results are consistent with the existing experimental results. Finally, we note that considered only localized stray field in this work. It will be interesting to study the behavior of the domain wall when the full nonlocal demagnetization field [11] and [12] is used .

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REFERENCES

[1] J. C. Slonczewski, *Current-driven excitation of magnetic multilayers*, J. Magn. Magn. Mat., **159** (1996), 1–7.

[2] L. Berger, *Emission of spin waves by a magnetic multilayer traversed by a current*, Phys. Rev. B, **54** (1996), 9353.

[3] L. Berger, *New origin for spin current and current-induced spin precession in magnetic multilayers*, J. Appl. Phys., **89** (2001), 5521.

[4] G. A. Prinz, *Spin-polarized transport*, Physics Today, April, (1995), 58–63.

[5] C. J. Garcia-Cervera and X. P. Wang, *Spin-Polarized currents in ferromagnetic multilayers*, J. Comp. Phys., **224** (2007), 699–711.

[6] C. J. Garcia-Cervera and X. P. Wang, *Spin-Polarized transport: Existence of weak solutions*, Discrete and Continuous Dynamical Systems (B) , **7** (2007), 87–100.

[7] S. Zhang and Z. Li, *Roles of nonequilibrium conduction electrons on the magnetization dynamics of ferromagnets*, Phys. Rev. Lett., **93** (2004), 127204.

[8] Z. Li and S. Zhang, *Domain wall dynamics driven by adiabatic spin-transfer torques*, Phys. Rev. B, **70** (2004), 024417.

[9] J. He, Z. Li and S. Zhang, *Current-driven domain-wall depinning*, J. Appl. Phys., **98** (2005), 016108.

[10] Luc Thomas, M. Hayashi, X. Jiang, R. Moriya, C. Rettner and S. P. Parkin, *Oscillatory dependence of current driven magnetic domain wall motion on current pulse length*, Nature Letters, **443** (2006), 197.

[11] Amikam Aharoni, "Introduction to the Theory of Ferromagnetism, International Series of Monographs on Physics," Oxford University Press, 1996.

[12] C. J. Garcia-Cervera, *One-dimensional magnetic domain walls*, Euro. Jnl. Appl. Math., **15** (2004), 451–486.

[13] Christof Melcher, *Domain wall motion in ferromagnetic layers*, Phys. D, **192** (2004), 249–264.

[14] Antonio Capella, Christof Melcher and Felix Otto, *Wave-type dynamics in ferromagnetic thin films and the motion of Neel walls*, Nonlinearity, **20** (2007), 2519–2537.

[15] Luc Thomas, M. Hayashi, X. Jiang, R. Moriya, C. Rettner and S. P. Parkin, *Resonant amplification of magnetic domain wall motion by a train of current pulses*, Science Reports, **315** (2007), 1553.

[16] N. L. Schryer and L. R. Walker, *The motion of 180 domain walls in uniform dc magnetic fields*, J. Appl. Phys. **45** (1974), 5406.

[17] L. Yang, "Current Induced Domain Wall Motion: Analysis and Simulations," Ph.D thesis, HKUST, 2008.

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