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SPIN-POLARIZED TRANSPORT: EXISTENCE OF WEAK SOLUTIONS

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Abstract. A system modeling spin-polarized transport in ferromagnetic multilayers is considered. In this model, the spin accumulation is described by a quasilinear diffusion equation with discontinuous, measurable coefficients. This equation is coupled to the Landau-Lifshitz-Gilbert equation, a nonlinear, nonlocal equation describing the precession of the magnetization in the ferromagnetic layers. The global existence of weak solutions is proved.

1. **Introduction.** Layered magnetic structures form an integral part of most magnetic recording devices [1]. Traditionally the recording process is performed by applying a magnetic field in order to orient the magnetization in a given direction. Since these systems are typically bistable, information can be stored in digital form.

A new mechanism for magnetization reversal in magnetic multilayers was introduced by Slonczewski [2] and by Berger [3, 4]. In their approach, a current flows perpendicular to the layers. The electron spins are polarized by the first layer, and when the current reaches the second layer, this polarization exerts an additional torque on the magnetization. The use of spin-polarized currents in semiconductor devices could potentially revolutionize the magnetic recording industry, and has been the subject of much research during the past few years (see [5, 6] for a review).

In the model introduced by Slonczweski and Berger, the spin accumulation is assumed to be uniform. However, spatial variations in the spin density have been found to be important in recent magneto-resistance experiments [7, 8, 9]. A new model for the spin-magnetization system that takes into account the diffusion process of the spin accumulation through the multilayer has been presented by Zhang *et al.* [10, 11], where only the one-dimensional case was considered.

In this article, we consider an extension to three dimensions of the model derived in [10], and we start the study of this model for spin-polarized transport.

We consider a magnetic multilayer consisting of two ferromagnetic films Ω_1 and Ω_2, of thickness d_1 and d_2, respectively, separated by a non-magnetic interlayer Ω_0

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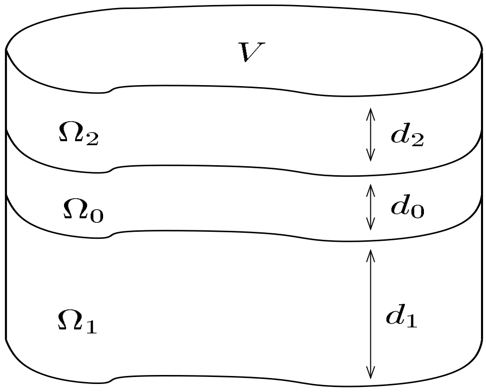

FIGURE 1. Schematic of a multilayer: Ω_1 and Ω_2 are ferromagnetic materials. Ω_0 is a non-magnetic conductor.

of thickness d_0 (see Fig. 1). The multilayer occupies the volume $\Omega = \Omega_0 \cup \Omega_1 \cup \Omega_2$, where

$$\begin{aligned} \Omega_1 &= V \times [0, d_1] \\ \Omega_0 &= V \times [d_1, d_1 + d_0] \\ \Omega_2 &= V \times [d_1 + d_0, d_1 + d_0 + d_2] \end{aligned}$$

The spin accumulation $\mathbf{s}$ is defined on Ω and the magnetization $\mathbf{m}$ is defined on the two magnetic layers $\Sigma = \Omega_1 \cup \Omega_2$, and extended as zero outside. We will assume that the boundary of V is smooth. Denote $Q_T = [0, T] \times \Omega$ and $R_T = [0, T] \times \Sigma$.

In the absence of spin currents, the relaxation process of the magnetization distribution is described by the Landau-Lifshitz-Gilbert equation (LLG) [12, 13]:

$$\frac{\partial \mathbf{m}}{\partial t} = -\mathbf{m} \times \mathbf{h} + \alpha \mathbf{m} \times \frac{\partial \mathbf{m}}{\partial t}, \quad \mathbf{x} \in \Sigma, \tag{1}$$

with Neumann boundary condition:

$$\frac{\partial \mathbf{m}}{\partial \nu} = 0, \quad \text{on } \partial\Sigma, \tag{2}$$

where ν represents the outward unit normal on $\partial\Sigma$. In (1), $\mathbf{m} : \Sigma \to \mathbb{R}^3$ is the magnetization field, and it satisfies $|\mathbf{m}| = 1$ a.e. The constant $\alpha > 0$ is the damping parameter, and the second term on the right hand side is usually referred to as Gilbert damping. The local field $\mathbf{h}$ can be derived from the Landau-Lifshitz energy:

$$F_{LL}[\mathbf{m}] = \int_\Sigma \Phi(\mathbf{m}) + \frac{1}{2} \int_\Sigma |\nabla \mathbf{m}|^2 - \frac{1}{2} \int_\Sigma \mathbf{h}_d \cdot \mathbf{m}, \tag{3}$$

and

$$\mathbf{h} = -\frac{\delta F_{LL}}{\delta \mathbf{m}} = -\nabla_{\mathbf{m}} \Phi + \Delta \mathbf{m} + \mathbf{h}_d. \tag{4}$$

In (3), the first and second terms are the anisotropy and exchange energies, respectively. We will only consider uniaxial materials with easy axis parallel to the OX-axis, for which $\Phi(\mathbf{m}) = m_2^2 + m_3^2$. The last term in (3) is the self-induced energy, and $\mathbf{h}_d = -\nabla u$ is the demagnetizing field. The magnetostatic potential, u, solves the differential equation

$$\Delta u = \text{div } (\mathbf{m} \chi_\Sigma) \text{ in } \mathbb{R}^3, \tag{5}$$

in the sense of distributions. The solution to this equation is

$$u(\mathbf{x}) = \int_\Sigma \nabla N(\mathbf{x}-\mathbf{y}) \cdot \mathbf{m}(\mathbf{y})\, d\mathbf{y}, \tag{6}$$

where $N(\mathbf{x}) = -\frac{1}{4\pi|\mathbf{x}|}$ is the Newtonian potential in $\mathbb{R}^3$ [14]. For simplicity, all material constants have been set equal to one.

Using the vector identity

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}, \tag{7}$$

equation (1) may be written as

$$\frac{\partial \mathbf{m}}{\partial t} = -\frac{1}{1+\alpha^2}\mathbf{m} \times \mathbf{h} - \frac{\alpha}{1+\alpha^2}\mathbf{m} \times (\mathbf{m} \times \mathbf{h}), \quad \mathbf{x} \in \Sigma. \tag{8}$$

The second term on the right hand side is the phenomenological damping term introduced by Landau and Lifshitz in 1935 [12].

If $\alpha = 0$, and neglecting the anisotropy and self-induced fields, equation (1) becomes

$$\frac{\partial \mathbf{m}}{\partial t} = -\mathbf{m} \times \Delta \mathbf{m}. \tag{9}$$

Equation (9) describes the symplectic flow of harmonic maps to the unit sphere, and is usually referred to as the Schrödinger map equation [15, 16, 17].

On the other hand, in the limit $\alpha \to \infty$, equation (8) reduces to [18]

$$\frac{\partial \mathbf{m}}{\partial t} = -\mathbf{m} \times (\mathbf{m} \times \Delta \mathbf{m}) = \Delta \mathbf{m} + |\nabla \mathbf{m}|^2 \mathbf{m}. \tag{10}$$

This equation describes the heat flow of harmonic maps [19, 20, 21].

The spin polarized effect is described by the following Cauchy problem:

$$\begin{cases} \frac{\partial \mathbf{s}}{\partial t} = -\mathrm{div}\, \mathbf{J_s} - D_0(\mathbf{x})\mathbf{s} - D_0(\mathbf{x})\mathbf{s} \times \mathbf{m}, \\ \frac{\partial \mathbf{m}}{\partial t} = -\mathbf{m} \times (\mathbf{h} + \mathbf{s}) + \alpha \mathbf{m} \times \frac{\partial \mathbf{m}}{\partial t}, \\ \mathbf{s}(x,0) = \mathbf{s}_0(x), \\ \mathbf{m}(x,0) = \mathbf{m}_0(x), \\ \frac{\partial \mathbf{s}}{\partial \nu}|_{\partial\Omega} = 0, \\ \frac{\partial \mathbf{m}}{\partial \nu}|_{\partial\Sigma} = 0, \end{cases} \tag{11}$$

where $\mathbf{s}$ is the *spin accumulation*, $\mathbf{J_s}$ is the spin current, and $D_0(\mathbf{x})$ is the diffusion coefficient. The spin current is

$$\mathbf{J_s} = \mathbf{m} \otimes \mathbf{J}_e - D_0(\mathbf{x}) \left[\nabla \mathbf{s} - \beta \mathbf{m} \otimes (\nabla \mathbf{s} \cdot \mathbf{m})\right]. \tag{12}$$

where $\mathbf{J}_e$ is the applied electric current, and $0 < \beta < 1$ is the spin-polarization parameter. The additional term in the effective field in the LLG equation corresponds to the interaction

$$F_s[\mathbf{s}, \mathbf{m}] = -\int_\Sigma \mathbf{m} \cdot \mathbf{s}\, d\mathbf{x}. \tag{13}$$

Equation (11) results from considering the transport in a multilayer as a diffusive process [10, 11]. The last term in the spin equation represents the interaction between the spin accumulation and the background magnetization, and is responsible for the transfer of angular momentum between them.

There is by now a large body of literature regarding the LLG equation (see [18, 22, 23, 24, 25] and the references therein). However, the spin-transfer system (11) poses some new difficulties, since the magnetization is zero outside Σ, and the diffusion coefficient is discontinuous at the interface between the magnetic and non-magnetic layers. The numerical solution of (11) has been considered by the authors in [26].

If the magnetization is smooth, equation (1) is equivalent to

$$\frac{\partial \mathbf{m}}{\partial t} = -\text{div }(\mathbf{m}\times\nabla\mathbf{m}) - \mathbf{m}\times(-\nabla_{\mathbf{m}}\Phi + \mathbf{h}_d + \mathbf{s}) + \alpha\mathbf{m}\times\frac{\partial \mathbf{m}}{\partial t}. \tag{14}$$

This formulation motivates the following definition of weak solutions:

Definition 1. Let $\mathbf{s}_0 \in H^1(\Omega)$, $\mathbf{m}_0 \in H^1(\Sigma)$, $|\mathbf{m}_0| = 1$ a.e. We say $(\mathbf{s}, \mathbf{m})$ is a global weak solution of equations (11) if

1. $\mathbf{m} \in L_\infty(\mathbb{R}^+; H^1(\Sigma))$, $\frac{\partial \mathbf{m}}{\partial t} \in L_2(\mathbb{R}^+ \times \Sigma)$, and $|\mathbf{m}| = 1$ a.e.
2. $\mathbf{s} \in L_\infty(\mathbb{R}^+; L_2(\Omega)) \cap L_2(\mathbb{R}^+; H^1(\Omega))$, $\frac{\partial \mathbf{s}}{\partial t} \in L_2(\mathbb{R}^+; H^{-1}(\Omega))$.
3. For all $\phi \in [H^1(Q_T)]^3$, $\psi \in [H^1(R_T)]^3$, we have

$$\begin{aligned} &\int_0^T \left\langle \frac{\partial \mathbf{s}}{\partial t}, \phi \right\rangle_{H^{-1}(\Omega)} \\ = &\int_{Q_T} \mathbf{J}_s \cdot \nabla\phi - \int_{Q_T} D_0(\mathbf{x})\mathbf{s}\cdot\phi - \int_{Q_T} D_0(\mathbf{x})\mathbf{s}\times\mathbf{m}\cdot\phi, \end{aligned} \tag{15}$$

$$\begin{aligned} &\int_{R_T} \frac{\partial \mathbf{m}}{\partial t}\cdot\psi - \alpha\int_{R_T} \mathbf{m}\times\frac{\partial \mathbf{m}}{\partial t}\cdot\psi \\ = &\int_{R_T} \mathbf{m}\times\nabla\mathbf{m}\cdot\nabla\psi - \int_{R_T} \mathbf{m}\times(-\nabla_{\mathbf{m}}\Phi + \mathbf{h}_d + \mathbf{s})\cdot\psi, \end{aligned} \tag{16}$$

   where

$$\mathbf{h}_d = -\nabla\left(\nabla N * \mathbf{m}\right). \tag{17}$$

4. $\mathbf{s}(0, \mathbf{x}) = \mathbf{s}_0(\mathbf{x})$, and $\mathbf{m}(0, \mathbf{x}) = \mathbf{m}_0(\mathbf{x})$ in the trace sense.

Remark 1. Note that the definition of weak solution implies that for all $T > 0$, $\mathbf{m} \in C^{1/2}([0, T]; L^2(\Omega))$ [27].

The remainder of the article is organized as follows: In section 2, an approximate solution to (11) is constructed based on a Galerkin approximation, and the necessary a priori estimates in order to guarantee convergence are obtained. Finally, the following theorem is proved in section 3:

Theorem 1 (Existence of weak solutions)**.** *Let $D_0 : \Omega \to \mathbb{R}^+$ be a measurable function, and assume that there exist positive constants such that*

$$0 < c \le D_0(\mathbf{x}) \le C, \quad a.e. \mathbf{x} \in \Omega. \tag{18}$$

Assume further that $\mathbf{J}_e \in (H^1(\mathbb{R}^+ \times \Sigma))^3$. Given $\mathbf{m}_0 \in (H^1(\Sigma))^3$, and $\mathbf{s}_0 \in (H^1(\Omega))^3$, there exists a weak solution $(\mathbf{s}, \mathbf{m})$ to problem (11).

Remark 2 (Nonuniqueness of weak solutions)**.** Given a weak solution to the Landau-Lifshitz-Gilbert equation $\mathbf{m}$, it follows that $(0, \mathbf{m})$ is a solution to the system (11) with initial condition $(0, \mathbf{m}_0)$, and $\mathbf{J}_e = 0$. Therefore the nonuniqueness of solutions to (11) follows directly from the nonuniqueness theorem 1.6 in [18].

2. **Galerkin Approximation: A Priori Estimates.** Let $\{\omega_n(\mathbf{x})\}\,(n = 1, 2, ...)$ be the normalized eigenfunctions of

$$\Delta u + \lambda u = 0; \qquad \frac{\partial u}{\partial \nu}|_{\partial\Omega} = 0, \tag{19}$$

and let $0 = \lambda_1 \leq \dots \lambda_n \leq \dots$ be the corresponding eigenvalues. Similarly, let $\{\theta_n(\mathbf{x})\}\,(n = 1, 2, ...)$ be the normalized eigenfunctions of

$$\Delta u + \mu u = 0; \qquad \frac{\partial u}{\partial \nu}|_{\partial\Sigma} = 0, \tag{20}$$

and $0 = \mu_1 \leq \cdots \leq \mu_n \leq \dots$ be the corresponding eigenvalues. Then $\{\omega_n\}$, $\{\theta_n\}$ are smooth up to the boundary, and form bases of $H^1(\Omega)$ and $H^1(\Sigma)$, respectively [27]. We define the orthogonal projection

$$\Pi_N : (H^1(R_T))^3 \to \mathcal{S}_N = \text{span}\,\{\theta_1, \dots, \theta_N\} \subset (H^1(R_T))^3, \tag{21}$$

as

$$\Pi_N(\mathbf{u}) = \sum_{n=0}^{N} (\mathbf{u}, \theta_n)_{L^2(\Sigma)}\theta_n, \;\; \forall \mathbf{u} \in (H^1(R_T))^3. \tag{22}$$

Consider the approximate solutions

$$\mathbf{s}_N(\mathbf{x}, t) = \sum_{n=0}^{N} \alpha_{\mathbf{n}}(t)\omega_n(\mathbf{x}), \qquad \mathbf{m}_N(\mathbf{x}, t) = \sum_{n=0}^{N} \beta_{\mathbf{n}}(t)\theta_n(\mathbf{x}), \tag{23}$$

where α_n, β_n are three dimensional vector valued functions and are chosen such that

$$\int_\Omega \frac{\partial \mathbf{s}_N}{\partial t} \cdot \omega_n = \int_\Omega \mathbf{J}_{sN} \cdot \nabla \omega_n - \int_\Omega D_0(\mathbf{x})\mathbf{s}_N \cdot \omega_n - \int_\Omega D_0(\mathbf{x})\mathbf{s}_N \times \mathbf{m}_N \cdot \omega_n, \tag{24}$$

and

$$\begin{aligned} &\int_\Sigma \frac{\partial \mathbf{m}_N}{\partial t} \cdot \theta_n - \alpha \int_\Sigma \mathbf{m}_N \times \frac{\partial \mathbf{m}_N}{\partial t} \cdot \theta_n \\ = &\int_\Sigma \mathbf{m}_N \times \nabla \mathbf{m}_N \cdot \nabla \theta_n - \int_\Sigma \mathbf{m}_N \times \Pi_N\left(\mathbf{g}_N + \mathbf{s}_N|_\Sigma\right) \cdot \theta_n, \end{aligned} \tag{25}$$

for $n = 1, 2, ..., N$ and with the initial conditions

$$\int_\Omega \mathbf{s}_N(\mathbf{x}, 0) \cdot \omega_n(\mathbf{x}) = \int_\Omega \mathbf{s}_0(\mathbf{x}) \cdot \omega_n(\mathbf{x}), \quad n = 1, 2, ..., N \tag{26}$$

and

$$\int_\Omega \mathbf{m}_N(\mathbf{x}, 0) \cdot \theta_n(\mathbf{x}) = \int_\Omega \mathbf{m}_0(\mathbf{x}) \cdot \theta_n(\mathbf{x}), \quad n = 1, 2, ..., N. \tag{27}$$

In (24), $\mathbf{J}_{sN}$ is defined as

$$\mathbf{J}_{sN} = \frac{\sqrt{2}}{\sqrt{1 + |\mathbf{m}_N|^2}}\mathbf{m}_N \otimes \mathbf{J}_e - D_0(\mathbf{x})\left[\nabla \mathbf{s}_N - \frac{(1+\delta)\beta}{\delta + |\mathbf{m}_N|^2}\mathbf{m}_N \otimes (\nabla \mathbf{s}_N \cdot \mathbf{m}_N)\right], \tag{28}$$

where $\delta > 0$ is to be determined. In (25),

$$\mathbf{g}_N = -\nabla \Phi(\mathbf{m}_N) + \mathbf{h}_{dN}, \tag{29}$$

and

$$\mathbf{h}_{dN} = -\nabla\left(\nabla N * \mathbf{m}_N\right). \tag{30}$$

The local (in time) existence of solutions to the Cauchy problem (24)-(27) follows from a similar argument to the one presented in [18]: Define

$$\mathbf{z} = (\alpha_1^1, \alpha_1^2, \alpha_1^3, \ldots, \alpha_N^1, \alpha_N^2, \alpha_N^3, \beta_1^1, \beta_1^2, \beta_1^3, \ldots, \beta_N^1, \beta_N^2, \beta_N^3).$$

Then, the previous Cauchy problem may be written in the form

$$\begin{pmatrix} I_{3N} & 0_{3N} \\ 0_{3N} & I_{3N} - A(\mathbf{z}) \end{pmatrix} \frac{d\mathbf{z}}{dt} = F(\mathbf{z}), \tag{31}$$

where $A(\mathbf{z}) \in \mathbb{R}^{(3N)\times(3N)}$ is an antisymmetric matrix. Therefore one can solve for $\dot{\mathbf{z}}$ in (31), and existence of solutions follows from Picard's theorem [28].

Note that since both $\mathbf{s}_N(t,\cdot) \in C^\infty(\bar{\Omega})$ and $\mathbf{m}_N(t,\cdot) \in C^\infty(\bar{\Sigma})$, equation (25) may be rewritten as

$$\frac{\partial \mathbf{m}_N}{\partial t} - \alpha \Pi_N \left(\mathbf{m}_N \times \frac{\partial \mathbf{m}_N}{\partial t}\right) = -\Pi_N\left(\mathbf{m}_N \times \mathbf{h}_N\right), \tag{32}$$

where

$$\mathbf{h}_N = \Delta \mathbf{m}_N + \Pi_N(\mathbf{g}_N + \mathbf{s}_N|_\Sigma) \in \mathcal{S}_N. \tag{33}$$

Multiplying (25) by β_n and adding up to N, we get

$$\frac{d}{dt}\int_\Sigma |\mathbf{m}_N|^2 = 0, \tag{34}$$

so

$$\|\mathbf{m}_N(t,\cdot)\|_{L^2(\Sigma)} = \|\Pi_N(\mathbf{m}_0)\|_{L^2(\Sigma)}. \tag{35}$$

In order to take the limit as $N \to \infty$, we need to make sure that all the functions $\mathbf{s}_N$ and $\mathbf{m}_N$ are defined at least in a common interval $[0,T]$. This is a consequence of (35), and the following lemma:

Lemma 1. *Let $\mathbf{s}_N(x,t)$ and $\mathbf{m}_N(x,t)$ be the solution to the Cauchy problem (24)-(27). Then the interval of definition of $\mathbf{s}_N$ and $\mathbf{m}_N$ can be extended to $[0,\infty)$,*

$$\mathbf{s}_N \in L_\infty(\mathbb{R}^+; L_2(\Omega)) \cap L_2(\mathbb{R}^+; H^1(\Omega)), \tag{36}$$

$$\frac{\partial \mathbf{s}_N}{\partial t} \in L_2(\mathbb{R}^+; H^{-1}(\Omega)), \tag{37}$$

$$\mathbf{m}_N \in L_\infty(\mathbb{R}^+; H^1(\Sigma)), \tag{38}$$

and

$$\frac{\partial \mathbf{m}_N}{\partial t} \in L_2(\mathbb{R}^+ \times \Sigma), \tag{39}$$

and the sequences are uniformly bounded in the corresponding spaces.

Proof. Multiplying (24) by α_n and summing up to N, we have

$$\begin{aligned} \int_\Omega \frac{d\mathbf{s}_N}{dt}\cdot \mathbf{s}_N &= \int_\Omega \mathbf{J}_{sN}\cdot\nabla\mathbf{s}_N - \int_\Omega D_0(\mathbf{x})\mathbf{s}_N\cdot\mathbf{s}_N - \int_\Omega D_0(\mathbf{x})\mathbf{s}_N\times\mathbf{m}_N\cdot\mathbf{s}_N \\ &= \int_\Omega \mathbf{J}_{sN}\cdot\nabla\mathbf{s}_N - \int_\Omega D_0(\mathbf{x})|\mathbf{s}_N|^2. \end{aligned}$$

Now,

$$\begin{aligned}
&\int_\Omega \mathbf{J}_{sN} \cdot \nabla \mathbf{s}_N \\
= \quad & \sqrt{2}\int_\Omega \frac{1}{\sqrt{1+|\mathbf{m}_N|^2}}(\mathbf{m}_N \otimes \mathbf{J}_e)\cdot \nabla \mathbf{s}_N - \int_\Omega D_0 |\nabla \mathbf{s}_N|^2 \\
& (\delta+1)\beta \int_\Omega D_0(\mathbf{x}) \frac{1}{\delta + |\mathbf{m}_N|^2}(\nabla \mathbf{s}_N \cdot \mathbf{m}_N)^2 \\
\leq \quad & \frac{1}{\epsilon}\int_\Omega \frac{|\mathbf{J}_e|^2}{D_0} + \frac{\epsilon}{2}\int_\Omega D_0|\nabla \mathbf{s}_N|^2 - \int_\Omega D_0|\nabla \mathbf{s}_N|^2 + (1+\delta)\beta\int_\Omega D_0|\nabla \mathbf{s}_N|^2 \\
\leq \quad & \frac{1}{\epsilon}\int_\Omega \frac{|\mathbf{J}_e|^2}{D_0} + \Big(\frac{\epsilon}{2} - 1 + (1+\delta)\beta\Big)\int_\Omega D_0|\nabla \mathbf{s}_N|^2 \qquad (40)
\end{aligned}$$

We choose ϵ and δ such that

$$\Big(\frac{\epsilon}{2} - 1 + (1+\delta)\beta\Big) < 0 \tag{41}$$

For example, if $\epsilon = \frac{1-\beta}{2}$, and $\delta = \frac{1-\beta}{4\beta}$,

$$\frac{\epsilon}{2} - 1 + (1+\delta)\beta = -\frac{1-\beta}{2} < 0. \tag{42}$$

Therefore

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |\mathbf{s}_N|^2 + \int_\Omega D_0|\mathbf{s}_N|^2 + \frac{(1-\beta)}{2}\int_\Omega D_0(\mathbf{x})|\nabla \mathbf{s}_N|^2 \leq \frac{2}{1-\beta}\int_\Omega \frac{|\mathbf{J}_e|^2}{D_0(\mathbf{x})}. \tag{43}$$

Since $\mathbf{J}_e \in L_2(\mathbb{R}^+, L^2(\Omega)) \cap L_\infty(\mathbb{R}^+, L^2(\Omega))$, the functions $\mathbf{s}_N$ and $\mathbf{m}_N$ can be extended to $[0,\infty)$ and we get

$$\mathbf{s}_N \in L_\infty(\mathbb{R}^+, L_2(\Omega)) \cap L^2(\mathbb{R}^+, H^1(\Omega)) \tag{44}$$

with uniform bounds independent of T or N.

Multiplying (32) by $\frac{\partial \mathbf{m}_N}{\partial t}$ and $\mathbf{h}_N$ and integrating on Σ, we get

$$\int_\Sigma \left|\frac{\partial \mathbf{m}_N}{\partial t}\right|^2 - \alpha\int_\Sigma \Pi_N\left(\mathbf{m}_N \times \frac{\partial \mathbf{m}_N}{\partial t}\right)\cdot \frac{\partial \mathbf{m}_N}{\partial t} = -\int_\Sigma \Pi_N\left(\mathbf{m}_N \times \mathbf{h}_N\right)\cdot \frac{\partial \mathbf{m}_N}{\partial t}, \tag{45}$$

and

$$\int_\Sigma \frac{\partial \mathbf{m}_N}{\partial t}\cdot \mathbf{h}_N - \alpha\int_\Sigma \Pi_N\left(\mathbf{m}_N \times \frac{\partial \mathbf{m}_N}{\partial t}\right)\cdot \mathbf{h}_N = -\int_\Sigma \Pi_N\left(\mathbf{m}_N \times \mathbf{h}_N\right)\cdot \mathbf{h}_N. \tag{46}$$

Noting that $\frac{\partial \mathbf{m}_N}{\partial t}, \mathbf{h}_N \in \mathcal{S}_N$, and using the fact that

$$\int_\Sigma \Pi_N(\mathbf{h})\cdot \mathbf{g} = \int_\Sigma \mathbf{h}\cdot\mathbf{g},\ \forall \mathbf{h} \in (H^1(\Sigma))^3,\ \forall \mathbf{g} \in \mathcal{S}_N, \tag{47}$$

we get

$$\begin{aligned}
\int_\Sigma \left|\frac{\partial \mathbf{m}_N}{\partial t}\right|^2 &= \int_\Sigma \left(\mathbf{m}_N \times \frac{\partial \mathbf{m}_N}{\partial t}\right)\cdot \mathbf{h}_N, \qquad (48)\\
\int_\Sigma \frac{\partial \mathbf{m}_N}{\partial t}\cdot \mathbf{h}_N &= \alpha\int_\Sigma \left(\mathbf{m}_N \times \frac{\partial \mathbf{m}_N}{\partial t}\right)\cdot \mathbf{h}_N, \qquad (49)
\end{aligned}$$

From lemma 2 below,

$$\int_\Sigma \frac{\partial \mathbf{m}_N}{\partial t}\cdot \mathbf{h}_N = -\frac{1}{2}\frac{d}{dt}\int_\Sigma |\nabla \mathbf{m}_N|^2 - \frac{d}{dt}\int_\Sigma \Phi(\mathbf{m}_N) -\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3}|\mathbf{h}_{dN}|^2 + \int_\Sigma \frac{\partial \mathbf{m}_N}{\partial t}\cdot \mathbf{s}_N. \tag{50}$$

Therefore

$$\alpha\int_\Sigma \left|\frac{\partial \mathbf{m}_N}{\partial t}\right|^2 + \frac{1}{2}\frac{d}{dt}\int_\Sigma |\nabla \mathbf{m}_N|^2 + \frac{d}{dt}\int_\Sigma \Phi(\mathbf{m}_N) + \frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3}|\mathbf{h}_{dN}|^2 = \int_\Sigma \frac{\partial \mathbf{m}_N}{\partial t}\cdot \mathbf{s}_N \le \frac{\alpha}{2}\int_\Sigma \left|\frac{\partial \mathbf{m}_N}{\partial t}\right|^2 + \frac{1}{2\alpha}\int_\Omega |\mathbf{s}_N|^2 \tag{51}$$

From (43), $\int_\Omega |\mathbf{s}_N|^2 \in L_2(\mathbb{R}^+)$, and therefore,

$$\mathbf{m}_N \in L_\infty(\mathbb{R}^+, H^1(\Sigma)), \tag{52}$$

$$\frac{\partial \mathbf{m}_N}{\partial t} \in L_2(\mathbb{R}^+\times\Sigma), \tag{53}$$

$$\Phi(\mathbf{m}_N) \in L_\infty(\mathbb{R}^+, L_1(\Sigma)), \tag{54}$$

$$\mathbf{h}_{dN} \in L_\infty(\mathbb{R}^+, L^2(\mathbb{R}^3)). \tag{55}$$

From (24), it follows that

$$\int_\Omega \frac{\partial \mathbf{s}_N}{\partial t}\cdot\omega_n = \int_\Omega \mathbf{J}_{sN}\cdot\nabla\omega_n - \int_\Omega D_0(\mathbf{x})\mathbf{s}_N\cdot\omega_n - \int_\Omega \mathbf{s}_N\times\mathbf{m}_N\cdot\omega_n \le C||\omega_n||_{H^1}. \tag{56}$$

Therefore,

$$\left\{\frac{\partial \mathbf{s}_N}{\partial t}\right\} \quad \text{is uniformly bounded in} \quad L_2(\mathbb{R}^+, H^{-1}(\Omega)). \tag{57}$$

This completes the proof of lemma 1. □

Equation (50) follows from the following lemma:

Lemma 2. *Consider $\mathbf{m}_N$, $\mathbf{h}_{dN}$, and $\mathbf{h}_N$ defined as before. Then:*

1. *For any $N > 0$,*

$$\int_\Sigma \frac{\partial \mathbf{m}_N}{\partial t}\cdot \mathbf{h}_N = -\frac{1}{2}\frac{d}{dt}\int_\Sigma |\nabla \mathbf{m}_N|^2 - \frac{d}{dt}\int_\Sigma \Phi(\mathbf{m}_N) -\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3}|\mathbf{h}_{dN}|^2 + \int_\Sigma \frac{\partial \mathbf{m}_N}{\partial t}\cdot \mathbf{s}_N. \tag{58}$$

2. *For any $N > 0$, $T \ge 0$,*

$$\int_{\mathbb{R}^3}|\mathbf{h}_{dN}|^2 \le \int_\Sigma |\mathbf{m}_N|^2. \tag{59}$$

3. *For any $N, M > 0$, and $T \ge 0$, we have*

$$\int_{\mathbb{R}^3}|\mathbf{h}_{dN}-\mathbf{h}_{dM}|^2 \le \int_\Sigma |\mathbf{m}_N-\mathbf{m}_M|^2. \tag{60}$$

Proof. The field $\mathbf{h}_N$ is defined as

$$\mathbf{h}_N = \Delta\mathbf{m}_N + \Pi_N(-\nabla_{\mathbf{m}}\Phi(\mathbf{m}_N) + \mathbf{h}_{dN} + \mathbf{s}_N|_\Sigma) \in \mathcal{S}_N. \tag{61}$$

Therefore,

$$\int_\Sigma \frac{\partial \mathbf{m}_N}{\partial t} \cdot \Delta\mathbf{m}_N = -\int_\Sigma \nabla\frac{\partial \mathbf{m}_N}{\partial t} \cdot \nabla\mathbf{m}_N = -\frac{1}{2}\frac{d}{dt}\int_\Sigma |\nabla\mathbf{m}_N|^2, \tag{62}$$

and

$$\int_\Sigma \frac{\partial \mathbf{m}_N}{\partial t} \cdot \Pi_N(-\nabla_{\mathbf{m}}\Phi(\mathbf{m}_N)) = -\int_\Sigma \frac{\partial \mathbf{m}_N}{\partial t} \cdot \nabla_{\mathbf{m}}\Phi(\mathbf{m}_N) = -\frac{d}{dt}\int_\Sigma \Phi(\mathbf{m}_N). \tag{63}$$

The field $\mathbf{h}_{dN}$ is defined as

$$\mathbf{h}_{dN} = -\nabla\left(\nabla N * \mathbf{m}_N\right) = -\nabla u_N, \tag{64}$$

where u_N is the solution to

$$\Delta u_N = \text{div }(\mathbf{m}_N\chi_\Sigma). \tag{65}$$

Multiplying (65) by $v \in H^1(\mathbb{R}^3)$ and integrating by parts, we get that

$$\int_{\mathbb{R}^3} \nabla u_N \cdot \nabla v = \int_\Sigma \mathbf{m}_N \cdot \nabla v. \tag{66}$$

Taking a derivative with respect to time in (65), we get

$$\Delta\frac{\partial u_N}{\partial t} = \text{div }\left(\frac{\partial \mathbf{m}_N}{\partial t}\chi_\Sigma\right), \tag{67}$$

and therefore for any $v \in H^1(\mathbb{R}^3)$,

$$\int_{\mathbb{R}^3} \nabla\frac{\partial u_N}{\partial t} \cdot \nabla v = \int_\Sigma \frac{\partial \mathbf{m}_N}{\partial t} \cdot \nabla v. \tag{68}$$

Then,

$$\frac{d}{dt}\int_{\mathbb{R}^3} |\nabla u_N|^2 = \frac{d}{dt}\int_\Sigma \nabla u_N \cdot \mathbf{m}_N = \int_\Sigma \nabla\frac{\partial u_N}{\partial t} \cdot \mathbf{m}_N + \int_\Sigma \nabla u_N \cdot \frac{\partial \mathbf{m}_N}{\partial t}. \tag{69}$$

Taking $v = \frac{\partial u_N}{\partial t}$ in (66),

$$\int_\Sigma \nabla\frac{\partial u_N}{\partial t} \cdot \mathbf{m}_N = \int_{\mathbb{R}^3} \nabla u_N \cdot \nabla\frac{\partial u_N}{\partial t}, \tag{70}$$

and taking $v = u_N$ in (68) it follows that

$$\int_\Sigma \nabla\frac{\partial u_N}{\partial t} \cdot \mathbf{m}_N = \int_\Sigma \frac{\partial \mathbf{m}_N}{\partial t} \cdot \nabla u_N. \tag{71}$$

Therefore,

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3} |\nabla u_N|^2 = \frac{1}{2}\frac{d}{dt}\int_\Sigma \nabla u_N \cdot \mathbf{m}_N = \int_\Sigma \nabla u_N \cdot \frac{\partial \mathbf{m}_N}{\partial t}, \tag{72}$$

from which (58) follows.

Taking $v = u_N$ in (66),

$$\int_{\mathbb{R}^3} |\nabla u_N|^2 = \int_\Sigma \nabla u_N \cdot \mathbf{m}_N \le \left(\int_{\mathbb{R}^3} |\nabla u_N|^2\right)^{\frac{1}{2}} \cdot \left(\int_\Sigma |\mathbf{m}_N|^2\right)^{\frac{1}{2}}, \tag{73}$$

from which (59) follows.

Finally, for any N, M, we have that

$$\int_{\mathbb{R}^3} \nabla(u_N - u_M) \cdot \nabla v = \int_\Sigma (\mathbf{m}_N - \mathbf{m}_M) \cdot \nabla v. \tag{74}$$

Therefore (60) follows similarly to (59). □

3. **Proof of Theorem 1.**

Proof. It follows from lemma 1 that there exist subsequences of $\{\mathbf{s}_N\}$, $\{\mathbf{m}_N\}$ (not relabeled) such that:

$$\mathbf{s}_N \rightharpoonup \mathbf{s} \quad \text{in } L^2(\mathbb{R}^+ \times \Omega), \tag{75}$$

$$\mathbf{s}_N \to \mathbf{s} \quad \text{in } L^2([0,T] \times \Omega), \tag{76}$$

$$\nabla \mathbf{s}_N \rightharpoonup \nabla \mathbf{s} \quad \text{in } L^2(\mathbb{R}^+ \times \Omega), \tag{77}$$

$$\frac{\partial \mathbf{s}_N}{\partial t} \rightharpoonup \frac{\partial \mathbf{s}}{\partial t} \quad \text{in } L_2(\mathbb{R}^+; H^{-1}(\Omega), \tag{78}$$

$$\mathbf{m}_N \rightharpoonup \mathbf{m} \quad \text{in } H^1(\mathbb{R}^+ \times \Sigma), \tag{79}$$

$$\mathbf{m}_N \to \mathbf{m} \quad \text{in } L^2([0,T] \times \Sigma), \tag{80}$$

and

$$\mathbf{m}_N \to \mathbf{m} \quad \text{point wise a.e. in } \mathbb{R}^+ \times \Sigma, \tag{81}$$

for some functions $\mathbf{s}$ and $\mathbf{m}$.

From lemma 2 it follows that

$$\mathbf{h}_{dN} \to \mathbf{h}_d \quad \text{in } L_\infty([0,T], L^2(\Sigma)). \tag{82}$$

To show that $\mathbf{s}, \mathbf{m}$ are weak solutions, consider $\phi \in [H^1(Q_T)]^3$, $\psi \in [H^1(R_T)]^3$. Define

$$\phi_N(t, \mathbf{x}) = \sum_{n=0}^{N} (\phi, \omega_n)_{L^2(\Omega)} \omega_n, \tag{83}$$

and

$$\psi_N(t, \mathbf{x}) = \sum_{n=0}^{N} (\psi, \theta_n)_{L^2(\Sigma)} \theta_n. \tag{84}$$

Then, ϕ_N and ψ_N converge uniformly in $[H^1(Q_T)]^3$ and $[H^1(R_T)]^3$ to ϕ and ψ, respectively, and

$$\int_{Q_T} \frac{\partial \mathbf{s}_N}{\partial t} \cdot \phi_N = \int_{Q_T} \mathbf{J}_{sN} \cdot \nabla \phi_N - \int_{Q_T} D_0(\mathbf{x}) \mathbf{s}_N \cdot \phi_N - \int_{Q_T} D_0(\mathbf{x}) \mathbf{s}_N \times \mathbf{m}_N \cdot \phi_N, \tag{85}$$

$$\begin{aligned} \int_{R_T} \frac{\partial \mathbf{m}_N}{\partial t} \cdot \psi_N \quad &= \quad \alpha \int_{R_T} \mathbf{m}_N \times \frac{\partial \mathbf{m}_N}{\partial t} \cdot \psi_N + \int_{R_T} \mathbf{m}_N \times \nabla \mathbf{m}_N \cdot \nabla \psi_N \\ &\quad - \int_{R_T} \mathbf{m}_N \times \Pi_N \left(\mathbf{g}_N + \mathbf{s}_N|_\Sigma \right) \cdot \psi_N. \end{aligned} \tag{86}$$

We need to show the above two equations converge to (15), (16) respectively.

Since $\{\psi_N\}$ converges strongly in $H^1(R_T)$, $\{\mathbf{m}_N\}$ converges strongly in $L^2(R_T)$ and weakly in $H^1(R_T)$, and $\{\mathbf{s}_N\}$ converges strongly in $L^2(Q_T)$:

$$\mathbf{m}_N \psi_N \to \mathbf{m} \psi \quad \text{in } L^2(R_T), \tag{87}$$

$$\mathbf{m}_N \times \nabla \mathbf{m}_N \rightharpoonup \mathbf{m} \times \nabla \mathbf{m} \quad \text{in } L^2(R_T), \tag{88}$$

$$\Pi_N \left(\mathbf{g}_N + \mathbf{s}_N|_\Sigma \right) \to -\nabla_{\mathbf{m}} \Phi(\mathbf{m}) + \mathbf{h}_d + \mathbf{s} \quad \text{in } L^2(R_T), \tag{89}$$

$$\int_{R_T} \frac{\partial \mathbf{m}_N}{\partial t} \cdot \psi_N \to \int_{R_T} \frac{\partial \mathbf{m}}{\partial t} \cdot \psi, \tag{90}$$

$$\begin{aligned}&\int_{R_T} \mathbf{m}_N \times \frac{\partial \mathbf{m}_N}{\partial t} \cdot \psi_N \\ =\ & \int_{R_T} (\psi_N \mathbf{m}_N) \times \frac{\partial \mathbf{m}_N}{\partial t} \to \int_{R_T} (\psi \mathbf{m}) \times \frac{\partial \mathbf{m}}{\partial t} = \int_{R_T} \mathbf{m} \times \frac{\partial \mathbf{m}}{\partial t} \cdot \psi, \end{aligned} \tag{91}$$

$$\int_{R_T} \mathbf{m}_N \times \nabla \mathbf{m}_N \cdot \nabla \psi_N \to \int_{R_T} \mathbf{m} \times \nabla \mathbf{m} \cdot \nabla \psi, \tag{92}$$

and

$$\begin{aligned}&\int_{R_T} \mathbf{m}_N \times \Pi_N\left(\mathbf{g}_N + \mathbf{s}_N|_\Sigma\right) \cdot \psi_N = \int_{R_T} (\psi_N \mathbf{m}_N) \times \Pi_N\left(\mathbf{g}_N + \mathbf{s}_N|_\Sigma\right) \\ &\to \int_{R_T} (\psi \mathbf{m}) \times (-\nabla_{\mathbf{m}} \Phi(\mathbf{m}) + \mathbf{h}_d + \mathbf{s}) = \int_{R_T} \mathbf{m} \times (-\nabla_{\mathbf{m}} \Phi(\mathbf{m}) + \mathbf{h}_d + \mathbf{s}) \cdot \psi. \end{aligned} \tag{93}$$

Therefore,

$$\begin{aligned}\int_{R_T} \frac{\partial \mathbf{m}}{\partial t} \cdot \psi \quad &= \quad \alpha \int_{R_T} \mathbf{m} \times \frac{\partial \mathbf{m}}{\partial t} \cdot \psi + \int_{R_T} \mathbf{m} \times \nabla \mathbf{m} \cdot \nabla \psi \\ &\quad - \int_{R_T} \mathbf{m} \times (-\nabla_{\mathbf{m}} \Phi(\mathbf{m}) + \mathbf{h}_d + \mathbf{s}) \cdot \psi. \end{aligned} \tag{94}$$

From this it follows, taking $\psi = \mathbf{m}\psi_0$, where $\psi_0 \in C_0^\infty(\Sigma)$, that

$$\frac{d}{dt} \int_{R_T} |\mathbf{m}|^2 \psi_0 = 0, \ \forall \psi_0 \in C_0^\infty(\Sigma), \tag{95}$$

and therefore

$$|\mathbf{m}(t, \mathbf{x})| = |\mathbf{m}_0(\mathbf{x})| = 1, \ a.e.\Sigma. \tag{96}$$

Consider now equation (85):

$$\int_{Q_T} \frac{\partial \mathbf{s}_N}{\partial t} \cdot \phi_N = \int_{Q_T} \frac{\partial \mathbf{s}_N}{\partial t} \cdot (\phi_N - \phi) + \int_{Q_T} \frac{\partial \mathbf{s}_N}{\partial t} \cdot \phi \tag{97}$$

From (56), we have

$$\left| \int_{Q_T} \frac{\partial \mathbf{s}_N}{\partial t} \cdot (\phi_N - \phi) \right| \leq C \int_0^T ||(\phi_N - \phi)||_{H^1(\Omega)} \to 0, \tag{98}$$

and from (78) it follows that

$$\int_{Q_T} \frac{\partial \mathbf{s}_N}{\partial t} \cdot \phi_N = \int_0^T \left\langle \frac{\partial \mathbf{s}_N}{\partial t}(t, \cdot), \phi \right\rangle_{H^{-1}(\Omega)} \to \int_0^T \left\langle \frac{\partial \mathbf{s}}{\partial t}(t, \cdot), \phi \right\rangle_{H^{-1}(\Omega)}. \tag{99}$$

For the terms on the right hand side of (85), we have

$$\begin{aligned}\int_{Q_T} \mathbf{J}_{sN} \cdot \nabla \phi_N \quad &= \quad \int_{Q_T} \frac{\sqrt{2}}{\sqrt{1 + |\mathbf{m}_N|^2}} (\mathbf{m}_N \otimes \mathbf{J}_e) \cdot \nabla \phi_N - \int_{Q_T} D_0(\mathbf{x}) \nabla \mathbf{s}_N \cdot \nabla \phi_N \\ &\quad + \beta(1 + \delta) \int_{Q_T} \frac{1}{\delta + |\mathbf{m}_N|^2} D_0(\mathbf{x}) \mathbf{m}_N \otimes (\nabla \mathbf{s}_N \cdot \mathbf{m}_N) \cdot \nabla \phi_N \\ &= \quad I - II + \beta(1 + \delta)\, III \end{aligned}$$

Note that

$$\frac{\sqrt{2}\mathbf{m}_N}{\sqrt{1 + |\mathbf{m}_N|^2}} \to \mathbf{m} \ \text{ in } L^2(Q_T), \tag{100}$$

and point wise a.e. Similarly,

$$\frac{\mathbf{m}_N}{\sqrt{\delta+|\mathbf{m}_N|^2}} \to \frac{1}{\sqrt{1+\delta}}\mathbf{m} \ \text{ in } L^2(Q_T), \tag{101}$$

and pointwise a.e.

We have

$$I = \int_{Q_T} \frac{\sqrt{2}}{\sqrt{1+|\mathbf{m}_N|^2}}(\mathbf{m}_N \otimes \mathbf{J}_e)\cdot\nabla(\phi_N - \phi) + \int_{Q_T} \frac{\sqrt{2}}{\sqrt{1+|\mathbf{m}_N|^2}}(\mathbf{m}_N \otimes \mathbf{J}_e)\cdot\nabla\phi. \tag{102}$$

Now,

$$\int_{Q_T} \frac{\sqrt{2}}{\sqrt{1+|\mathbf{m}_N|^2}}(\mathbf{m}_N \otimes \mathbf{J}_e)\cdot\nabla(\phi_N - \phi) \le \sqrt{2}\|\mathbf{J}_e\|_{L^2(Q_T)}\|\phi_N - \phi\|_{H^1(Q_T)} \to 0. \tag{103}$$

The second term satisfies

$$\begin{aligned}\int_{Q_T} \frac{\sqrt{2}}{\sqrt{1+|\mathbf{m}_N|^2}}(\mathbf{m}_N \otimes \mathbf{J}_e)\cdot\nabla\phi &= \int_{Q_T} \frac{\sqrt{2}\mathbf{m}_N}{\sqrt{1+|\mathbf{m}_N|^2}}(\mathbf{J}_e\cdot\nabla\phi) \to \int_{Q_T} \mathbf{m}\,(\mathbf{J}_e\cdot\nabla\phi) \\ &= \int_{Q_T} (\mathbf{m}\otimes\mathbf{J}_e)\cdot\nabla\phi, \end{aligned} \tag{104}$$

as a consequence of (81) and the Dominated Convergence Theorem.

Similarly, we have

$$II = \int_{Q_T} D_0(\mathbf{x})\nabla\mathbf{s}_N\cdot\nabla\phi_N \to \int_{Q_T} D_0(\mathbf{x})\nabla\mathbf{s}\cdot\nabla\phi, \tag{105}$$

because of the weak convergence of $\{\mathbf{s}_N\}$ and the strong convergence of $\{\phi_N\}$. Finally,

$$\begin{aligned} III &= \int_{Q_T} \frac{D_0}{\delta+|\mathbf{m}_N|^2}\mathbf{m}_N\otimes(\nabla\mathbf{s}_N\cdot\mathbf{m}_N)\cdot\nabla\phi_N \\ &= \int_{Q_T} \frac{D_0}{\delta+|\mathbf{m}_N|^2}\mathbf{m}_N\otimes(\nabla\mathbf{s}_N\cdot\mathbf{m}_N)\cdot\nabla(\phi_N-\phi) \\ &\quad + \int_{Q_T} \frac{D_0}{\delta+|\mathbf{m}_N|^2}(\mathbf{m}_N-\mathbf{m})\otimes(\nabla\mathbf{s}_N\cdot\mathbf{m}_N)\cdot\nabla\phi \\ &\quad + \int_{Q_T} \frac{D_0}{\delta+|\mathbf{m}_N|^2}\mathbf{m}\otimes(\nabla(\mathbf{s}_N-\mathbf{s})\cdot\mathbf{m}_N)\cdot\nabla\phi \\ &\quad + \int_{Q_T} \frac{D_0}{\delta+|\mathbf{m}_N|^2}\mathbf{m}\otimes(\nabla\mathbf{s}\cdot\mathbf{m}_N)\cdot\nabla\phi \end{aligned}$$

The first three terms tend to zero as $N \to \infty$ because of (36), (75)-(81), and (100)-(101). The last term converges to

$$\frac{1}{1+\delta}\int_{Q_T} D_0(\mathbf{x})\mathbf{m}\otimes(\nabla\mathbf{s}\cdot\mathbf{m})\cdot\nabla\phi \tag{106}$$

Therefore, taking the limit $N \to \infty$ in (85) we get:

$$\int_{Q_T} \frac{\partial\mathbf{s}}{\partial t}\cdot\phi = \int_{Q_T} \mathbf{J}_s\cdot\nabla\phi - \int_{Q_T} D_0(\mathbf{x})\mathbf{s}\cdot\phi - \int_{Q_T} D_0(\mathbf{x})\mathbf{s}\times\mathbf{m}\cdot\phi, \tag{107}$$

and $(\mathbf{s},\mathbf{m})$ is a global weak solution to (11). □

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