Choosing a proper risk measure is an important regulatory issue, as exemplified in governmental regulations such as Basel II accord [1] and its recent revision [2], which use Value-at-Risk (VaR) with scenario analysis as the risk measure for setting capital requirement for market risk. The main motivation of this article is to investigate whether VaR with scenario analysis are good risk measures for external regulation. By using the notion of comonotonic random variables studied in decision theory literatures such as Refs 3–8, we shall propose a new class of risk measures satisfying a new set of axioms. The new class of risk measures includes VaR with scenario analysis, in particular the current and recently revised Basel II risk measures [1,2] as special cases, and therefore provides a theoretical basis for using VaR with scenario analysis as robust risk measures for the purpose of external, regulatory risk measurement.

Broadly speaking, a risk measure attempts to assign a single numerical value to a random financial loss. Obviously, it can be problematic to use one number to summarize the whole statistical distribution of the financial loss. Therefore, one shall avoid doing this if it is at all possible. However, in many cases there is no other choice.

Examples of such cases include regulatory capital requirements, insurance risk premiums, and margin requirements in financial trading. Consequently, how to choose a good risk measure becomes a problem of great practical importance.

In this article, we only study static risk measures, that is, one period risk measures. Mathematically, let $\Omega$ be the set of all the possible states of nature at the end of an observation period, $\lambda$ be the set of financial losses under consideration, in which each financial loss is a random variable defined on $\Omega$. Then a risk measure $\rho$ is a mapping from $\lambda$ to the real line $\mathbb{R}$. The multiperiod or dynamic risk measures are related to dynamic consistency or time consistency for preferences [9–13].

An important property of risk measures is robustness. The concept of robustness has been well developed in economics and statistics. In the theory of decision or policy making in economics, the robustness of preferences or policy rules refers to the property that they can accommodate model misspecification error and would perform well across a set of alternative models. This notion of robustness has a close connection with ambiguity aversion and model uncertainty. A widely used class of robust preferences that model ambiguity and explain the famous Ellsberg Paradox [14] are the multiple priors preferences, also known as maxmin expected utility preferences, proposed by Gilboa and Schmeidler [15]. An agent with this kind of preference ranks a pay off profile $Y$ by $\min_{P \in \mathcal{P}} \{E_P[u(Y)]\}$, where $\mathcal{P}$ is the set of subjective probabilities, or a set of priors, held by the agent, and $E_P[u(Y)]$ is the expected utility of $Y$ under the subjective probability $P$. The ambiguity is reflected by the set of priors $\mathcal{P}$. Multiple priors preferences have been used to incorporate ambiguity in asset pricing [16,17]. An alternative class of robust preferences are the multiplier preferences [18–20], which also postulate that agents have multiple prior probability measures. In statistics, robustness has been studied comprehensively in the robust statistics

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literature, which is mainly concerned with the distributional robustness, that is, a statistic is robust if it is insensitive to small deviation of the true underlying distribution from the assumed distribution [21,22]. Loosely speaking, a robust statistic is insensitive to small changes in the data, which are either small changes in all the observations or large changes in a few observations (outliers).

A risk measure is said to be robust if (i) it can accommodate model misspecification (possibly by incorporating multiple priors); and (ii) it is insensitive to small changes in the data (by using robust statistics).

Another important issue related to risk measures, which has not been well addressed in the existing literature, is the objective of a risk measure. In terms of objectives, risk measures can be classified into two categories: internal risk measures used for internal risk management at individual institutions, and external risk measures used for external regulation and imposed for all the relevant institutions. Internal risk measures are proposed for the interests of institutions’ shareholders or managers, while external risk measures are used by regulatory agencies to maintain safety and soundness of the financial system. One risk measure may be suitable for internal management, but not for external regulation, and vice versa. There is no reason to believe that one unique risk measure can fit all needs. For example, variance is an internal risk measure that is commonly used for hedging and asset allocation in institutions. However, variance is not a good risk measure for calculating capital requirements in external regulation, because it does not help to achieve the regulators’ objective of ensuring that very high confidence level, for example, 99.9% institutions have enough capital to absorb their losses. In contrast, VaR, which summarizes the worst loss of a portfolio at a given confidence level, better suits the needs of regulators and, therefore, is more suitable for external regulation than variance.

Another reason for the existence of internal and external risk measures lies in the fact that the information available for institutions and regulators are different. Institutions have more information and hence could tailor their risk measures to incorporate all of it. External regulators have access to a subset of information only. Therefore, their way of measuring risk is different. No amount of disclosure requirements can eliminate private information that institutions have.

In this article, we shall focus on external risk measures from the viewpoint of regulatory agencies. We will show in the sections titled “Legal Motivation” and “The Main Reason to Relax Subadditivity: Robustness” that an external risk measure should be robust but coherent risk measures are generally not robust. We will then propose a new class of risk measures that includes a subclass of robust ones in the section titled “Main Results: Natural Risk Statistics.”

**VaR WITH SCENARIO ANALYSIS AND RISK STATISTICS**

**Value-at-Risk**

One of the most widely used risk measures in financial regulation and risk management is VaR, which is a quantile at some predefined probability level. More precisely, let $F(\cdot)$ be the distribution function of $X$, then for given $\alpha \in (0, 1)$, VaR of the loss variable $X$ at level $\alpha$ is defined as the $\alpha$-quantile of $X$, that is,

$$ \text{VaR}_\alpha(X) := \inf\{x \mid F(x) \geq \alpha\} = F^{-1}(\alpha). \quad (1) $$

In practice, VaR is usually estimated from samples of the random loss $X$, that is, a data set $\hat{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$. Let $(x_{(1)}, \ldots, x_{(n)})$ be the sample order statistics and $F_n(x) := \frac{1}{n} \sum_{i=1}^{n} I_{[x_i \leq x]}$ be the empirical distribution function. The most commonly used estimator for $\text{VaR}_\alpha(X)$ is the sample quantile $x_{[\lceil an \rceil]} = F_n^{-1}(\alpha)$, where $\lceil an \rceil$ is the smallest integer that is no less than $an$. The sample quantile is strongly consistent and asymptotically normal [23]. See Refs 24 and 25 for more comprehensive discussion of VaR.

**VaR with Scenario Analysis and Basel II Risk Measures**

VaR with scenario analysis is the class of risk measures that involve the calculation and
ROBUST EXTERNAL RISK MEASURES

The Basel II Accord [1] uses VaR with scenario analysis to calculate capital requirement for market risk. More precisely, the Basel II Accord specifies that the market risk capital requirement at any particular day \( t \) for banks using the internal models approach is calculated as

\[
c_t = \max \left\{ \text{VaR}_{t-1}, k \cdot \frac{1}{60} \sum_{i=1}^{60} \text{VaR}_{t-i} \right\},
\]

where \( \text{VaR}_{t-1} \) is the value of VaR calculated under scenario \( i \), \( i = 1, 2 \).

Basel II Accord [1] uses VaR with scenario analysis to calculate capital requirement for market risk. More precisely, Basel II Accord specifies that the market risk capital requirement at any particular day \( t \) for banks using the internal models approach is calculated as

\[
c_t = \max \left\{ \text{VaR}_{t-1}, k \cdot \frac{1}{60} \sum_{i=1}^{60} \text{VaR}_{t-i} \right\},
\]

where \( k \) is a constant that is no less than 3; \( \text{VaR}_{t-i} \) is the 10-day VaR at 99% confidence level calculated on day \( t - i \), \( i = 1, \ldots, 60 \). \( \text{VaR}_{t-i} \) is usually estimated from a data set \( \tilde{x} = (x_1', x_2', \ldots, x_n') \in \mathbb{R}^n \), which are based on the most recent one year observation of market variables up to day \( t - i \) and can be generated by historical simulation or Monte Carlo simulation [24,25]. Hence, \( \text{VaR}_{t-i} \) corresponds to the scenario based on the information available on the day \( t - i \). Therefore, the risk measure defined in Equation (3) is a VaR with scenario analysis that involves 60 scenarios.

During the financial crisis that started in late 2007, it was found that most banks’ actual losses were significantly higher than the capital requirement calculated by Equation (3) [2]. In addition, the risk measure (Eq. 3) has been criticized for its procyclicality [26]. Therefore, the Basel committee recently revised the Basel II market risk framework [2]. The revised market risk capital requirement is defined as

\[
\begin{align*}
c_t &= \max \left\{ \text{VaR}_{t-1}, k \cdot \frac{1}{60} \sum_{i=1}^{60} \text{VaR}_{t-i} \right\} \\
&\quad + \max \left\{ s\text{VaR}_{t-1}, \ell \cdot \frac{1}{60} \sum_{i=1}^{60} s\text{VaR}_{t-i} \right\},
\end{align*}
\]

where \( \text{VaR}_{t-i} \) is defined the same as in Equation (3); \( k \) and \( \ell \) are constants no less than 3; \( s\text{VaR}_{t-i} \) is called the stressed VaR on day \( t - i \), which is calculated under the scenario that the financial market is under significant stress such as the one that happened during the period from 2007 to 2008. The additional capital requirement based on stressed VaR helps to reduce the procyclicality of the original risk measure (Eq. 3).

Interestingly, the risk measures defined in Equations (2), (3), and (4) do not belong to any existing theoretical framework of risk measures in the literature. In this article, we are going to define a new class of risk measures that include these risk measures as special cases and thus provide theoretical basis for the use of VaR with scenario analysis in external regulation.

Risk Statistics: Data-based Risk Measures

In external regulation, the random loss \( X \) under consideration corresponds to the loss of the whole trading book of a financial institution, which contains all kinds of assets across many sectors. Specifying accurate models for \( X \) (under different scenarios) involves modeling the joint distribution of a large number of correlated risk factors and is hence usually very difficult. Therefore, the behavior of \( X \) under different scenarios is usually represented by different sets of data observed or generated under different scenarios, and the risk measurement of the loss is directly estimated from the data.

More precisely, suppose the behavior of the random loss \( X \) is represented by a collection of data \( \tilde{x} = (x_1', x_2', \ldots, x_m') \in \mathbb{R}^m \), where \( \tilde{x} = (x_1', x_2', \ldots, x_n') \in \mathbb{R}^n \) is the data subset that corresponds to the \( i \)-th scenario; \( n_i \) is the sample size of \( \tilde{x} \); \( n_1 + n_2 + \cdots + n_m = n \); for each \( i = 1, \ldots, m \), \( \tilde{x}^i \) can be a data set based on historical observations, or a data set simulated according to a well-defined procedure or model, or a mixture of historical and simulated data. The random loss \( X \) can be discrete or continuous. For example, the data used in the calculation of the revised Basel II risk measure defined in Equation (4) comprise 120 data subsets that correspond to 120 different scenarios.
A risk statistic $\hat{\rho}$ is simply a mapping from $\mathbb{R}^n$ to $\mathbb{R}$. It is a data-based risk measure that maps $\hat{x}$, the data representation of the random loss $X$, to $\hat{\rho}(\hat{x})$, the risk measurement of $X$.

In this article, we will define a new set of axioms for risk statistics instead of risk measures, because risk statistics directly measure risk from data without specifying subjective models, which greatly reduces model misspecification error.

### REVIEW OF EXISTING RISK MEASURES

There are mainly two families of static risk measures suggested in the literature, coherent and convex risk measures, and insurance risk measures. Both are based on axiomatic approaches, by which some axioms are postulated first, and then all the risk measures satisfying the axioms are identified.

#### Coherent and Convex Risk Measures

- **Axioms.** Coherent risk measures [27] satisfy the following axioms:

  1. **Axiom A1.** Translation invariance: $\rho(X + b) = \rho(X) + b$, $\forall b \in \mathbb{R}, \forall X \in \mathcal{X}$.

  2. **Axiom A2.** Positive homogeneity: $\rho(aX) = a\rho(X)$, $\forall a \geq 0, \forall X \in \mathcal{X}$.

  3. **Axiom A3.** Monotonicity: $\rho(X) \leq \rho(Y)$, if $X \leq Y$.

  4. **Axiom A4.** Subadditivity: $\rho(X + Y) \leq \rho(X) + \rho(Y)$, $\forall X, Y \in \mathcal{X}$.

  What is questionable lies in the subadditivity requirement in Axiom A4, which basically means that “a merger does not create extra risk” [27, p. 209]. We will discuss the controversies related to this axiom in the section titled “Other Reasons to Relax Subadditivity.”

  It is shown in Refs 22, 27, and 28 that a risk measure $\rho$ is coherent if and only if there exists a family $\mathcal{Q}$ of probability measures, such that $\rho(X) = \sup_{Q \in \mathcal{Q}} \mathbb{E}^Q[X]$, $\forall X \in \mathcal{X}$, where $\mathbb{E}^Q[X]$ is the expectation of $X$ under the probability measure $Q$. Therefore, measuring risk by a coherent risk measure amounts to computing maximal expectation under a set of prior probability measures.

- **Axioms.** Coherent risk measures [27] satisfy the following axioms:

  5. **Axiom A5.** Coherent risk measures satisfy the following axioms:

    - **Axiom A1.** Translation invariance: $\rho(X + b) = \rho(X) + b$, $\forall b \in \mathbb{R}, \forall X \in \mathcal{X}$.
    - **Axiom A2.** Positive homogeneity: $\rho(aX) = a\rho(X)$, $\forall a \geq 0, \forall X \in \mathcal{X}$.
    - **Axiom A3.** Monotonicity: $\rho(X) \leq \rho(Y)$, if $X \leq Y$.
    - **Axiom A4.** Subadditivity: $\rho(X + Y) \leq \rho(X) + \rho(Y)$, $\forall X, Y \in \mathcal{X}$.
    - **Axiom A5.** Coherent risk measures satisfy the following axioms:

      - **Axiom A5.** Law invariance: $\rho(X) = \rho(Y)$, if $X$ and $Y$ have the same distribution.


      Convex risk measures proposed in Refs 34 and 35 are risk measures that satisfy Axiom A1, A3, and the following convexity axiom A6, which relaxes the positive homogeneity and subadditivity axioms in coherent risk measures.

      - **Axiom A6.** Convexity: $\rho(\lambda X + (1 - \lambda)Y) \leq \lambda \rho(X) + (1 - \lambda)\rho(Y)$, $\forall X, Y \in \mathcal{X}, \forall \lambda \in [0, 1]$.

      **Main Drawbacks of Coherent and Convex Risk Measures.** A main drawback of coherent risk measures is that in general, they are not robust (see the section titled “Comparison with Coherent Risk Measures”). For example, TCE is too sensitive to the modeling assumption for the tails of loss distributions, and it is sensitive to outliers in the data (see the section titled “Tail Conditional Median: A Robust Risk Measure”). However, robustness is an indispensable requirement for external risk measures, as we will point out in the section titled “The Main Reason to Relax Subadditivity: Robustness.” Hence, coherent
A main drawback of insurance risk measures is that neither can they incorporate multiple priors of the probability distributions on the set of scenarios, nor can they incorporate multiple approaches to measure risk under each scenario (see the section titled “Comparison with Insurance Risk Measures” for more detailed discussion).

The main reason that insurance risk measures cannot incorporate multiple priors is that they require comonotonic additivity. To incorporate multiple priors, we shall relax the comonotonic additivity to comonotonic subadditivity in the section titled “Main Results: Natural Risk Statistics.”

The mathematical concept of comonotonic subadditivity is also studied independently in Refs 38 and 39, which give representations of the functionals satisfying comonotonic subadditivity or comonotonic convexity from a mathematical perspective. There are several differences between Refs 38 and 39 and this article. See Ref. 40 for the details.

LEGAL MOTIVATION

In this section, we shall discuss two basic concepts of law, that is, legal realism and legal positivism, which motivate us to argue that: (i) External risk measures should be robust, because robustness is essential for law enforcement; (ii) External risk measures should have consistency with people’s behavior, because law should reflect society norms.

Legal Realism and Robustness of Law

Legal realism is the viewpoint that the legal decision of a court regarding a case is determined by the actual practices of judges, rather than the law set forth in statutes and precedents. All the legal rules contained in statutes and precedents have uncertainty due to the uncertainty in human language and that human beings are unable to anticipate all the possible future circumstances [41, p. 128]. Hence, a law is only a guideline that human beings are unable to anticipate all the possible future circumstances.

A main drawback of insurance risk measures is that they rule out VaR, which does not satisfy subadditivity or convexity universally [27]. This posts a serious inconsistency between the academic theory and governmental practice. The inconsistency is due to the subadditivity Axiom A4. By relaxing this axiom and requiring subadditivity only for comonotonic random variables, we are able to define a new class of risk measures that include VaR and more generally, VaR with scenario analysis, thus eliminating the inconsistency.

Insurance Risk Measures

Insurance risk measures [8] satisfy the following axioms:

**Axiom B1.** Law invariance: the same as Axiom A5.

**Axiom B2.** Monotonicity: $\rho(X) \leq \rho(Y)$, if $X \leq Y$.

**Axiom B3.** Comonotonic additivity: $\rho(X + Y) = \rho(X) + \rho(Y)$, if $X$ and $Y$ are comonotonic. ($X$ and $Y$ are *comonotonic* if $(X(\omega_1) - X(\omega_2))Y(\omega_1) - Y(\omega_2)) \geq 0$ holds almost surely for $\omega_1$ and $\omega_2$ in $\Omega$.)

**Axiom B4.** Continuity: $\lim_{d \to 0} \rho(X - d) = \rho(X)$, $\lim_{d \to -\infty} \rho(\min(X, d)) = \rho(X)$, $\forall X$. ($x^+ := \max(x, 0), \forall x \in \mathbb{R}$).

**Axiom B5.** Scale normalization: $\rho(1) = 1$.

The notion of comonotonic random variables is studied in Refs 3, 5, and 36. The psychological motivation of comonotonic random variables comes from alternative models to expected utility theory including the prospect theory (see the section titled “Theory of Choice under Uncertainty and Risk”). Dhaene et al. [37] give a recent review of risk measures and comonotonicity.

VaR with scenario analysis, such as the Basel II risk measures (Eqs 3 and 4), are not insurance risk measures, although VaR itself is an insurance risk measure (see Ref. 36 for the proof that VaR satisfies Axiom B3). An insurance risk measure does not necessarily satisfy subadditivity.
such a way that different judges will reach similar conclusions when they implement the law.

In particular, external risk measures imposed in banking regulation should be robust with respect to underlying models and data. However, coherent risk measures generally lack robustness, as manifested in Theorem 3.

Legal Positivism and Social Norm

Legal positivism is the thesis that the existence and content of law depend on social norms and not on their merits. It is based on the argument that if a system of rules are to be imposed by force in the form of law, there must be a sufficient number of people who accept it voluntarily; without their voluntary cooperation, the coercive power of law and government cannot be established [41, pp. 201–204].

Therefore, external risk measures imposed in banking regulations should also reflect most people's behavior. However, decision theory suggests that most people's decision can violate the subadditivity Axiom A4 (see the section titled "Theory of Choice under Uncertainty and Risk" for details).

THE MAIN REASON TO RELAX SUBADDITIVITY: ROBUSTNESS

Robustness is Indispensable for External Risk Measures

In determining capital requirement, regulators impose a risk measure and allow institutions to use their own internal risk models and private data in the calculation. For example, Basel II's internal models approach allows institutions to use their own internal models to calculate the capital requirement for market risk.

There are two issues rising from the use of internal models and private data in external regulation: (i) the data can be noisy, flawed, or unreliable; (ii) there can be several statistically indistinguishable models for the same asset or portfolio due to limited amount of available data.

For example, the heaviness of tail distributions cannot be identified in many cases. Although it is accepted that stock returns have tails heavier than those of normal distributions, one school of thought believes that the tails are exponential-type, while another believes that the tails are power-type. Heyde and Kou [42] show that it is very difficult to distinguish between exponential-type and power-type tails with 5,000 observations (about 20 years of daily observations). This is mainly because the quantiles of exponential-type distributions and power-type distributions may overlap. Hence, the tail behavior may be a subjective issue depending on people's modeling preferences.

To address the two issues above, external risk measures should demonstrate robustness with respect to model misspecification and small changes in the data. From a regulator's viewpoint, an external risk measure must be unambiguous, stable, and be implemented consistently across all the relevant institutions, no matter what internal beliefs or internal models each individual institution may have. In situations when the correct model cannot be identified, two institutions that have exactly the same portfolio can use different internal models, both of which can obtain the approval of the regulator. From the regulator's viewpoint, the same portfolio should incur the same or almost the same amount of regulatory capital. Therefore, the external risk measure should be robust, otherwise different institutions can be required to hold very different regulatory capital for the same risk exposure, which makes the risk measure unacceptable to most institutions.

In addition, if the external risk measure is not robust, institutions can take regulatory arbitrage by choosing a model that significantly reduces the capital requirement or by manipulating the input data.

Coherent Risk Measures are not Robust

The robustness of coherent risk measures is questionable in two aspects.

First, the theory of coherent risk measures suggests to use TCE to measure risk. However, as we will show in the section titled "Tail Conditional Median: A Robust Risk Measure," TCE is sensitive to modeling
assumptions of heaviness of tail distribution and to outliers in the data.

Second, in general, coherent risk measures are not robust. We will prove in Theorem 3 that every empirically law-invariant coherent risk measure puts larger weights on larger observations, and hence is obviously sensitive to small changes in the data.

Tail Conditional Median: A Robust Risk Measure

We propose a more robust risk measure, tail conditional median (TCM), which is a special case of the new class of risk measures to be defined in the section titled “Main Results: Natural Risk Statistics.” TCM of loss \( X \) at level \( \alpha \) is defined as

\[
\text{TCM}_\alpha(X) = \text{VaR}_{1-\alpha}(X) + \frac{\alpha}{2} \mu(X).
\]

It is called conditional median, because for \( X \) with a continuous distribution,

\[
\text{TCM}_\alpha(X) = \text{VaR}_{1-\alpha}(X) + \frac{\alpha}{2} \mu(X) = \text{median}[X \mid X \geq \text{VaR}_\alpha(X)],
\]

that is, the conditional median of \( X \) given that \( X \geq \text{VaR}_\alpha(X) \). For \( X \) with a general distribution involving discontinuity, \( \text{VaR}_{1-\alpha}(X) \) can be viewed as a regularized version of \( \text{median}[X \mid X \geq \text{VaR}_\alpha(X)] \).

The equality (Eq. 6) shows that if one wants to measure the size of loss beyond the confidence level \( \alpha \), one can use \( \text{VaR} \) at a higher level \( \frac{1-\alpha}{2} \), which gives the conditional median of the size of loss beyond level \( \alpha \). This contradicts the criticism on \( \text{VaR} \) that \( \text{VaR} \) cannot provide information about the size of loss beyond certain confidence level.

There are many examples in the existing literature showing that \( \text{VaR} \) violates subadditivity, for example, the examples on p. 216 and p. 217 of Ref. 27. However, it is straightforward to verify that in those examples subadditivity will not be violated if one replaces \( \text{VaR}_\alpha \) by \( \text{TCM}_\alpha \).

TCM can be shown to be more robust than TCE by at least four standard tools in robust statistics [21,22]: (i) influence functions, (ii) asymptotic breakdown points, (iii) continuity of statistical functional, and (iv) finite sample breakdown points. For example, TCE has an unbounded influence function but TCM has a bounded influence function, which implies that TCM is more robust than TCE with respect to outliers in the data. See Ref. 40 for more detailed discussion.

TCE is highly model-dependent, in particular, sensitive to modeling assumption of extreme tails of loss distribution, as illustrated in Fig. 1. The left panel of the figure shows the value of TCE with respect to \( \log(1-\alpha) \) for Laplace and T distributions with degree of freedom 3, 5, and 12, which are normalized to have mean 0 and variance 1, where \( \alpha \) is in the range [95%, 99.9%]. The right panel of the figure shows the value of TCM for those distributions. Apparently, the range of variation of TCM with respect to change of modeling assumption is much smaller than that of TCE. For example, with \( \alpha = 99.6\% \), the variation of TCE is 1.44, whereas the variation of TCM is only 0.75. In the context of capital requirement, suppose a bank’s trading portfolio loss is \( \sigma X \), where \( \sigma = \$1 \) billion and \( X \) has mean 0 and variance 1, which can be modeled to have one of the four distributions. At confidence level \( \alpha = 99\% \), if TCE is used to calculate capital requirement, the capital requirement for banks using one of the four models will be in the range of [2.95, 4.05] billion; in contrast, if TCM is used, the range will be [2.79, 3.38] billion. Hence, TCM results in much more comparable capital requirements across banks when they may use different models.

It is worth noticing that TCE is not necessarily larger than TCM. Eling and Tibiletti [43] compare TCE and TCM for a set of capital market data and find that although TCE is on average about 10% higher than TCM at standard confidence levels, TCM is higher than TCE in about 10% of the studied cases.

Robust Risk Measures Versus Conservative Risk Measures

A risk measure is said to be more conservative than another, if it generates higher risk measurement than the other for the
Figure 1. The left panel shows TCE for Laplacian distribution and T distributions with degree of freedom 3, 5, 12, which are normalized to have mean 0 and variance 1. The right panel shows TCM for the same set of distributions. The x-axis is log(1 − α) where α ∈ [0.95, 0.999]. TCM is less sensitive to modeling assumption of the underlying distribution than TCE, as the change of its value with respect to change of underlying distributions has a narrower range.

The same underlying risk exposure. The use of more conservative risk measures in external regulation is desirable from a regulator’s viewpoint, since it generally increases the safety of the financial system (Of course, risk measures should not be too conservative to retard the economic growth).

There is no contradiction between the robustness and the conservativeness of external risk measures. Robustness addresses the issue of whether a risk measure can be implemented consistently, so it is a requisite property of an external risk measure. Conservativeness addresses the issue of how stringently an external risk measure should be implemented, given that it can be implemented consistently. In other words, being more conservative is a further desirable property of an external risk measure. An external risk measure should be robust in the first place before one can consider the issue of how to implement it in a conservative way.

By the following approaches, one can construct an external risk measure that is both robust and conservative:

1. More scenarios can be included to incorporate the possible effects of extreme events that lie outside normal market conditions. For example, the revised Basel II risk measure Equation (4) takes into account 60 more stressed scenarios compared to the Basel II risk measure Equation (3).

2. Regulators can simply multiply the original risk measure by a larger constant to obtain a more conservative risk measure. For example, one can use larger \( k \) in Equation (3). This is equivalent to choosing a larger \( s \) in Axiom C1 for the new risk measures (see the
section titled “Main Results: Natural Risk Statistics”.

3. One can use more weights or more prior probability measures to define the external risk measure. See the section titled “Main Results: Natural Risk Statistics” for more detailed discussion.

OTHER REASONS TO RELAX SUBADDITIVITY

Diversification and Tail Subadditivity of VaR

The subadditivity axiom for coherent risk measures and the convexity axiom for convex risk measures conform to the idea that diversification does not increase risk. There are two main motivations for diversification. One is based on the simple observation that $\text{SD}(X + Y) \leq \text{SD}(X) + \text{SD}(Y)$, for any two random variables $X$ and $Y$ with finite second moments, where $\text{SD}(\cdot)$ denotes standard deviation. The other is based on expected utility theory. Samuelson [44] shows that any investor with a strictly concave utility function will uniformly diversify among independently and identically distributed risks with finite second moments (see Refs 45–48 for the discussion on whether diversification is beneficial when the asset returns are dependent). Both of the two motivations require finiteness of second moments of the risks.

Is diversification still preferable for risks with infinite second moments? The answer can be no. Ibragimov and Walden [49] show that diversification is not preferable for unbounded extremely heavy-tailed distributions, in the sense that the expected utility of the diversified portfolio is smaller than that of the undiversified portfolio. They also show that, investors with certain S-shaped utility functions would prefer nondiversification, even for bounded risks. An S-shaped utility function is convex in the domain of losses, which is supported by experimental results and prospect theory [6,50]. See the section titled “Theory of Choice under Uncertainty and Risk” for more discussion on prospect theory.

VaR has been criticized because it does not satisfy subadditivity universally, but this criticism is unreasonable because even diversification is not universally preferable. In fact, Danielsson et al. [51] show that VaR is subadditive in the tail region, provided that the tail of the joint distribution is not extremely fat (with tail index less than one). The simulations that they carry out also show that $\text{VaR}_\alpha$ is indeed subadditive when $\alpha \in [95\%, 99\%]$ for most practical applications.

To summarize, there is no conflict between the use of VaR and diversification. When the risks do not have extremely heavy tails, diversification is preferred and VaR satisfies subadditivity in the tail region; when the risks have extremely heavy tails, diversification may not be preferable and VaR may fail to satisfy subadditivity.

Asset returns with tail index less than one have very fat tails. They are hard to find and easy to identify. Danielsson et al. [51] argue that they can be treated as special cases in financial modeling. Even if one encounters an extremely fat tail and insists on tail subadditivity, Garcia et al. [52] show that when tail thickness causes violation of subadditivity, a decentralized risk management team may restore the subadditivity for VaR by using proper conditional information.

Does A Merger Always Reduce Risk

Subadditivity basically means that “a merger does not create extra risk” [27, p. 209]. However, Dhaene et al. [53] point out that a merger may increase risk, particularly due to bankruptcy protection for institutions. For example, it is better to split a risky trading business into a separate subsidiary institution. This way, even if the loss from the subsidiary institution is enormous, the parent institution can simply let the subsidiary institution go bankrupt, thus confining the loss to that one subsidiary institution. Therefore, creating subsidiary institutions may incur less risk and a merger may increase risk.

For example, the collapse of Britain’s Barings Bank in February 1995, due to the failure of a single trader (Nick Leeson) in Singapore clearly indicates that a merger may increase risk. Had Barings Bank setup a separate institution for its Singapore unit, the bankruptcy in that unit would not have sunk the entire bank.
Theory of Choice under Uncertainty and Risk

Risk measures have a close connection with the psychological and economic theory of people’s choices under uncertainty and risk. Tversky and Kahneman [6,50] point out that people’s choices under risky prospects are inconsistent with the basic tenets of expected utility theory and propose an alternative model, called prospect theory, which can explain a variety of preference anomalies. The axiomatic analysis of prospect theory is presented in Refs 6 and 7. Many other people have studied alternative models to expected utility theory, such as Refs 3–5, 54.

Risk measures have a close connection with Theory of Choice under Uncertainty and Risk.

**Main Results: Natural Risk Statistics**

In this section, we will define and fully characterize the new data-based risk measures, which we call the natural risk statistics.

As discussed in the section titled “Risk Statistics: Data-Based Risk Measures,” we are going to measure the risk from a collection of data \( \mathbf{x} = (x^1, x^2, \ldots, x^m) \in \mathbb{R}^n \) that represents the behavior of the random loss \( X \), where \( x^i = (x_{1i}, x_{2i}, \ldots, x_{ni}) \) is the data subset that corresponds to scenario \( i \). The risk measurement will be given by \( \hat{\rho}(\mathbf{x}) \), where \( \hat{\rho} : \mathbb{R}^n \to \mathbb{R} \) is a risk statistic.

**Axioms and Representations for Natural Risk Statistics**

At first, we define the concept of scenario-wise comonotonicity for two data sets.

**Definition 1.** Two data sets \( \tilde{x} = (\tilde{x}^1, \tilde{x}^2, \ldots, \tilde{x}^m) \in \mathbb{R}^n \) and \( \tilde{y} = (\tilde{y}^1, \tilde{y}^2, \ldots, \tilde{y}^m) \in \mathbb{R}^n \) are scenario-wise comonotonic, if for \( \forall i, \forall 1 \leq j, k \leq n_i \),

\[
(\tilde{x}_j^i - \tilde{x}_k^i)(\tilde{y}_j^i - \tilde{y}_k^i) \geq 0.
\]

Then we postulate the following axioms for a risk statistic \( \hat{\rho} \).

**Axiom C1.** Positive homogeneity and translation scaling: \( \hat{\rho}(ax + b) = a\hat{\rho}(\tilde{x}) + s \cdot b, \forall \tilde{x} \in \mathbb{R}^n, \forall a \geq 0, \forall b \in \mathbb{R} \), where \( s > 0 \) is a constant, \( \mathbf{1} = (1, 1, \ldots, 1) \in \mathbb{R}^n \).

**Axiom C2.** Monotonicity: \( \hat{\rho}(\tilde{x}) \leq \hat{\rho}(\tilde{y}) \) for any \( \tilde{x} \leq \tilde{y} \), where \( \tilde{x} \leq \tilde{y} \) means \( x^i_j \leq y^i_j, \forall j, i \).

The two axioms above (with \( s = 1 \)) have been proposed for coherent risk measures.

**Axiom C3.** Scenario-wise comonotonic subadditivity: \( \hat{\rho}(\tilde{x} + \tilde{y}) \leq \hat{\rho}(\tilde{x}) + \hat{\rho}(\tilde{y}) \) for any \( \tilde{x} \) and \( \tilde{y} \) that are scenario-wise comonotonic.

In Axiom C3, we relax the subadditivity requirement in coherent risk measures so that the subadditivity is only required for comonotonic random variables. This also relaxes the comonotonic additivity requirement in insurance risk measures.

**Axiom C4.** Scenario-wise permutation invariance:

\[
\hat{\rho}\left((x^1_1, x^2_1, \ldots, x^m_1), x^{1}_{p_1, n_1}, x^{2}_{p_2, n_2}, \ldots, x^{m}_{p_m, n_m}\right) = \hat{\rho}\left((x^1_{p_1, 1}, \ldots, x^{1}_{p_1, n_1}, x^{2}_{p_2, 1}, \ldots, x^{2}_{p_2, n_2}, \ldots, x^{m}_{p_m, 1}, \ldots, x^{m}_{p_m, n_m})\right),
\]
for any permutation \((p_{i1}, \ldots, p_{in})\) of \((1, 2, \ldots, n)\), \(i = 1, \ldots, m\).

This axiom can be considered as the counterpart of the law invariance Axiom A5 in the empirical setting. It means that if two data set \(\tilde{x}\) and \(\tilde{y}\) have the same empirical distribution under each scenario, then \(\tilde{x}\) and \(\tilde{y}\) should give the same measurement of risk.

**Definition 2.** A risk statistic \(\hat{\rho} : \mathbb{R}^n \to \mathbb{R}\) is called a natural risk statistic if it satisfies Axiom C1–C4.

The following representation theorem fully characterizes the natural risk statistics.

**Theorem 1.** (i) For a given constant \(s > 0\) and an arbitrarily given set of weights \(W = \{\tilde{w}\} \subset \mathbb{R}^n\) with each \(\tilde{w} = (w_{1}^1, \ldots, w_{n}^1, \ldots, w_{n}^m)\) \(\in \mathbb{W}\) satisfying the following conditions:

\[
\sum_{j=1}^{n_1} w_j^1 + \sum_{j=1}^{n_2} w_j^2 + \cdots + \sum_{j=1}^{n_m} w_j^m = 1, \quad (7)
\]

\[
w_j^i \geq 0, \quad j = 1, \ldots, n_i; \quad i = 1, \ldots, m, \quad (8)
\]

define a risk statistic \(\hat{\rho} : \mathbb{R}^n \to \mathbb{R}\) as follows:

\[
\hat{\rho}(\tilde{x}) := s \cdot \sup_{\tilde{w} \in \mathbb{W}} \left\{ \sum_{j=1}^{n_1} w_j^1 x_{(j)}^1 + \sum_{j=1}^{n_2} w_j^2 x_{(j)}^2 + \cdots + \sum_{j=1}^{n_m} w_j^m x_{(j)}^m \right\},
\]

\[
\forall \tilde{x} = (\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_m) \in \mathbb{R}^n, \quad (9)
\]

where \(x_{(1)}, \ldots, x_{(n)}\) is the order statistics of \(\bar{x} = (x_1, \ldots, x_n)\) with \(x_{(n)}\) being the largest, \(i = 1, \ldots, m\). Then the risk statistic defined in Equation (9) is a natural risk statistic.

(ii) If \(\hat{\rho}\) is a natural risk statistic, then there exists a set of weights \(W = \{\tilde{w} = (\tilde{w}^1, \tilde{w}^2, \ldots, \tilde{w}^m)\} \subset \mathbb{R}^n\) with each \(\tilde{w} \in \mathbb{W}\) satisfying condition (7) and (8), such that

\[
\rho(\tilde{x}) = s \cdot \sup_{\tilde{w} \in \mathbb{W}} \left\{ \sum_{j=1}^{n_1} w_j^1 x_{(j)}^1 + \sum_{j=1}^{n_2} w_j^2 x_{(j)}^2 + \cdots + \sum_{j=1}^{n_m} w_j^m x_{(j)}^m \right\},
\]

\[
\forall \tilde{x} = (\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_m) \in \mathbb{R}^n. \quad (10)
\]

**Proof.** See Ref. 40 (for the case of \(m = 1\), see also Ref. 55). Q.E.D.

The natural risk statistics can also be characterized via acceptance sets as in the case of coherent risk measures. See Ref. 40 for the details.

The Basel II risk measures, and other VaR with scenario analysis, are special cases of natural risk statistics. More precisely, we have the following theorem:

**Theorem 2.** The Basel II risk measure defined in Equation (3) and the revised Basel II risk measure defined in Equation (4) are both natural risk statistics.

**Proof.** See Ref. 40. Q.E.D.

**Robust Natural Risk Statistics**

Natural risk statistics include a subclass of risk statistics that are robust in two aspects: (i) they can accommodate model misspecification by incorporating multiple prior probability measures on the set of scenarios; (ii) they are insensitive to small changes in the data by using robust statistics for each scenario.

Let \(\hat{\rho}\) be a natural risk statistic defined in Equation (9) that corresponds to the set of weights \(W\). Define a map \(\phi : \mathbb{W} \to \mathbb{R}^m \times \mathbb{R}^n\) such that for each \(\tilde{w} \in \mathbb{W}\), \(\phi(\tilde{w}) := (\hat{\rho}, \hat{q})\), where \(\hat{\rho} := (\rho^1, \ldots, \rho^m), \rho^i := \sum_{j=1}^{n_i} w_j^i, \)
and \( q_i := (q_{i1}, \ldots, q_{m_i}), \) \( q_j := (q_{j1}, \ldots, q_{m_j}) \), \( q_j^1 := 1_{(p_j > 0)}w_j/p_j \). Then \( \hat{\rho} \) can be written as

\[
\hat{\rho}(\tilde{x}) = s \cdot \sup_{(\tilde{\rho}, \tilde{\hat{\rho}}) \in \phi(W)} \left\{ \sum_{j=1}^{m} \rho_j^1(\tilde{x}_j^i) \right\}, \quad \text{where}
\]

\[
\hat{\rho}^i(\tilde{x}^i) := \sum_{j=1}^{n_i} q_j^i x_{ji}^i. \quad (11)
\]

Thus, each weight \( \tilde{w} \in W \) specifies: (i) the prior probability measure \( \tilde{\rho} \) on the set of scenarios; and (ii) the subsidiary risk statistic \( \hat{\rho}^i \) for each scenario \( i \), which measures risk from data subset \( \tilde{x}^i \).

Hence, \( \hat{\rho} \) can be robust to model mis-specification by incorporating multiple prior probability measures on the set of scenarios. And \( \hat{\rho} \) can be robust to small changes in the data if each subsidiary risk statistic \( \hat{\rho}^i(\tilde{x}^i) \) is a robust statistic, for example, VaR, or other robust L-statistics [22, Chapter 3].

The Basel II risk measures defined in Equations (3) and (4) incorporate multiple priors of probability distributions on the set of scenarios and use VaR as subsidiary risk statistics. Hence, they are robust natural risk statistics.

**Comparison with Coherent and Insurance Risk Measures**

**Comparison with Coherent Risk Measures**

To compare with coherent risk measures, we first define the law-invariant coherent risk statistics, the data-based versions of law-invariant coherent risk measures.

**Definition 3.** A risk statistic \( \hat{\rho} : \mathbb{R}^n \rightarrow \mathbb{R} \) is called a law-invariant coherent risk statistic, if it satisfies Axiom C1, C2, C4, and the following axiom E3.

**Axiom E3.** Subadditivity: \( \hat{\rho}(\tilde{x} + \tilde{y}) \leq \hat{\rho}(\tilde{x}) + \hat{\rho}(\tilde{y}), \forall \tilde{x}, \tilde{y} \in \mathbb{R}^n. \)

We have the following representation theorem for law-invariant coherent risk statistics.

**Theorem 3.** (i) For a given constant \( s > 0 \) and an arbitrarily given set of weights \( W = \{\tilde{w}\} \subset \mathbb{R}^n \) with each \( \tilde{w} = (w_{1}, \ldots, w_{n}), \ldots, w_{n_m} \} \in W \) satisfying the following conditions:

\[
\sum_{j=1}^{n_1} w_{j1} + \sum_{j=1}^{n_2} w_{j2} + \cdots + \sum_{j=1}^{n_m} w_{jm} = 1, \quad (12)
\]

\[
w_j \geq 0, j = 1, \ldots, n_i; i = 1, \ldots, m, \quad (13)
\]

\[
w_1 \leq w_2 \leq \cdots \leq w_m, \quad i = 1, \ldots, m, \quad (14)
\]

define a risk statistic

\[
\hat{\rho}(\tilde{x}) := s \cdot \sup_{\tilde{w} \in W} \left\{ \sum_{j=1}^{n_1} w_{j1} x_{j1}^1 + \sum_{j=1}^{n_2} w_{j2} x_{j2}^2 + \cdots + \sum_{j=1}^{n_m} w_{jm} x_{jm}^m \right\},
\]

\[
\forall \tilde{x} = (x_{11}, \ldots, x_{nm}) \in \mathbb{R}^n, \quad (15)
\]

where \( (x_{1i}, \ldots, x_{ni}) \) is the order statistics of \( \tilde{x} = (x_{1i}, \ldots, x_{ni}) \) with \( x_{nj} \) being the largest, \( i = 1, \ldots, m \). Then the risk statistic defined in Equation (15) is a law-invariant coherent risk statistic.

(ii) If \( \hat{\rho} \) is a law-invariant coherent risk statistic, then there exists a set of weights \( W = \{\tilde{w}\} \subset \mathbb{R}^n \) with each \( \tilde{w} \in W \) satisfying Equations (12), (13), and (14), such that

\[
\hat{\rho}(\tilde{x}) = s \cdot \sup_{\tilde{w} \in W} \left\{ \sum_{j=1}^{n_1} w_{j1} x_{j1}^1 + \sum_{j=1}^{n_2} w_{j2} x_{j2}^2 + \cdots + \sum_{j=1}^{n_m} w_{jm} x_{jm}^m \right\},
\]

\[
\forall \tilde{x} = (\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_m) \in \mathbb{R}^n. \quad (16)
\]

**Proof.** See Ref. 40 (for the case of \( m = 1 \), see also Ref. 55). Q.E.D.

By Theorem 1 and 3, we see the main differences between natural risk statistics and coherent risk measures:

1. Coherent risk measures are in general not robust, because every law-invariant
Theorem 4. \( \tilde{\rho} \) is an insurance risk statistic if and only if there exists a single weight \( \tilde{w} = (w_1, \ldots, w_1, \ldots, w_n, \ldots, w_n) \in \mathbb{R}^n \) with \( w_j \geq 0 \) for all \( i \) and \( \sum_{j=1}^{m} w_j = 1 \), such that

\[
\tilde{\rho}(\tilde{x}) = \sum_{j=1}^{n_1} w_1^j x_{(1)}^j + \sum_{j=1}^{n_2} w_2^j x_{(2)}^j + \cdots + \sum_{j=1}^{n_m} w_m^j x_{(m)}^j,
\]

where \( (x_{(1)}^1, \ldots, x_{(m)}^m) \) is the order statistics of \( \tilde{x} = (x_1^1, \ldots, x_n^m), i = 1, \ldots, m \).

Proof. See Ref. 40. Q.E.D.

By Theorem 1 and 4, we see the main differences between natural risk statistics and insurance risk statistics:

1. Insurance risk statistics are less robust to model misspecification than natural risk statistics, because they do not incorporate multiple priors of probability distributions on the set of scenarios, and they do not incorporate multiple subsidiary risk statistics for each scenario.
2. Insurance risk statistics do not include VaR with scenario analysis such as the ones defined in Equations (2), (3), and (4), although they include VaR as a special case. However, VaR with scenario analysis are special cases of natural risk statistics.
3. Insurance risk statistics, which incorporate only one prior, are a subclass of natural risk statistics.

CONCLUSION

We propose new data-based risk measures called natural risk statistics that are characterized by a new set of axioms. The new axioms only require subadditivity for comonotonic random variables, which is consistent with prospect theory in economics. Natural risk statistics include (i) TCM, which is more robust than TCE suggested by coherent risk measures; (ii) VaR with scenario analysis, in particular, the Basel II risk measures, as special cases. Therefore, natural risk statistics provide a theoretical basis for using VaR with scenario analysis in external regulation.

We emphasize that an external risk measure should be robust with respect to model misspecification and small changes in the data in order for the consistent implementation of the risk measure across all the
relevant institutions. In contrast to coherent risk measures which are generally not robust, natural risk statistics include a subclass of robust risk statistics that are suitable for external regulation.

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