CreditRisk+

Download document from CSFB web site: http://www.csfb.com/creditrisk/
Features of CreditRisk+

- Applies an actuarial science framework to the derivation of the loss distribution of a bond/loan portfolio.

- Only default risk is modelled, not downgrade risk.

- Default risk is not related to the capital structure of the firm. No assumption is made about the causes of default. An obligor A is either in default with probability $P_A$ and that not in default with probability $1 - P_A$. 
Assumptions on probability of default

The probability distribution for the number of defaults during a given period, is approximated by *Poisson distribution*

\[ P(n \text{ defaults}) = \frac{\mu^n e^{-\mu}}{n!} \quad \text{for} \quad n = 0,1,2,\ldots \]

where

\[ \mu = \text{average number of defaults over the period} = \sum_A P_A. \]

- Stationary assumption on the probability of default.
- Number of defaults that occur in any given period is independent of the number of defaults that occur in any other period.
CreditRisk+ risk measurement framework

Input

Stage One

- default rates
- default rates/volatilities

What is the frequency of defaults?

What is the severity of the losses?

Stage Two

- exposure
- recovery rates

Distribution of default losses
## Frequency of default events

<table>
<thead>
<tr>
<th>Credit Rating</th>
<th>One year default rate</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Average (%)</td>
<td>Standard deviation (%)</td>
</tr>
<tr>
<td>Aaa</td>
<td>0.00</td>
<td>0.0</td>
</tr>
<tr>
<td>Aa</td>
<td>0.03</td>
<td>0.1</td>
</tr>
<tr>
<td>A</td>
<td>0.01</td>
<td>0.0</td>
</tr>
<tr>
<td>Baa</td>
<td>0.13</td>
<td>0.3</td>
</tr>
<tr>
<td>Ba</td>
<td>1.42</td>
<td>1.3</td>
</tr>
<tr>
<td>B</td>
<td>7.62</td>
<td>5.1</td>
</tr>
</tbody>
</table>

Source: Carty and Lieberman (1996)

Rating B: standard deviation of default rate $\sqrt{7.62} = 2.76$ versus 5.1

CreditRisk\(^+\) assumes that the mean default rate is Gamma distributed.
Distribution of default events

Source: CreditRisk+
# Historical default rates of corporate bond issuers

1920-1996

<table>
<thead>
<tr>
<th>Seniority and security</th>
<th>Average</th>
<th>Standard Derivation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Senior secured bank loans</td>
<td>71.18</td>
<td>21.09</td>
</tr>
<tr>
<td>Senior secured public debt</td>
<td>63.45</td>
<td>26.21</td>
</tr>
<tr>
<td>Senior unsecured public debt</td>
<td>47.54</td>
<td>26.29</td>
</tr>
<tr>
<td>Senior subordinated public debt</td>
<td>38.28</td>
<td>24.74</td>
</tr>
<tr>
<td>Subordinated public debt</td>
<td>28.29</td>
<td>20.09</td>
</tr>
<tr>
<td>Junior subordinated public debt</td>
<td>14.66</td>
<td>8.67</td>
</tr>
</tbody>
</table>
Severity of the losses

In CreditRisk+, the exposure for each obligor is adjusted by the anticipated recovery rate in order to produce a loss given default (exogenous to the model).

Example of data set

<table>
<thead>
<tr>
<th>Obligor</th>
<th>Exposure</th>
<th>Credit rating</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>358,475</td>
<td>H</td>
</tr>
<tr>
<td>2</td>
<td>1,082,473</td>
<td>A</td>
</tr>
<tr>
<td>3</td>
<td>2,124,342</td>
<td>B</td>
</tr>
<tr>
<td>4</td>
<td>135,423</td>
<td>G</td>
</tr>
<tr>
<td>5</td>
<td>278,477</td>
<td>D</td>
</tr>
<tr>
<td>Credit rating</td>
<td>Mean default rate</td>
<td>Standard deviation</td>
</tr>
<tr>
<td>---------------</td>
<td>-------------------</td>
<td>--------------------</td>
</tr>
<tr>
<td>A</td>
<td>1.50%</td>
<td>0.75%</td>
</tr>
<tr>
<td>B</td>
<td>1.65%</td>
<td>0.80%</td>
</tr>
<tr>
<td>C</td>
<td>3.00%</td>
<td>1.50%</td>
</tr>
<tr>
<td>D</td>
<td>5.00%</td>
<td>2.50%</td>
</tr>
<tr>
<td>E</td>
<td>7.50%</td>
<td>3.75%</td>
</tr>
<tr>
<td>F</td>
<td>10.00%</td>
<td>5.00%</td>
</tr>
<tr>
<td>G</td>
<td>15.00%</td>
<td>7.50%</td>
</tr>
</tbody>
</table>
Division into exposure bands

Losses (exposures, net of recovery) are divided into bands, with the level of exposure in each band being approximated by a single number.

**Notation**

**Obligor**

| Obligor | A |

**Exposure (net of recovery)**

| Exposure | $L_A$ |

**Probability of default**

| Probability | $P_A$ |

**Expected loss**

<p>| Expected loss | $\lambda_A = L_A \times P_A$ |</p>
<table>
<thead>
<tr>
<th>Obligor A</th>
<th>Exposure ($) (loss given default)</th>
<th>Exposure $V_j$ (in $100,000)</th>
<th>Round-off exposure $\alpha_j$ (in $100,000$)</th>
<th>Band $j$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>150,000</td>
<td>1.5</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>460,000</td>
<td>4.6</td>
<td>5</td>
<td>5</td>
</tr>
<tr>
<td>3</td>
<td>435,000</td>
<td>4.35</td>
<td>5</td>
<td>5</td>
</tr>
<tr>
<td>4</td>
<td>370,000</td>
<td>3.7</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>5</td>
<td>190,000</td>
<td>1.9</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>6</td>
<td>480,000</td>
<td>4.8</td>
<td>5</td>
<td>5</td>
</tr>
</tbody>
</table>

* $L = $100,000


<table>
<thead>
<tr>
<th>$\nu_j$</th>
<th>number of obligors</th>
<th>$\varepsilon_j$</th>
<th>$\mu_j$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>30</td>
<td>1.5</td>
<td>1.5</td>
</tr>
<tr>
<td>2</td>
<td>40</td>
<td>8</td>
<td>4</td>
</tr>
<tr>
<td>3</td>
<td>50</td>
<td>6</td>
<td>2</td>
</tr>
<tr>
<td>4</td>
<td>70</td>
<td>25.2</td>
<td>6.3</td>
</tr>
<tr>
<td>5</td>
<td>100</td>
<td>35</td>
<td>7</td>
</tr>
<tr>
<td>6</td>
<td>60</td>
<td>14.4</td>
<td>2.4</td>
</tr>
</tbody>
</table>

$\nu_j = \text{common exposure in band } j \text{ in units of } L$

$\varepsilon_j = \text{expected loss in band } j \text{ in units of } L$

$\mu_j = \text{expected number of defaults in band } j$
• Expected loss for obligor $A$ in units of $L = e_A = \frac{\lambda_A}{L}$.

• The expected loss over one-year in band $j$
  
  $$\varepsilon_j = \sum_{A: v_A = v_j} e_A.$$ 

• The expected number of default in band $j$
  
  $$\mu_j = \frac{\varepsilon_j}{v_j}.$$ 

• The portfolio is divided into $m$ exposure bands to simplify calculations.

• The expected number of defaults in the portfolio
  
  $$\mu = \sum_{j=1}^{m} \mu_j.$$
Probability generating functions

The probability generating function of a discrete random variable $K$ is a function of the auxiliary variable $z$ such that the probability that $K = n$ is given by the coefficient of $z^n$ in the polynomial expansion of the probability generating function.

- The pgf of the sum $K_1 + K_2$ is simply the product of the two pgf’s.

**Example**

For a single obligor, $F_A(z) = (1 - P_A) + P_Az$.

For the whole portfolio, $F(z) = \sum_{n=0}^{\infty} p(n \text{ defaults})z^n$. 
Probability generating function of the distribution of loss amounts for band $j$

$$G_j(z) = \sum_{n=0}^{\infty} P(\text{loss} = nL) z^n = \sum_{n=0}^{\infty} P(n \text{ defaults}) z^{nv_j}$$

$$= \sum_{n=0}^{\infty} e^{-\mu_j} \frac{\mu_j^n}{n!} z^{nv_j} = \exp(-\mu_j + \mu_j z^{v_j})$$

Probability generating function for the entire portfolio

$$G(z) = \prod_{j=1}^{m} \exp(-\mu_j + \mu_j z^{v_j})$$

We then have

$$P(\text{loss of } nL) = \left. \frac{1}{n!} \frac{d^n G(z)}{dz^n} \right|_{z=0}, \quad n = 1, 2, \ldots$$
From $G(z) = \exp \left( -\sum_{j=1}^{m} \mu_j + \sum_{j=1}^{m} \mu_j z^{v_j} \right)$,

$$\sum_{j=1}^{m} \epsilon_j z^{v_j}$$

we write $p(z) = \frac{\sum_{j=1}^{m} \epsilon_j z^{v_j}}{\sum_{j=1}^{m} v_j}$,

then $G(z) = e^{\mu[p(z)-1]}$.

Here, $\mu$ gives the Poisson randomness of the incidence of default events and $p(z)$ gives the variability of exposure amounts within the portfolio.
Correlation in defaults

- Observed default probabilities are volatile over time, even for obligors having comparable credit quality. The variability of default probabilities can be related to underlying variability in a number of background factors, like the state of the economy.

- Two obligors are sensitive to the same set of background factors (with differing weights), their default probabilities will move together. These co-movements in probabilities give rise to correlations in defaults.

- CreditRisk+ does not attempt to model correlations explicitly but captures the same concentration effects through the use of default rate volatilities and sector analysis.
Sector Analysis

- Write $S_k, k = 1, \ldots, n$ for the sectors, each of which should be thought of as a subset of the set of obligors.

- Each sector is driven by a single underlying factor, which explains the variability over time in the average total default rate measured for that sector.

- The underlying factor influences the sector through the total average rate of defaults in that sector, which is modeled as a random variable $x_k$ with mean $\mu_k$ and standard deviation $\sigma_k$. 
Let $x_k$ be Gamma distributed with mean $\mu_k$ and standard deviation $\sigma_k$:

$$p(x < x_k \leq x + dx) = f(x)dx = \frac{1}{\beta^\alpha \Gamma(\alpha)} e^{-\frac{x}{\beta}} x^{\alpha - 1} dx.$$ 

Now,

$$F_k(z) = \sum_{n=0}^{\infty} p(n \text{ defaults}) z^n$$

$$= \sum_{n=0}^{\infty} \int_{x=0}^{\infty} p(n \text{ defaults}|x) f(x)dx$$

$$= \int_{0}^{\infty} e^{x(z-1)} \frac{e^{-\frac{x}{\beta}} x^{\alpha - 1}}{\beta^\alpha \Gamma(\alpha)} dx = \left( \frac{1 - p_k}{1 - p_k z} \right)^{\alpha_k}$$

where $\alpha_k = \frac{\mu_k^2}{\sigma_k^2}$, $\beta_k = \frac{\sigma_k^2}{\mu_k^2}$ and $p_k = \frac{\beta_k}{1 + \beta_k}$. 

The pgf for default events from the whole portfolio

$$F(z) = \prod_{k=1}^{n} F_k(z) = \prod_{k=1}^{n} \left( \frac{1 - p_k}{1 - p_k z} \right)^{\alpha_k}.$$
General sector analysis

The default rate of an individual obligor depends on more than one factors.

**Sector decomposition**
Assignment of $\theta_{Ak}$ represents the judgement of the extent to which the state of sector $k$ influence the fortunes of obligor $A$.

$$\mu_k = \sum_A \theta_{Ak} \mu_A$$

$$\sigma_k = \sum_A \theta_{Ak} \sigma_A.$$

**Pairwise correlation**
$\rho_{AB} = \rho(I_A, I_B)$ where $I_A = \begin{cases} 1 & \text{if obligor A defaults within time horizon} \\ 0 & \text{otherwise} \end{cases}$

$$\rho_{AB} = (\mu_A \mu_B)^{\frac{1}{2}} \sum_{k=1}^{n} \theta_{Ak} \theta_{Bk} \left( \frac{\sigma_k}{\mu_k} \right)^2.$$ 

If obligors $A$ and $B$ have no sector in common, then $\rho_{AB} = 0$. 
Risk contribution

The risk contribution of obligor A having exposure $E_A$ is the marginal effect of the presence of $E_A$ on the standard deviation of the distribution of the portfolio credit loss

$$RC_A = E_A \frac{\partial \sigma_p}{\partial E_A}$$

$$= \frac{E_A \mu_A}{\sigma_p} \left[ E_A + \sum_k \left( \frac{\sigma_k}{\mu_k} \right)^2 \epsilon_k \theta_{Ak} \right]$$
Advantages and limitations of CreditRisk+

- Closed form expressions are derived for the probability of portfolio band/loan losses. Marginal risk contributions by obligor can be easily computed.

- Focuses only on default, requiring relatively few inputs to estimate.

- Assumes no market risk.

- Ignores migration risk so that the exposure for each obligor is fixed and does not depend on eventual changes in credit quality.

- Credit exposures are taken to be constant.
CreditMetrics

- Methodology and dataset
- Costly CreditManager software
- Simulation-based portfolio approach
- Based on probabilities of ratings transitions and correlations of these probabilities
- Generates skewed loss distribution for calculation of expected loss, unexpected loss and risk capital

CreditRisk+

- Methodology
- Implemented in spreadsheet
- Analytic-based portfolio approach
- Based on default rate associated with ratings and the volatility of these rates
- same
Extension I

To incorporate the effects of ratings changes in CreditRisk+ (without the need of Monte Carlo simulation as in CreditMetrics).

- Profits and losses are used as net exposures.
- Default rate corresponds to the migration rate.

Extension II

Correlated credit events such as defaults can be studied and analyzed in a closed form fashion without the need of simulations.

Data set

$N$: The number of different types of exposures. The type of the exposure could be based on the rating of the exposure (A or BB, for example), the sector of the exposure (banking or aerospace, for example), or the geographical region or a combination of all three.

$n_i$: The number of exposures of the $i^{th}$ asset type.

e_{ik}: The dollar amount of the $k^{th}$ exposure of the $i^{th}$ asset type.

$p_i$: The probability of default for assets of $i^{th}$ type.

c_{ij}: The default correlation between exposures of $i^{th}$ type and $j^{th}$ type.

$q_{ij}$: The joint default probability of an exposure of type $i$ and an exposure of type $j$. Given the definitions above, the following identity holds:

$$q_{ij} = p_i p_j + c_{ij} \sqrt{p_i (1 - p_i) p_j (1 - p_j)}.$$
If the portfolio is made up of $N$ asset types, then the loss distribution under correlated defaults can be obtained by combining loss distribution of at most $2N$ scenarios.

References

Review articles

• S. Paul-Choudhury, “Choosing the right box of credit tricks,” *Risk* (Nov. 1997).
