Portfolio Loss Distribution
Risky assets in loan portfolio

- highly illiquid assets
- “hold-to-maturity” in the bank’s balance sheet

Outstandings
The portion of the bank asset that has already been extended to borrowers.

Commitment
A commitment is an amount the bank has committed to lend. Should the borrower encounter financial difficulties, it would draw on this committed line of credit.
Adjusted exposure and expected loss

Let $\alpha$ be the amount of drawn down or usage given default.

\[
\text{Asset value at later time } H, V_H = \text{Outstanding} + \alpha \times \text{commitment, Risky} \left(1-\alpha\right) \times \text{commitment, Riskless}
\]

Adjusted exposure is the risky part of $V_H$.

Expected loss $= \text{adjusted exposure} \times \text{loss given default} \times \text{probability of default}$

* Normally, practitioners treat the uncertain draw-down rate as a known function of the obligor’s end-of-horizon credit class rating.
### Example calculation of expected loss

<table>
<thead>
<tr>
<th>Commitment</th>
<th>$10,000,000</th>
</tr>
</thead>
<tbody>
<tr>
<td>Outstanding</td>
<td>$3,000,000</td>
</tr>
<tr>
<td>Internal risk rating</td>
<td>3</td>
</tr>
<tr>
<td>Maturity</td>
<td>1 year</td>
</tr>
<tr>
<td>Type</td>
<td>Non-secured</td>
</tr>
<tr>
<td>Unused drawn-down on default</td>
<td>65%</td>
</tr>
<tr>
<td>(for internal rating = 3)</td>
<td></td>
</tr>
<tr>
<td>Adjusted exposure on default</td>
<td>$8,250,000</td>
</tr>
<tr>
<td>EDF for internal rating = 3</td>
<td>0.15%</td>
</tr>
<tr>
<td>Loss given default for non-secured asset</td>
<td>50%</td>
</tr>
<tr>
<td>Expected loss</td>
<td>$6,188</td>
</tr>
</tbody>
</table>
Unexpected loss

Unexpected loss is the estimated volatility of the potential loss in value of the asset around its expected loss.

\[ UL = AE \times \sqrt{EDF \times \sigma^2_{LGD} + LGD^2 \times \sigma^2_{EDF}} \]

where

\[ \sigma^2_{EDF} = EDF \times (1 - EDF). \]

**Assumptions**

* The random risk factors contributing to an obligor’s default (resulting in EDF) are statistically independent of the severity of loss (as given by LGD).

* The default process is two-state event.
Example on unexpected loss calculation

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Adjusted exposure</td>
<td>$8,250,000</td>
</tr>
<tr>
<td>EDF</td>
<td>0.15%</td>
</tr>
<tr>
<td>$\sigma_{EDF}$</td>
<td>3.87%</td>
</tr>
<tr>
<td>LGD</td>
<td>50%</td>
</tr>
<tr>
<td>$\sigma_{LGD}$</td>
<td>25%</td>
</tr>
<tr>
<td>Unexpected loss</td>
<td>$178,511</td>
</tr>
</tbody>
</table>

* The calculated unexpected loss is 2.16% of the adjusted exposure, while the expected loss is only 0.075%.
Comparison between expected loss and unexpected loss

* The higher the recovery rate (lower LGD), the lower is the percentage loss for both EL and UL.
* EL increases linearly with decreasing credit quality (with increasing EDF)
* UL increases much faster than EL with increasing EDF.
Assets with varying terms of maturity

* The longer the term to maturity, the greater the variation in asset value due to changes in credit quality.

* The two-state default process paradigm inherently ignores the credit losses associated with defaults that occur beyond the analysis horizon.

* To mitigate some of the maturity effect, banks commonly adjust a risky asset’s internal credit class rating in accordance with its terms to maturity.
Portfolio expected loss

\[ EL_p = \sum_i EL_i = \sum_i AE_i \times LGD_i \times EDF_i \]

where \( EL_p \) is the expected loss for the portfolio,
\( AE_i \) is the risky portion of the terminal value of the \( i \)th asset
to which the bank is exposed in the event of default.

We may write

\[ \frac{EL_p}{AE_p} = \sum_i w_i \frac{EL_i}{AE_i} \]

where the weights refer to

\[ w_i = \frac{AE_i}{\sum_i AE_i} = \frac{AE_i}{AE_p}. \]
\[
\frac{EL_p}{AE_p} = \sum \frac{EL_i}{AE_i} = \sum \frac{AE_i}{AE_p} \frac{EL_i}{AE_i} = w_i \frac{EL_i}{AE_i}
\]

<table>
<thead>
<tr>
<th>(i)</th>
<th>(AE_i)</th>
<th>(w_i)</th>
<th>(EL_i)</th>
<th>(EL_i/AE_i)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$10M$</td>
<td>0.5</td>
<td>$1$</td>
<td>0.1</td>
</tr>
<tr>
<td>2</td>
<td>$4M$</td>
<td>0.2</td>
<td>$0.5$</td>
<td>0.125</td>
</tr>
<tr>
<td>3</td>
<td>$6M$</td>
<td>0.3</td>
<td>$0.6$</td>
<td>0.1</td>
</tr>
</tbody>
</table>

\[\sum AE_i = $20M \quad \sum w_i = 1\]

\[
\frac{EL_p}{AE_p} = 0.5 \times 0.1 + 0.2 \times 0.125 + 0.3 \times 0.1 = 0.105
\]
Portfolio unexpected loss

portfolio unexpected loss  = $\text{UL}_p = \sqrt{\sum_i \sum_j \rho_{ij} w_i w_j \text{UL}_i \text{UL}_j}$

where

$$\text{UL}_i = \text{AE}_i \times \sqrt{\text{EDF}_i \times \sigma_{\text{LGD}_i}^2 + \text{LGD}_i^2 \times \sigma_{\text{EDF}_i}^2}$$

and $\rho_{ij}$ is the correlation of default between asset $i$ and asset $j$. Due to diversification effect, we expect

$$\text{UL}_p << \sum_i \text{UL}_i.$$
Risk contribution

The risk contribution of a risky asset $i$ to the portfolio unexpected loss is defined to be the *incremental risk* that the exposure of a single asset contributes to the portfolio’s total risk.

$$RC_i = UL_i \left( \frac{\partial UL_p}{\partial UL_i} \right)$$

and it can be shown that

$$RC_i = UL_i \sum_j UL_j \rho_{ij} \frac{1}{UL_p}.$$
Undiversifiable risk

The risk contribution is a measure of the *undiversifiable risk* of an asset in the portfolio – the amount of credit risk which cannot be diversified away by placing the asset in the portfolio.

\[
UL_p = \sum_i RC_i
\]

To incorporate industry correlation, using \( i \rightarrow \text{industry } \alpha \) and \( j \rightarrow \text{industry } \beta \)

\[
RC_i = \frac{UL_i}{UL_p} \left[ UL_{i\in\alpha} (1 - \rho_{\alpha\alpha}) + \sum_{\beta \not= \alpha} \left( \sum_{k \in \beta} UL_k \right) \rho_{\alpha\beta} \right].
\]
Calculation of EL, UL and RC for a two-asset portfolio

\( \rho \) default correlation between the two exposures

\( \text{EL}_p \) portfolio expected loss

\( \text{EL}_p = \text{EL}_1 + \text{EL}_2 \)

\( \text{UL}_p \) portfolio unexpected loss

\[
\text{UL}_p = \sqrt{\text{UL}_1^2 + \text{UL}_2^2 + 2 \rho \text{UL}_1 \text{UL}_2}
\]

\( \text{RC}_1 \) risk contribution from Exposure 1

\[
\text{RC}_1 = \frac{\text{UL}_1 (\text{UL}_1 + \rho \text{UL}_2)}{\text{UL}_p}
\]

\( \text{RC}_2 \) risk contribution from Exposure 2

\[
\text{RC}_2 = \frac{\text{UL}_2 (\text{UL}_2 + \rho \text{UL}_1)}{\text{UL}_p}
\]

\[
\text{UL}_p = \text{RC}_1 + \text{RC}_2
\]

\[
\text{UL}_p \ll \text{UL}_1 + \text{UL}_2
\]
Fitting of loss distribution

The two statistical measures about the credit portfolio are

- portfolio expected loss;
- portfolio unexpected loss.

At the simplest level, the *beta distribution* may be chosen to fit the portfolio loss distribution.

*Reservation* A beta distribution with only two degrees of freedom is perhaps insufficient to give an adequate description of the tail events in the loss distribution.
Beta distribution

The density function of a beta distribution is

\[
F(x, \alpha, \beta) = \begin{cases} 
\frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1} (1 - x)^{\beta-1}, & 0 < x < 1 \\
0 & \text{otherwise} \\
\end{cases} 
\alpha > 0, \beta > 0
\]

Mean \( \mu = \frac{\alpha}{\alpha + \beta} \) and variance \( \sigma^2 = \frac{\alpha\beta}{(\alpha + \beta)^2(\alpha + \beta + 1)} \).
Economic Capital

If $X_T$ is the random variable for loss and $z$ is the percentage probability (confidence level), what is the quantity $v$ of minimum economic capital EC needed to protect the bank from insolvency at the time horizon $T$ such that

$$\Pr[X_T \leq v] = z.$$

Here, $z$ is the desired debt rating of the bank, say, 99.97% for an AA rating.
Capital multiplier

Given a desired level of $z$, what is $EC$ such that

$$\Pr[X_T - \text{EL}_p \leq EC] = z.$$ 

Let CM (capital multiplier) be defined by

$$EC = CM \times UL_p$$

then

$$\Pr\left[ \frac{X_T - \text{EL}_p}{UL_p} \leq CM \right] = z.$$
Monte Carol simulation of loss distribution of a portfolio

<table>
<thead>
<tr>
<th>1. Estimate default and losses</th>
<th>2. Estimate asset correlation between obligors</th>
</tr>
</thead>
<tbody>
<tr>
<td>Assign risk ratings to loss facilities and determine their default probability</td>
<td>Determine pairwise asset correlation whenever possible</td>
</tr>
<tr>
<td>+ Assign LGD and $\sigma_{LGD}$</td>
<td>OR Assign obligors to industry groupings, then determine industry pair correlation</td>
</tr>
</tbody>
</table>
3. Generate random loss given default

Determine stochastic loss given default

4. Generate correlated default events

Correlated default events + Decomposition of covariance matrix + Simulate default point
5. **Loss calculation**

Calculate facility loss for each scenario and obtain portfolio loss

6. **Loss distribution**

Construct simulated portfolio loss distribution
Generation of correlated default events

- Generate a set of random numbers drawn from a standard normal distribution.

- Perform a decomposition (Cholesky, SVD or eigenvalue) on the asset correlation matrix to transform the independent set of random numbers (stored in the vector $e$) into a set of correlated asset values (stored in the vector $e'$). Here, the transformation matrix is $M$, where

  \[ e' = M e. \]

  The covariance matrix $\Sigma$ and $M$ are related by

  \[ M^T M = \Sigma. \]
Calculation of the default point

The default point threshold, $DP$, of the $i^{th}$ obligor can be defined as $DP = N^{-1}(EDF_i, 0, 1)$. The criterion of default for the $i^{th}$ obligor is

\[
\begin{align*}
\text{default} & \quad \text{if} \quad e_i' < DP_i \\
\text{no default} & \quad \text{if} \quad e_i' \geq DP_i.
\end{align*}
\]
# Generate loss given default

The LGD is a stochastic variable with an unknown distribution.

A typical example may be

<table>
<thead>
<tr>
<th>Recovery rate (%)</th>
<th>LGD (%)</th>
<th>$\sigma_{\text{LGD}}$ (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>secured</td>
<td>65</td>
<td>35</td>
</tr>
<tr>
<td>unsecured</td>
<td>50</td>
<td>50</td>
</tr>
</tbody>
</table>

\[
\text{LGD}_i = \text{LGD}_s + f_i \times \sigma^s_{\text{LGD}}
\]

where $f_i$ is drawn from a uniform distribution whose range is selected so that the resulting LGD has a standard deviation that is consistent with historical observation.
Calculation of loss

Summing all the simulated losses from one single scenario

\[
\text{Loss} = \sum_{\text{Obligors in default}} \text{Adjusted exposure}_i \times \text{LGD}
\]

Simulated loss distribution

The simulated loss distribution is obtained by repeating the above process sufficiently number of times.
Features of portfolio risk

- The variability of default risk within a portfolio is substantial.
- The correlation between default risks is generally low.
- The default risk itself is dynamic and subject to large fluctuations.
- Default risks can be effectively managed through diversification.
- Within a well-diversified portfolio, the loss behavior is characterized by lower than expected default credit losses for much of the time, but very large losses which are incurred infrequently.