Jarrow-Lando-Turnbull model
Characteristics

- Credit rating dynamics is represented by a Markov chain.
- Default is modelled as the first time a continuous time Markov chain with $K$ states hitting the absorbing state $K$ (default state).
- LGD is characterized as a fraction of an otherwise similar default-free claim.
Markov chain model

- To describe the dynamics of bond credit ratings

Let $X_t$ represent the credit rating at time $t$ of a bond, and $X = \{X_t, t = 0, 1, 2, \ldots\}$ is a time-homogeneous Markov chain on the state space $N = \{1, 2, \ldots, K, K + 1\}$, $K + 1$ designates default (absorbing state)

$$Q = \begin{pmatrix} q_{11} & \cdots & q_{1k} & q_{1,K+1} \\
\vdots & \ddots & \vdots & \vdots \\
q_{k1} & q_{kK} & q_{K,K+1} \\
0 & \cdots & 0 & 1 \end{pmatrix}$$

$q_{ij} = \Pr[X_{t+1} = j | X_t = i], \ i, j \in N, t = 0, 1, 2, \ldots$
Assumptions on the risk premia

The credit rating process $\tilde{X}$ after risk neutralization is not necessarily Markovian

$$\tilde{q}_{ij}(t, t+1) = \pi_{ij}(t)q_{ij}$$

where $q_{ij}$ is the actual transitional probabilities of the observed time-homogeneous Markov chain $X$,

$$\pi_{ij}(t)$$ are the risk premium adjustments.

JLT model assumes $\pi_{ij}(t) = \pi_i(t)$ for $j \neq i$, and they are deterministic functions of $t$. Some structure is imposed to improve analytical tractability.

The assumption is imposed to facilitate statistical information since the historical $q_{ij}$ can be used in the inference process.
Risk neutralized transition matrix

Risk neutralized process $\tilde{X}$ becomes a non-homogeneous Markov chain with transitional probabilities

$$\tilde{q}_{ij}(t, t+1) = \begin{cases} 
\pi_i(t)q_{ij} & i \neq j \\
1 - \pi_i(t)(1 - q_{ij}) & i = j 
\end{cases}$$

- $\tilde{X}$ and $\{r(t)\}$ (spot rate process) are assumed to be mutually independent under the risk neutral measure (accuracy may deteriorate for speculative-grade bonds).
Price of risky discount bonds

Risky discount bond in the $j^{th}$ credit rating class

$$
\nu_j(t,T) = \tilde{E}_t \left\{ e^{-\int_r^T dr(s) ds} \left[ 1_{\{\tau_j > T\}} + \delta 1_{\{\tau_j \leq T\}} \right] \right\}
$$

$$
= \tilde{E}_t \left[ e^{-\int_r^T dr(s) ds} \tilde{E}_t \left[ 1_{\{\tau_j > T\}} + \delta 1_{\{\tau_j \leq T\}} \right] \right] \text{ (independence)}
$$

$$
= \nu_0(t,T) \left[ \delta + (1 - \delta) \tilde{P}_t \{\tau_j > T\} \right]
$$

where $\tau_j$ is the absorption (default time) of $\tilde{X}$ when $\tilde{X}_t = j$.

$$
\tilde{P}_t \{\tau_j > T\} = \sum_{k=1}^K \tilde{q}_{j,k}(t,T) = 1 - \tilde{q}_{j,K+1}(t,T).
$$
Numerical implementation of Jarrow-Lamdo-Turnbull model

- modelling default and credit migration in preference to modelling recovery rate

\[
\begin{align*}
I & = \begin{pmatrix} 0.90 & 0.05 & 0.05 \end{pmatrix} \\
d & = J \begin{pmatrix} 0.10 & 0.80 & 0.10 \end{pmatrix} \\
D & = \begin{pmatrix} 0 & 0 & 1.00 \end{pmatrix}
\end{align*}
\]

\( I = \text{investment grade, } J = \text{junk grade and } D = \text{default (absorbing)} \)

\[
\begin{pmatrix} r_{01} \\ r_{02} \end{pmatrix} = \begin{pmatrix} 0.08 \\ 0.09 \end{pmatrix}, \quad \begin{pmatrix} s_{I,01} \\ s_{I,02} \end{pmatrix} = \begin{pmatrix} 0.01 \\ 0.015 \end{pmatrix}, \quad \begin{pmatrix} s_{J,01} \\ s_{J,02} \end{pmatrix} = \begin{pmatrix} 0.02 \\ 0.03 \end{pmatrix}
\]

- assume that there is no correlation between rating migration and interest rate
Pricing of risky debt of maturity one and two periods are

\[ B_I(0,1) = \frac{1}{1.09}, \quad B_I(0,2) = \frac{1}{1.105^2}, \quad B_J(0,1) = \frac{1}{1.10}, \quad B_J(0,2) = \frac{1}{1.12^2}. \]

Assume recovery rate \( \phi = 0.40 \) so that payoff vector \( \begin{pmatrix} 1 \\ \phi \end{pmatrix} \).

Suppose currently at state \( I \),

\[ d_I = \begin{pmatrix} 0.90 \\ 0.05 \\ 0.05 \end{pmatrix}. \]

Transform \( d_I \) into the risk neutral vector \( q_I \) with adjustment \( \pi_I \)

\[ q_I = \begin{pmatrix} 1 - 0.10\pi_I \\ 0.05\pi_I \\ 0.05\pi_I \end{pmatrix}. \]
We find $\pi_I$ by making the expected value of discounted cash flow equal to the traded price of bond

$$B_I(1,2) = \frac{1}{1+r_{0I}} C^T q_I$$

$$\frac{1}{1.09} = \frac{1}{1.08} (1 \ 1 \ 0.4) \begin{pmatrix} 1-0.10\pi_I \\ 0.05\pi_I \\ 0.05\pi_I \end{pmatrix}$$ giving $\pi_I = 0.30581$.

Similarly,

$$B_J(1,2) = \frac{1}{1+r_{0J}} C^T q_J$$

$$\frac{1}{1.10} = \frac{1}{1.08} (1 \ 1 \ 0.4) \begin{pmatrix} 1-0.10\pi_J \\ 0.20\pi_{JJ} \\ 0.10\pi_{JJ} \end{pmatrix}$$ giving $\pi_J = 0.30303$.

Risky neutral transition matrix

$$I = \begin{pmatrix} 0.9694 & 0.0153 & 0.0153 \\ 0.0303 & 0.9394 & 0.0303 \\ 0 & 0 & 1.00 \end{pmatrix}$$
Duffie-Singleton model
Formulation in Duffie-Singleton model

- Treat default as an unpredictable event involving a sudden loss in market value.

- Default is assumed to occur at a risk-neutral hazard rate $h_t$; default over time $\Delta t$, given no default before time $t$, is approximately $h_t \Delta t$.

\[
V_{\text{risky}} = \tilde{E}_0 \left[ \exp \left( - \int_0^t R_s ds \right) X \right]
\]

where $\tilde{E}_t$ denotes risk-neutral expectation, $R_t$ is the default-adjusted short-rate process, $X$ is the value of contingent claim at maturity.
**Default-adjusted short-rate process**

\[ R_t = r_t + h_t L_t + \ell_t \]

where \( r_t \) is the default-free short rate,

\( L_t \) is the *fractional loss* given default,

\( \ell_t \) represents the *fractional carrying costs* of the defaultable claim (liquidity premium can be included),

\( h_t \) is the *arrival intensity* at time \( t \) (under risk neutral process)

of a Poisson process whose first jump occurs at default.
Difficulties in parameter estimation

In order to disentangle the separate contributions of $h$ and $L$, one would need additional data on
(i) default recovery values,
(ii) frequency of default of bonds of a given class.

- The exogeneity of $h$ and $L$ can misspecify some contractual features in some cases.

- The expected loss rate can switch from one regime to another in swap contracts depending on the reciprocal “moneyness” and “non-moneyness” of both counterparties through time.
Default probability density and hazard rate

\( \tau^* \) – random variable representing the default time of a credit event

\[ Q(t) = P[\tau^* \leq t] = \text{probability distribution function of } \tau^* \]

\[ q(t) = \frac{dQ}{dt} = P[t < \tau^* < t + dt]. \]

\( q(t) \) is not the same as the hazard (default intensity) rate, \( h(t) \).

Indeed, \( h(t) \ dt \) is the probability of default between time \( t \) and \( t + dt \) as seen as time \( t \), assuming no default between time zero and time \( t \).
Define $G(t) = \text{survival function } = P[\tau > t] = 1 - Q(t)$, then $h(t) = \frac{G'(t)}{G(t)}$ and $G(0) = 1$.

Solving for $G(t)$, we obtain

$$G(t) = \exp\left( - \int_0^t h(s) \, ds \right)$$

Default probability density and hazard rate function are related by

$$q(t) = -G'(t) = h(t) \exp\left( - \int_0^t h(s) \, ds \right)$$
Valuation of risky bonds

\[ v_i(t, T) = \text{value of a defaultable zero-coupon bond of a firm that} \]
\[ \text{currently has credit rating } i \text{ at time } t, \text{ maturing at } T \]

\[ v_i(t, T) = P(t, T)[\phi + (1 - \phi)q_i(t, T)] \]

where \( \phi = \text{recovery ratio} \)

\[ q_i(t, T) = \text{probability of a default occuring after } T, \text{ given that} \]
\[ \text{the debt has credit rating } i \text{ as of time } t. \]

\[ P(t, T) = \text{value of a default-free zero-coupon bond} \]
Madan-Unal model
Formulation of Madan-Unal model

- Decomposes risky debt into two embedded securities:
  - *survival security*: paying a dollar if there is no default and nothing otherwise
  - *default security*: paying the rate of recovery in bankruptcy if default occurs and nothing otherwise
Evaluation of payoffs

- Default occurs with probability $\phi_1$.
- Recovery conditional on default is high ($H_1$) or low ($L_1$) with probabilities $q_{H1}$ and $q_{L1}$.
- If there is no default in the first period, then second period outcomes depend on the evolution of firm specific information $x$ and interest rates $r$.

\[
\begin{align*}
V & \\
1 - \phi_1 & \\
\phi_1 & \\
H_1 & \\
L_1 & \\
q_{up} & (x_{2up}, r_{2up}) \\
q_{un} & (x_{2un}, r_{2un}) \\
q_{dp} & (x_{2dp}, r_{2dp}) \\
q_{dn} & (x_{2dn}, r_{2dn}) \end{align*}
\]
Risk of recovery in default

Use the option components of junior and senior debt to extract information on the *default payout distribution* from the market prices of these debt instruments.

- Merton model – *predetermined* distribution given by the shortfall of firm value relative to the promised payment.

- Most other models assume a *constant* payout rate conditional on default.
**Structural models require**

1. Issuer's asset value process and issuer's capital structure
2. Loss given default; terms and conditions of the debt issue
3. Default-risk-free interest rate process
4. Correlation between the default-risk-free interest rate and the asset price

**Reduced form models require**

1. Issuer's default (bankruptcy) process
2. Loss given default (can be specified as a stochastic process)
3. Default-risk-free interest rate process
4. Correlation between the default-risk-free interest rate and the asset price

*Different models measure the same risk but*

- impose different restrictions and distributional assumptions
- different techniques for calibration and solution.