Optimal Shouting Policies of Options with Shouting Rights

presented by

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* Joint work with Min Dai and Lixin Wu
The holders are allowed to **reset** certain terms of the derivative contract according to some *pre-specified rules*.

Terms that are resettable
- strike prices
- maturities

*Voluntary* (shouting) or *automatic* resets; other constraints on resets may apply.
Examples

1. *S & P 500 index bear market put warrants* with a 3-month reset (started to trade in 1996 in CBOE and NYSE)
   - original exercise price of the warrant, $X$
     = closing index level on issue date
   - exercise price is reset at the closing index level $S_t$ on the reset date if $S_t > X$ (automatic reset)

Reset-strike warrants are available in Hong Kong and Taiwan markets.
2. *Reset feature in Japanese convertible bonds*

- reset downward on the conversion price

**Sumitomo Banks 0.75% due 2001**

<table>
<thead>
<tr>
<th><strong>Issue date</strong></th>
<th>May 96</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>First reset date</strong></td>
<td>31 May 1997</td>
</tr>
<tr>
<td><strong>Annual reset date thereafter</strong></td>
<td>31 May</td>
</tr>
<tr>
<td><strong>Reset calculation period</strong></td>
<td>20 business day period, excluding holidays in Japan, ending 15 trading days before the reset day</td>
</tr>
<tr>
<td><strong>Calculation type</strong></td>
<td>Simple average over calculation period</td>
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3. *Executive stock options*
   - resetting the strike price and maturity

4. *Corporate debts*
   - strong incentive for debtholders to extend the maturity of a defaulting debt if there are liquidation costs

5. *Canadian segregated fund*
   - Guarantee on the return of the fund (protective floor); guarantee level is simply the strike price of the embedded put.
   - Two opportunities to reset per year (at any time in the year) for 10 years. Multiple resets may involve sequentially reduced guarantee levels.
   - Resets may require certain fees.
Objectives of our work

Examine the *optimal shouting policies* of options with *voluntary* reset rights.

*Free boundary value problems*

- Critical asset price level to shout;
- Characterization of the optimal shouting boundary for one-shout and multi-shout models (analytic formulas, numerical calculations and theoretical analyses)
**Resettable put option**
The strike price is reset to be the prevailing asset price at the shouting moment chosen by the holder.

Terminal payoff = \[
\begin{cases}
\max(X - S_T, 0) & \text{if no shout occurs} \\
\max(S_t - S_T, 0) & \text{if shouting occurs at time } t
\end{cases}
\]

**Shout call option**
Terminal payoff = \[
\begin{cases}
\max(S_T - X, S_t - X) & \text{if shouting occurs at time } t, S_t > X \\
\max(S_T - X, 0) & \text{if no shout occurs}
\end{cases}
\]

**Shout floor** (protective floor is not set at inception)
Shout to install a protective floor on the return of the asset.

Terminal payoff = \[
\begin{cases}
\max(S_t - S_T, 0) & \text{if shouting occurs at time } t \\
0 & \text{if no shout occurs}
\end{cases}
\]
Relation between the resettable put option and the shout call option

Since
\[
\begin{cases}
\max(S_T - X, S_t - X) = (S_T - X) + \max(X - S_T,0) \\
\max(S_T - X,0) = \max(S_t - S_T,0)
\end{cases}
\]

so

price of one-shout shout-call option
\[= \text{price of one-shout resettable put option} + \text{forward contract}.\]

- Both options share the same optimal shouting policy.
- Same conclusion applied to multi-shout options.
Formulation as free boundary value problems

- Both one-shout resettable put option and one-shout shout floor become an at-the-money put option upon shouting.
- Price function of an at-the-money put option is $SP_1(\tau)$, where

$$P_1(\tau) = e^{-r\tau} N(-d_2) - e^{-q\tau} N(-d_1)$$

where

$$d_1 = \frac{r - q + \frac{\sigma^2}{2}}{\sigma} \sqrt{\tau} \quad \text{and} \quad d_2 = d_1 - \sigma \sqrt{\tau}.$$ 

Linear complementarity formulation of the pricing function $V(S, \tau)$

$$\frac{\partial V}{\partial \tau} - \frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} - (r - q) S \frac{\partial V}{\partial S} + r V \geq 0,$$

$$V(S, \tau) \geq SP_1(\tau),$$

$$\left[ \frac{\partial V}{\partial \tau} - \frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} - (r - q) S \frac{\partial V}{\partial S} + r V \right] [V - SP_1(\tau)] = 0$$

$$V(S,0) = \begin{cases} \max(X - S,0) & \text{for resettable put} \\ 0 & \text{for shout floor} \end{cases}.$$
Properties of $P_1(\tau)$

(i) If $r \leq q$, then
\[
\frac{d}{d\tau} \left[ e^{q\tau} P_1(\tau) \right] > 0 \quad \text{for} \quad \tau \in (0, \infty).
\]

(ii) If $r > q$, then there exists a unique critical value $\tau_1^* \in (0, \infty)$ such that
\[
\left. \frac{d}{d\tau} \left[ e^{q\tau} P_1(\tau) \right] \right|_{\tau = \tau_1^*} = 0,
\]
and
\[
\frac{d}{d\tau} \left[ e^{q\tau} P_1(\tau) \right] > 0 \quad \text{for} \quad \tau \in (0, \tau_1^*)
\]
\[
\frac{d}{d\tau} \left[ e^{q\tau} P_1(\tau) \right] < 0 \quad \text{for} \quad \tau \in (\tau_1^*, \infty)
\]
Price of the one-shout shout floor, \( R_1(S, \tau) \)

\[
R_1(S, \tau) = S \ g(\tau)
\]

Substituting into the linear complementarity formulation:

\[
\frac{d}{d\tau} \left[ e^{q\tau} q(\tau) \right] \geq 0, \quad g(\tau) \geq P_1(\tau)
\]

\[
\frac{d}{d\tau} \left[ e^{q\tau} g(\tau) \right] \left[ g(\tau) - P_1(\tau) \right] = 0
\]

\[
g(0) = 0.
\]

(i) \( r \leq q \)

\[
\frac{d}{d\tau} \left[ e^{q\tau} P_1(\tau) \right] > 0 \quad \text{for} \quad \tau > 0 \quad \text{and} \quad P_1(0) = 0;
\]

so

\[
g(\tau) = P_1(\tau), \quad \tau \in (0, \infty).
\]
(ii) \( r > q \)

\[ g(\tau) = P_1(\tau) \quad \text{for} \quad \tau \in (0, \tau^*_1]. \]

When \( \tau > \tau^*_1 \), we cannot have \( g(\tau) = P_1(\tau) \) since this leads to

\[ \frac{d}{d\tau}\left[e^{q\tau} g(\tau)\right] = \frac{d}{d\tau}\left[e^{q\tau} - P_1(\tau)\right] = 0, \]

a contradiction. Hence,

\[ \frac{d}{d\tau}\left[e^{q\tau} g(\tau)\right] = 0 \quad \text{for} \quad \tau \in (\tau^*_1, \infty). \]

Solving

\[ g(\tau) = e^{-q(\tau - \tau^*_1)}P_1(\tau^*_1) \quad \text{for} \quad \tau \in (\tau^*_1, \infty). \]
Optimal shouting policy of the shout floor

- does not depend on the asset price level (due to linear homogeneity in $S$)

- when \( \frac{d}{d\tau} \left[ e^{q\tau} P_1(\tau) \right] \geq 0, R_1(S, \tau) = SP_1(\tau), \) inferring that the holder should shout at once.

Summary
(i) \( r \leq q, \tau \in (0, \infty) \) or (ii) \( r > q, \tau \leq \tau^*_1 \)

holder should shout at once

- \( r > q, \tau \leq \tau^*_1 \)

holder should not shout at any asset price level.
Optimal shouting boundary for the resettable put option

Asymptotic behavior of \( S^*_1(\tau) \) as \( \tau \to 0^+ \)

\[ S^*_1(0^+) = X, \text{ independent of the ratio of } r \text{ and } q. \]

**Proof**

\[
D_1(S, \tau) = V_1(S, \tau) - SP_1(\tau)
\]

\[
\frac{\partial D_1}{\partial \tau} - \frac{\sigma^2}{2} S^2 \frac{\partial^2 D_1}{\partial S^2} - (r - q) S \frac{\partial D_1}{\partial S} + rD_1 = -S \left[ P_1'(\tau) + qP_1(\tau) \right]
\]

\[
0 < S < S^*_1(\tau), \ \tau > 0
\]

\[
D_1(0, \tau) = X e^{-r\tau} \text{ and } P_1(S^*_1(\tau), \tau) = 0
\]

\[
\frac{\partial D_1}{\partial S}(S^*_1(\tau), \tau) = 0, \quad D_1(S,0) = \max(X - S,0).
\]

Note that \( D_1(S, \tau) \geq 0 \) for all \( \tau; -S[P'(\tau) + qP_1(\tau)] \to -\infty \) as \( \tau \to 0^+ \).

Assuming \( S^*_1(0^+) > X \), then for \( S \in (X, S^*_1(0^+)) \), we have

\[
\frac{\partial D_1}{\partial \tau}(S,0^+) = -S \left[ P_1'(\tau) + qP_1(0) \right] < 0, \text{ a contradiction.}
\]

Financial intuition dictates that \( S^*_1(0^+) \geq X. \)
Asymptotic behaviour of $S_1^*(\tau)$ as $\tau \to \infty$

$S_{1,\infty}^*$ exists when $r < q$; this is linked with the existence of the limit:

$$\lim_{\tau \to \infty} e^{\epsilon \tau} P_1(\tau) = 1 \quad \text{for } r < q.$$ 

Write $W_1(S, \tau) = e^{\epsilon \tau} V(S, \tau); \quad W_1^\infty(S) = \lim_{\tau \to \infty} W_1(S, \tau)$.

$$\frac{\sigma^2}{2} S^2 \frac{d^2 W_1^\infty}{dS^2} + (r - q) S \frac{dW_1^\infty}{dS} = 0, \quad 0 < S < S_{1,\infty}^*$$

$W_1^\infty(0) = X, \quad W_1^\infty(S_{1,\infty}^*) = S_{1,\infty}^*$,

$$\frac{dW_1^\infty}{dS}(S_{1,\infty}^*) = 1.$$ 

$$W_1^\infty(S) = X + \frac{\alpha^\alpha}{(1 + \alpha)^{1+\alpha}} X^{-\alpha} S^{1+\alpha}, \quad 0 < S < S_{1,\infty}^*,$$

where $S_{1,\infty}^* = \left(1 + \frac{1}{\alpha}\right) X$ and $\alpha = \frac{2(q - r)}{\sigma^2} (\alpha$ becomes zero when $r = q)$. 
When \( r > q, V_1(S, \tau) \geq R_1(S, \tau) > SP_1(\tau) \) for \( \tau > \tau^*_1 \).

It is never optimal to shout at \( \tau > \tau^*_1 \).

**Lemma** For \( r > q \) and \( \tau_0 < \tau^*_1 \), there exists a critical asset price \( S^*_1(\tau_0) \) such that \( V_1(S, \tau_0) = SP_1(\tau_0) \) for \( S \geq S^*(\tau_0) \).

**Integral equation for \( S^*_1(\tau) \)**

\[
S^*_1(\tau)P_1(\tau) = P_E(S^*_1(\tau), \tau) + S^*_1(\tau)e^{-q\tau} \int_0^\tau N(d^*_1,\tau-u) \frac{d}{du}[e^{qu}P_1(u)] \, du
\]

where

\[
d^*_1,\tau-u = \frac{\ln \frac{S^*_1(\tau)}{S^*_1(u)} + (r - q + \frac{\sigma^2}{2})(\tau - u)}{\sigma \sqrt{\tau - u}}.
\]
Pricing formulation of the $n$-shout resettable put option

Terminal payoff $= \max(S_{t_\ell} - S_{T,0})$, where $t_\ell$ is the last shouting instant, $0 \leq \ell \leq n$.

Define $P_n(\tau) = V_{n-1}(1, \tau; X = 1)$, then

$$\frac{\partial V_n}{\partial \tau} - \frac{\sigma^2}{2} S^2 \frac{\partial^2 V_n}{\partial S^2} - (r - q) S \frac{\partial V_n}{\partial S} + r V_n \geq 0, \quad V_n(S, \tau) \geq S P_n(\tau),$$

$$\left[ \frac{\partial V_n}{\partial \tau} - \frac{\sigma^2}{2} S^2 \frac{\partial^2 V_n}{\partial S^2} - (r - q) S \frac{\partial V_n}{\partial S} + r V \right] [V_n - S P_n(\tau)] = 0$$

$V_n(S,0) = \max(X - S,0)$.

* The analytic form for $P_n(\tau), n > 1$. 

Properties of the price function and optimal shouting boundaries

(i) \( r < q \), \( S^*_n(\tau) \) is defined for \( \tau \in (0, \infty) \)

\[
S^*_n(\tau) < S^*_n(\tau), \quad n = 1, 2
\]

\[
S^*_n(0^+) = X
\]

\( S^*_n(\tau) \) is an increasing function of \( \tau \) with a finite asymptotic value as \( \tau \to \infty \).

\[
S^*_{2,\infty} = \frac{1 + \frac{1}{\alpha}}{1 + \frac{\alpha^\alpha}{(1+\alpha)^{1+\alpha}}}, \quad \text{and} \quad S^*_{3,\infty} = \frac{1 + \frac{1}{\alpha}}{1 + \frac{\alpha^\alpha}{(1+\alpha)^{1+\alpha}} \left[ 1 + \frac{\alpha^\alpha}{(1+\alpha)^{1+\alpha}} \right]}. 
\]

(ii) \( r < q \), \( S^*_n(\tau) \) is defined only for \( \tau \in (0, \tau^*_n) \), where \( \tau^*_n \) is given by the solution to

\[
\frac{d}{d\tau} \left[ e^{q\tau} P_n(\tau) \right] = 0.
\]
Summaries and conclusions

- The behaviors of the optimal shouting boundaries of the resettable put options depends on $r > q$, $r = q$ or $r < q$.

- Monotonic properties
  (i) an option with more shouting rights outstanding should have higher value;
  (ii) the holder shouts at a lower critical asset price with more shouting rights outstanding;
  (iii) the holder chooses to shout at a lower critical asset price for a shorter-lived option;
  (iv) the critical time earlier than which it is never optimal to shout increases with more shouting rights outstanding.

- Analytic price formula of the one-shout shout floor and integral representation of the shouting premium of the one-shout resettable put are obtained.