Default Correlation
Correlation of default

“If Obligor A’s credit quality deteriorates, how well does the credit quality of Obligor B correlate to Obligor A?”

Some empirical observations are

- Default correlations are general low though they decrease as ratings increase.
- Default correlations generally increase initially with time and then decrease as the horizon extends longer.
- Default among and between specific industries are inconclusive.
Formula for correlation of default

\[ \rho_{ij} = \frac{P(D_i D_j) - \text{EDF}_i \times \text{EDF}_j}{\sqrt{\text{EDF}_i(1 - \text{EDF}_i)} \sqrt{\text{EDF}_j(1 - \text{EDF}_j)}} \]

where the joint probability of default

\[ P(D_i D_j) = \text{EDF}_i + \text{EDF}_j - P(D_i + D_j) \]

and \( P(D_i + D_j) \) is the probability that at least one obligor has defaulted.
\[ P(D_1D_2) = EDF_1 \times EDF_2 + \rho_{12} \sqrt{EDF_1(1 - EDF_1)} \sqrt{EDF_2(1 - EDF_2)} \]

**Example**

Suppose \( EDF_1 = 5\% \) (firm 1 has a 5\% probability of default), and \( EDF_2 = 1\% \).

(i) If \( \rho_{12} = 0 \) (independence of defaults), then

\[
P(D_1D_2) = \text{joint default probability of both firms}
= 5\% \times 1\% = 0.05\%.
\]

(ii) If \( \rho_{12} = 0.2 \), then \( P(D_1D_2) = 0.48\% \).

This is almost 10 times as the case of \( \rho_{12} = 0 \).
Evaluation of a letter of credit (LOC) - backed debt

- Current rating agency is to rate LOC-backed debt with the same credit quality as the LOC bank.

— Two failures have to occur before the debtholder experiences a financial loss.

Let $\text{EDF}_{\text{LOC}} = 0.5\%$ and $\text{EDF}_{\text{obligor}} = 2.0\%$.

(i) If $\rho = 0.2$, then default probability of the LOC-backed debt $\approx 0.2\%$.

(ii) If $\rho = 0$, then the joint default probability is only $0.06\%$. 

Estimation of default correlation

1. Use *historical* default data and statistically calculate pairwise correlation of default.

2. Use *Monte Carlo simulation* on a specific choice of risky debt model and compute the covariance structure.

3. Given two obligors’ asset volatilities and their variance – covariance structure, calculate *analytically* their joint probability of default and compute the default correlation.
Difficulties in using historical data in estimating default correlations

1. There are not enough time-series data available to accurately estimate default correlations.

2. Do not use firm-specific information.

3. Default correlations are time-varying, so past history may not reflect the current reality.
Simpler method of calculating correlation of default

Assume that each default probability distribution is normal so that \( P(D_i D_j) \) is jointly bivariate normal.

Let the asset correlation between Asset A and Asset B be \( \rho = 0.19 \); and assume \( \text{EDF}_a = 0.0062 \) and \( \text{EDF}_b = 0.0025 \), asset volatilities are \( \sigma_a = 0.5 \) and \( \sigma_b = 0.6 \).

Joint default probability

\[
JDP = \int_{-\infty}^{D_b} \int_{-\infty}^{D_a} \frac{1}{2\pi\sigma_a\sigma_b\sqrt{1-\rho^2}} \exp \left( -\frac{1}{2(1-\rho^2)} \left[ \left( \frac{x}{\sigma_a} \right)^2 - 2\rho \frac{xy}{\sigma_a\sigma_b} + \left( \frac{y}{\sigma_b} \right)^2 \right] \right) dx \, dy
\]
The upper limits $D_a$ and $D_b$ are given by inverse normals,

$$D_a = -0.75 \quad \text{and} \quad D_b = -1.404,$$

giving $JDP = 6.504 \times 10^{-5}$.

Default correlation between $A$ and $B$

$$= \frac{JDP - EDF_a \times EDF_b}{\sqrt{EDF_a (1 - EDF_a) \times EDF_b (1 - EDF_b)}} = 0.013.$$

The calculated default correlation is significantly smaller than the asset correlation.
Asset based approach for default correlation

First passage time model

\[
\begin{pmatrix}
    d \ln V_1 \\
    d \ln V_2
\end{pmatrix} =
\begin{pmatrix}
    \mu_1 \\
    \mu_2
\end{pmatrix} dt + \Omega
\begin{pmatrix}
    dZ_1 \\
    dZ_2
\end{pmatrix}
\]

\(\mu_1\) and \(\mu_2\) are constant drift terms
\(Z_1\) and \(Z_2\) are two independent standard Brownian motions.

\[
\Omega \Omega' =
\begin{pmatrix}
    \sigma_1^2 & \rho \sigma_1 \sigma_2 \\
    \rho \sigma_1 \sigma_2 & \sigma_2^2
\end{pmatrix}
\]

\(V_i(t) > e^{\lambda_i t} K_i, i = 1, 2\), as the non-default requirements.
Assuming $\lambda_i = \mu_i$ (avoiding the drift terms), Zhou* (2001) obtained

$$P(D_1 + D_2) = 1 - \frac{2r_0}{\sqrt{2\pi t}} e^{-r_0^2/4t} \sum_{n=1,3,5} \frac{1}{n} \sin \frac{n\pi\theta_0}{\alpha}$$

$$\left[ I_{\frac{1}{2}\left(n\pi\alpha^{-1}\right)} \left( \frac{r_0^2}{4t} \right) + I_{\frac{1}{2}\left(n\pi\alpha^{-1}\right)} \left( \frac{r_0^2}{4t} \right) \right]$$

where $I_v(z)$ is the modified Bessel function with order $v$, $Z_i = \frac{b_i}{\sigma_i}$

$$\theta_0 = \begin{cases} 
\tan^{-1}\left( \frac{Z_2 \sqrt{1-\rho^2}}{Z_1 - \rho Z_2} \right) & \text{if } \frac{Z_2 \sqrt{1-\rho^2}}{Z_1 - \rho Z_2} > 0 \\
\pi + \tan^{-1}\left( \frac{Z_2 \sqrt{1-\rho^2}}{Z_1 - \rho Z_2} \right) & \text{otherwise}
\end{cases}$$

$$\alpha = \begin{cases} 
\tan^{-1}\left( -\frac{\sqrt{1-\rho^2}}{\rho} \right) & \text{if } \rho < 0 \\
\pi + \tan^{-1}\left( -\frac{\sqrt{1-\rho^2}}{\rho} \right) & \text{otherwise}
\end{cases}$$

Let \( P(D_i(t)) = P(\tau_i^* \leq t) = 2N\left(-\frac{\ln\frac{V_{i,0}}{K_i}}{\sigma_i \sqrt{t}}\right) \)

\( \rho_D(t) = \) Default correlation between firm 1 and firm 2 over \([0, t]\)
\( = \text{corr}(D_1(t), D_2(t)) \)
\( = \frac{P[(D_1(t)D_2(t)] - P[D_1(t)]P[D_2(t)]}{\sqrt{P(D_1(t))[1 - P(D_1(t))]} \sqrt{P(D_2(t))[1 - P(D_2(t))]} \}

\( P(D_1D_2) = P(D_1) + P(D_2) - P(D_1 + D_2) \)

where \( P(D_1 + D_2) = \) probability that at least one default has occurred.
Dynamic nature of default correlation

1. The default correlation and the underlying asset return correlation have the same sign. In general, the default correlation is lower than the underlying asset return correlation.
   * Firms in the same industry (region) often have higher default correlations.

2. Default correlations are generally very small over short investment horizons. They increase and then slowly decrease with time. The time to reach the peak default correlation depends on the credit quality of the underlying firms.
Relation between default correlation and time for various credit qualities
• Over a short investment horizon, default correlations are low because quick defaults are rare and are almost idiosyncratic (peculiar to that firm).

• Over a long time horizon, the default of a firm is virtually inevitable [non-default events become rare and idiosyncratic].
1. For long-term investments (say, 5 to 10 years), the default correlation can be quite a significant factor if the underlying firm values are highly correlated.

2. The dynamic nature of default correlations requires active management of the portfolio. Accordingly, the capital requirements must be adjusted throughout time. For example, if the default probability of each loan doubles, the probability of multiple defaults in a portfolio may be significantly more than doubled.
Default correlations in different sectors of the US (×100)

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